

# Vector Calculus (H.1)

## Green's Theorem

20151021

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# Green's Theorem

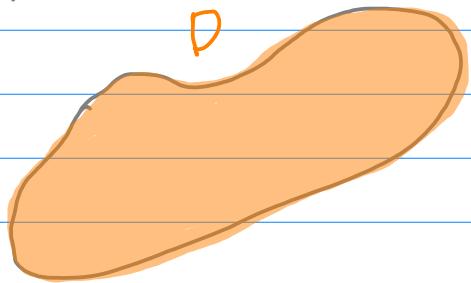
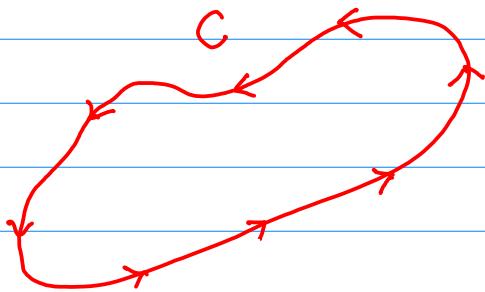
Simple closed curve  
positively oriented  
piecewise smooth

C

the region enclosed  
by C

D

$$\int_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



$$\int_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\int \frac{\partial P}{\partial y} \, dy \Rightarrow P$$

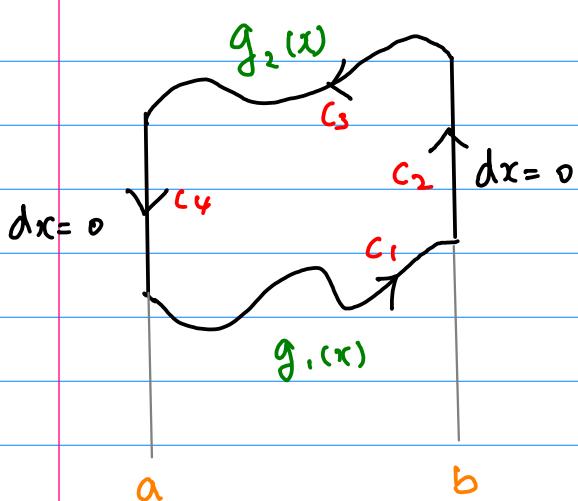
$$\int \frac{\partial Q}{\partial x} \, dx \Rightarrow Q$$

$$\boxed{\int f'(x) \, dx \Rightarrow f(x)}$$

$$\int_a^b f'(x) \, dx = f(b) - f(a)$$

Fundamental Theorem

# Type 1 regions and contours



double integral

$$\iint_D \left( \frac{\partial P}{\partial y} \right) dA$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} dy dx$$

$$= \int_a^b P(x, g_2(x)) - P(x, g_1(x)) dx$$

line integral

$$\int_C P dx = \int_{C_1 + C_2 + C_3 + C_4} P dx$$

$$= \int_{C_1} P dx - \int_{-C_3} P dx$$

$$= \int_a^b P(x, g_1(x)) dx - \int_a^b P(x, g_2(x)) dx$$

$$C_2, C_4: dx = 0$$

$$\int_{C_2} P dx = \int_{C_4} P dx = 0$$

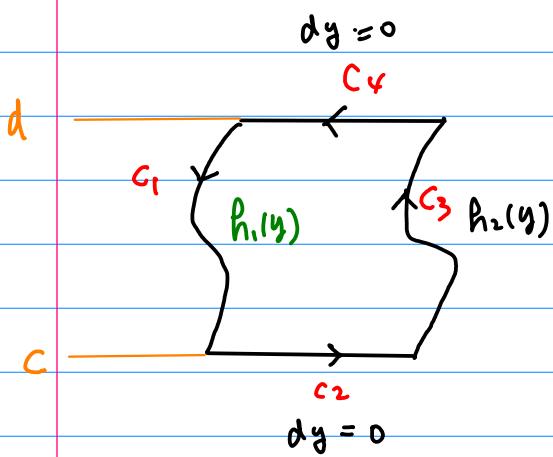
$$\int_C P dx$$

$$= - \iint_D \left( \frac{\partial P}{\partial y} \right) dA$$

line integral

double integral

# Type 2 regions and contours



double integral

$$\iint_D \left( \frac{\partial Q}{\partial x} \right) dA$$

$$= \int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial Q}{\partial y} dx dy$$

$$= \int_c^d Q(h_2(y), y) - Q(h_1(y), y) dy$$

line integral

$$\int_C Q dx = \int_{C_1 + C_2 + C_3 + C_4} Q dy$$

$$= - \int_{-c_1}^0 Q dy + \int_{c_3}^d Q dy$$

$$C_2, C_4: dy = 0$$

$$\int_{C_2} Q dy = \int_{C_4} Q dy = 0$$

$$= - \int_c^d Q(h_1(y), y) dy + \int_c^d Q(h_2(y), y) dy$$

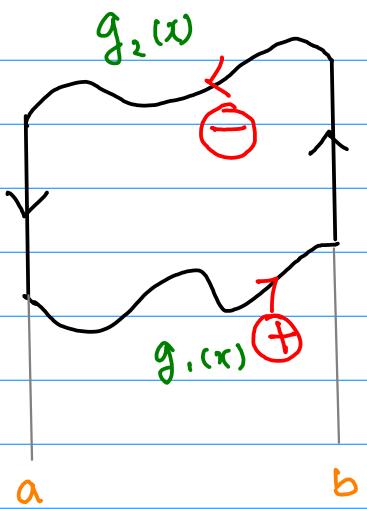
$$\int_C Q dy$$

$$+$$

$$\iint_D \left( \frac{\partial Q}{\partial x} \right) dA$$

line integral

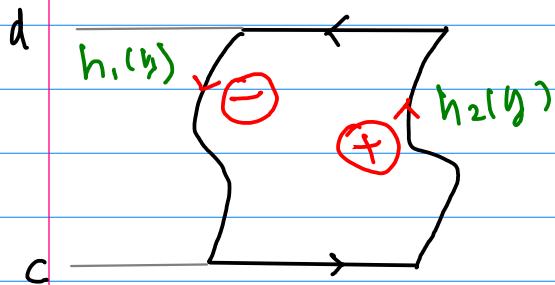
double integral



$$\int_C P \, dx = -$$

$$\iint_D \left( \frac{\partial P}{\partial y} \right) dA$$

$$+ g_1 - g_2$$

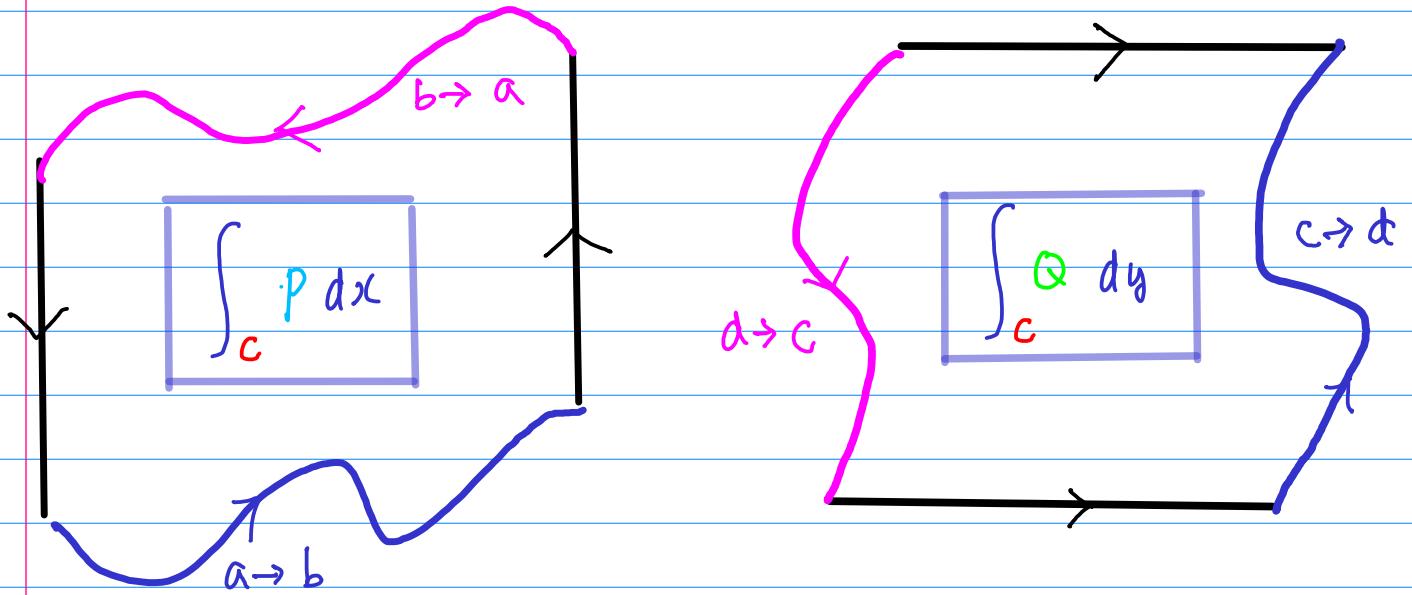


$$\int_C Q \, dy = +$$

$$\iint_D \left( \frac{\partial Q}{\partial x} \right) dA$$

$$+ h_2 - h_1$$

$$\int_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

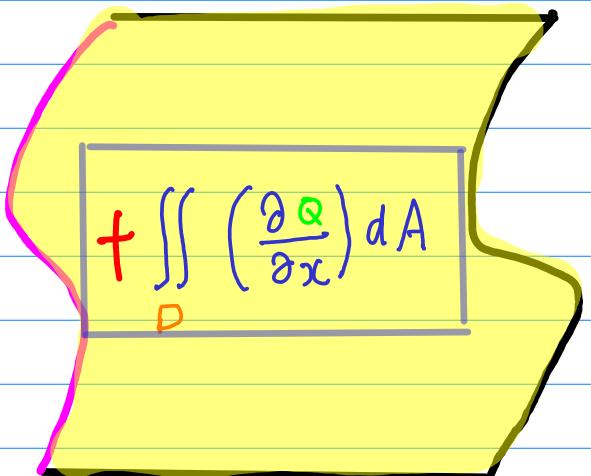
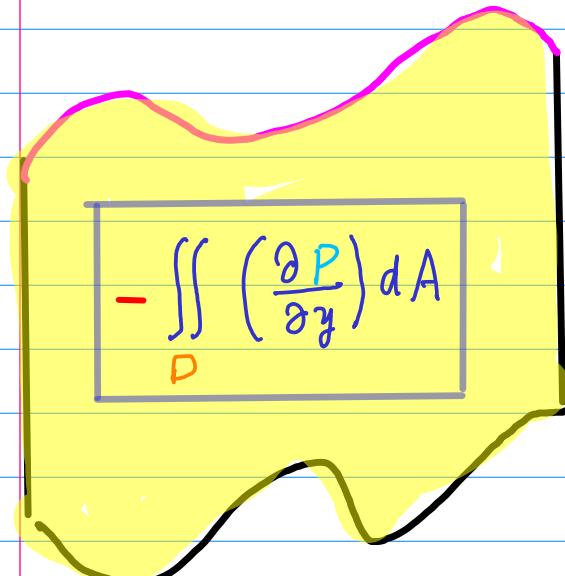


$$\int_a^b \left[ \int \left( \frac{\partial P}{\partial y} \right) dy \right] dx$$

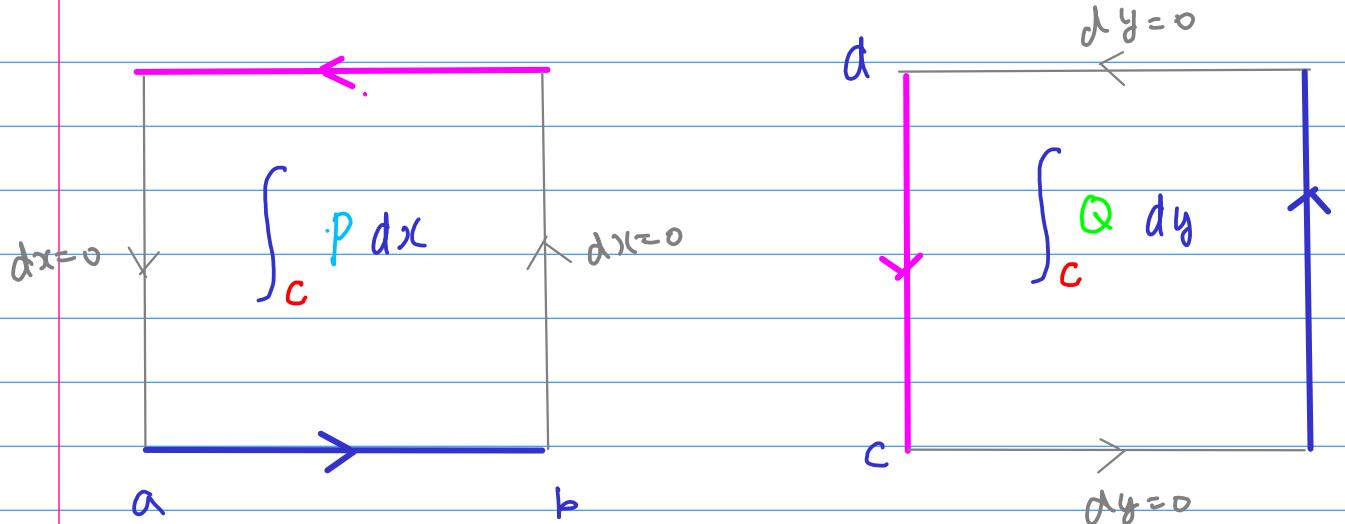
$$\int_a^b -(\text{TOP} - \text{BOTTOM}) dx$$

$$\int_c^d \left[ \int \left( \frac{\partial Q}{\partial x} \right) dx \right] dy$$

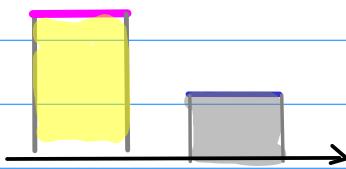
$$\int_c^d (\text{RIGHT} - \text{LEFT}) dy$$



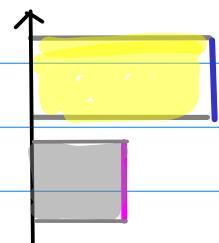
# Rectangular regions and contours



$$\int_a^b - (\text{TOP} - \text{BOTTOM}) dx$$



$$\int_c^d (\text{RIGHT} - \text{LEFT}) dy$$



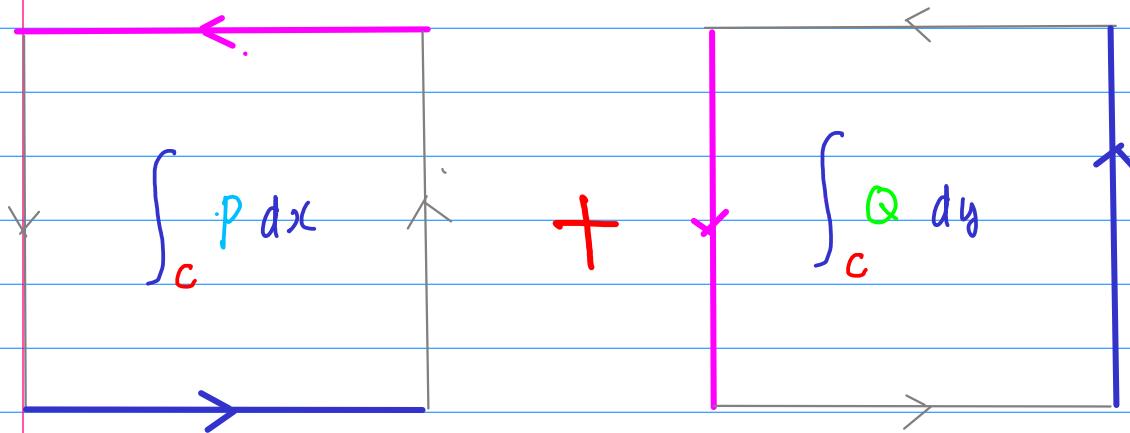
$$\int_a^b \left[ \int \left( \frac{\partial P}{\partial y} \right) dy \right] dx$$

$$\int_c^d \left[ \int \left( \frac{\partial Q}{\partial x} \right) dx \right] dy$$

A large rectangle divided into four quadrants. The bottom-left quadrant is shaded yellow. The formula  $-\iint_D \left( \frac{\partial P}{\partial y} \right) dA$  is written inside this yellow region.

A large rectangle divided into four quadrants. The bottom-right quadrant is shaded yellow. The formula  $+\iint_D \left( \frac{\partial Q}{\partial x} \right) dA$  is written inside this yellow region.

# General Contour



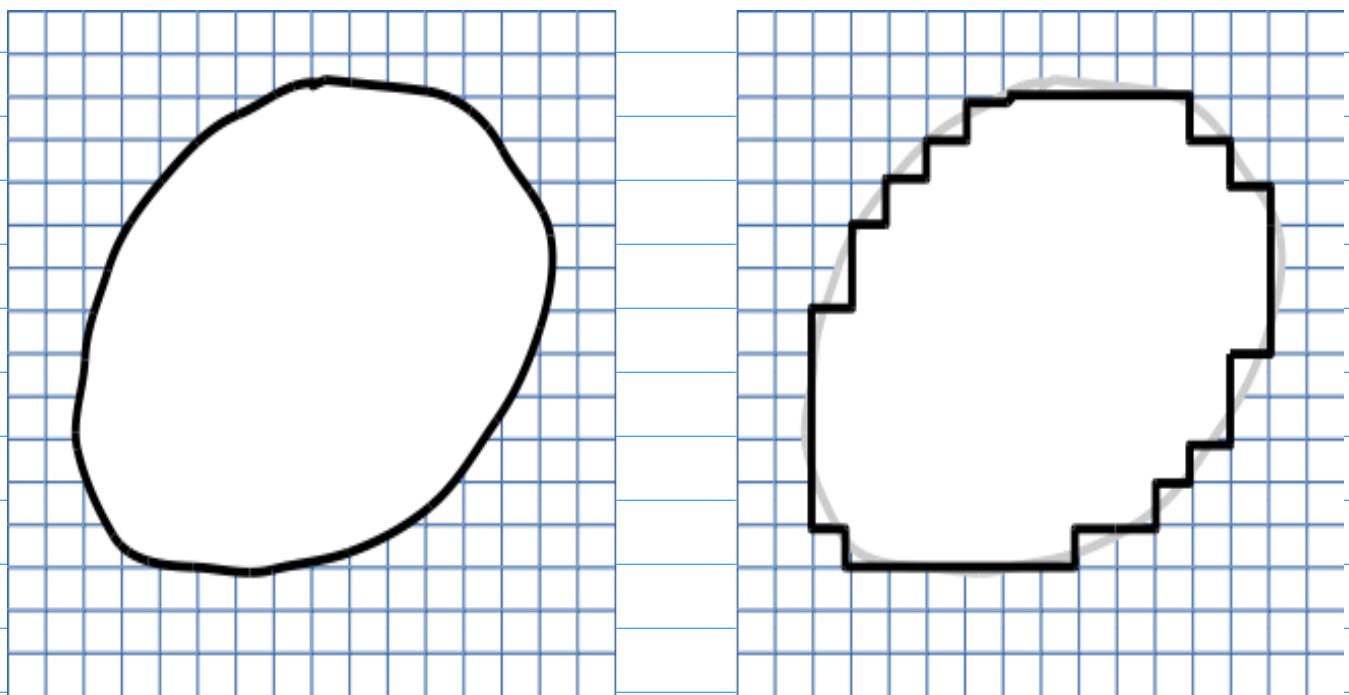
$\equiv$

A diagram showing the equivalence between the contour integral and a double integral. It consists of two yellow sticky notes. The left note contains the equation  $-\iint_D \left( \frac{\partial P}{\partial y} \right) dA$ . The right note contains the equation  $\iint_D \left( \frac{\partial Q}{\partial x} \right) dA$ . Between the two notes is a red plus sign ( $+$ ).

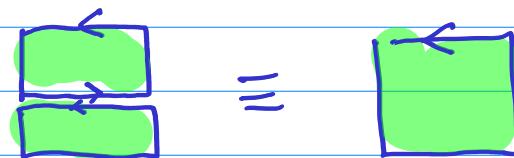
The final equation is enclosed in a blue rectangular border. It states that the contour integral  $\int_C P \, dx + Q \, dy$  is equivalent to the double integral  $\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$ .

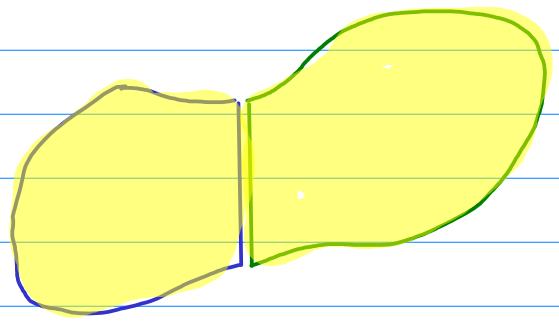
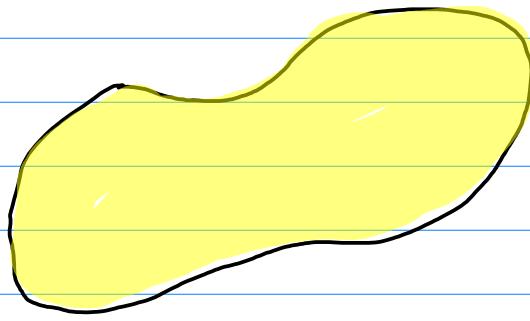
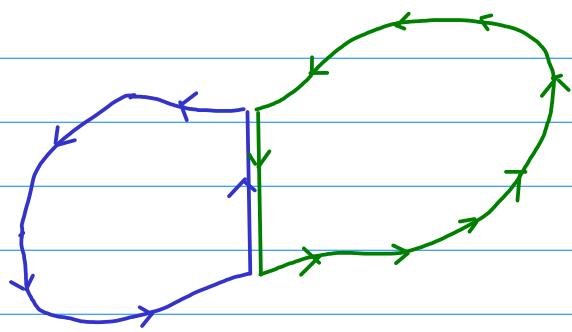
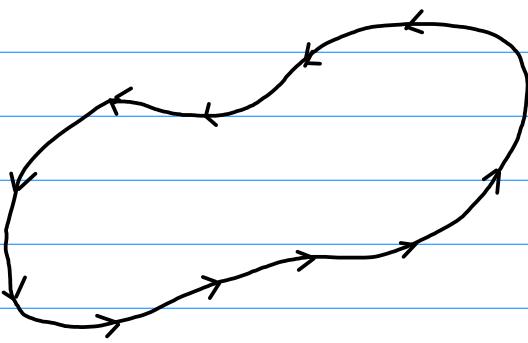
# Approximation

# (Digitization)

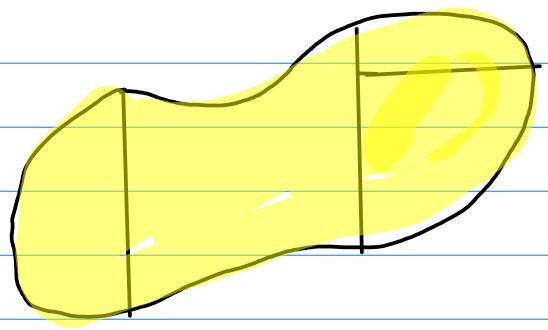
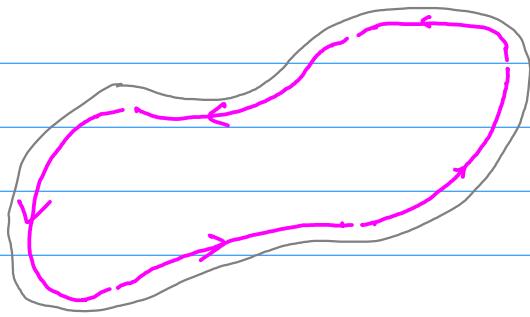
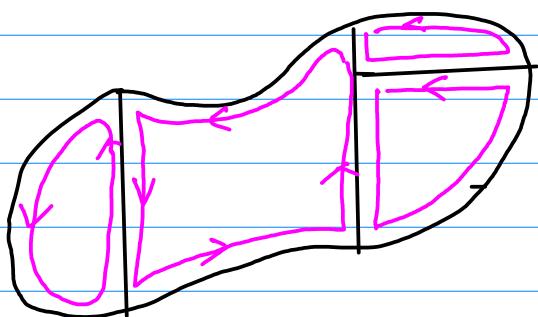
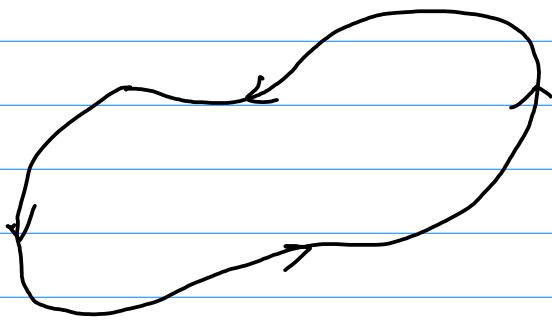


approximated by a collection of ~~rectangles~~

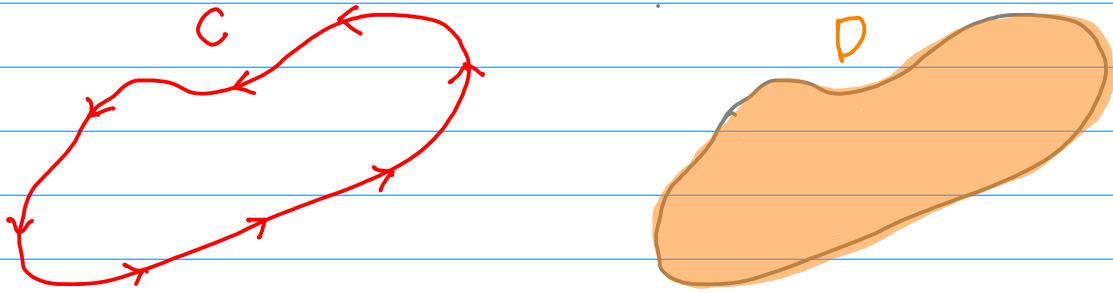




$$\int_C P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



$$\int_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



If  $P, Q$  are the  $(x), (y)$  component of a gradient vector field then there exists a potential function

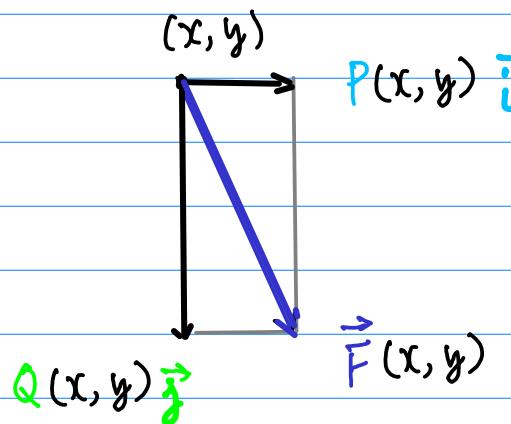
$\text{f}$

$$\begin{aligned}\nabla f &= \left( \frac{\partial f}{\partial x} \right) \vec{i} + \left( \frac{\partial f}{\partial y} \right) \vec{j} \\ &= P \vec{i} + Q \vec{j}\end{aligned}$$

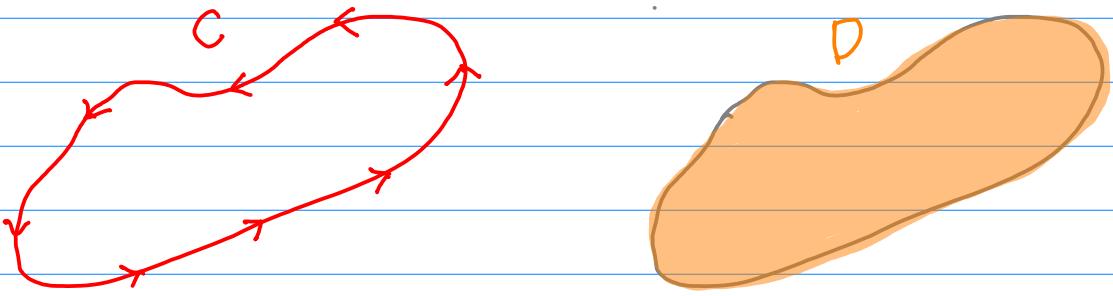
$$= \vec{F}$$

$$f(x, y) = \int P(x, y) \, dx$$

$$f(x, y) = \int Q(x, y) \, dy$$



$$\int_C P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



$P, Q$  can be  $x, y$  component of  
a gradient vector field of  $f(x, y)$   $\leftrightarrow$

$$P = \frac{\partial f}{\partial x}$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} \quad \text{)} \quad \text{(pink bracket)}$$

$$Q = \frac{\partial f}{\partial y}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} \quad \text{)} \quad \text{(pink bracket)}$$

conservative field

## Vector Field $\vec{F}(x, y)$

$$\vec{F}(x, y) = p(x, y) \vec{i} + q(x, y) \vec{j}$$

$$\vec{r}(t) = x(t) \vec{i} + y(t) \vec{j}$$

$$\begin{aligned}\vec{F}(\vec{r}(t)) &= \vec{F}(x(t), y(t)) \\ &= p(x, y) \vec{i} + q(x, y) \vec{j}\end{aligned}$$

$$\int \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot \vec{T} d\vec{s}$$

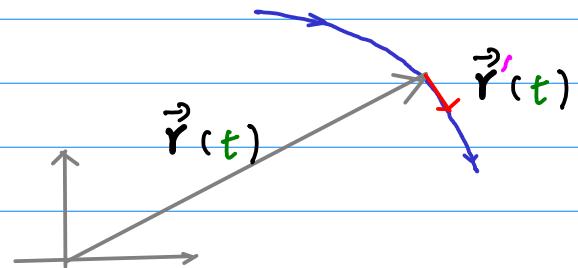
$$= \int \vec{F} \cdot \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} d\vec{s}$$

$$\vec{T} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

$$= \int \vec{F} \cdot \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \|\vec{r}'(t)\| dt$$

$$= \int \vec{F} \cdot \vec{r}'(t) dt$$

$$= \int \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$



## Vector Field $\vec{F}(x, y)$

$$\begin{aligned}
 \int_C \vec{F} \cdot d\vec{r} &= \int_a^b \vec{F} \cdot \vec{r}'(t) dt \\
 &= \int_a^b (P \vec{i} + Q \vec{j}) \cdot (x' \vec{i} + y' \vec{j}) dt \\
 &= \int_a^b P x' dt + \int_a^b Q y' dt \\
 &= \int_C P dx + \int_C Q dy
 \end{aligned}$$

$P, Q$  can be  $x, y$  component of

a gradient vector field

$\Rightarrow$  conservative field

any closed contour  $C$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy = \int_C \nabla f \cdot d\vec{r} = 0$$

$$\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t)) = P(x, y) \vec{i} + Q(x, y) \vec{j}$$

## Example

$$f(x, y) = x^2 + xy + y^2$$

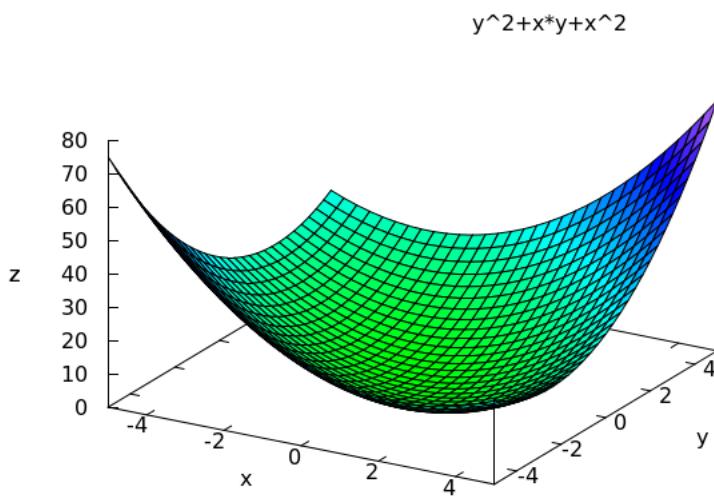
$$\frac{\partial f}{\partial x} = 2x + y \quad \frac{\partial f}{\partial y} = x + 2y$$

$$\nabla f = \underbrace{(2x+y)\vec{i}}_{\text{P}} + \underbrace{(x+2y)\vec{j}}_{\text{Q}}$$

$$\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x} = 1 \quad ; \text{Conservative Vector Field}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$



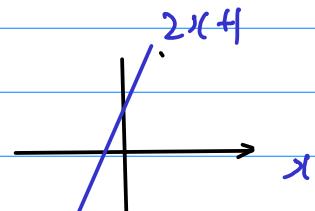
$$f(x, y) = x^2 + xy + y^2$$

$$\frac{\partial f}{\partial x} = 2x + y$$

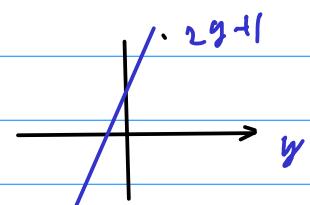
$$\frac{\partial f}{\partial y} = x + 2y$$

$$f(1, 1) = 1+1+1=3$$

$$y=1 \Rightarrow \frac{\partial f}{\partial x} = 2x + 1$$



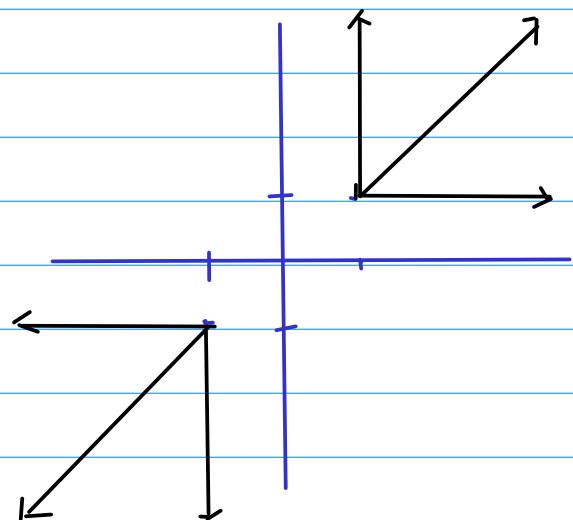
$$x=1 \Rightarrow \frac{\partial f}{\partial y} = 1+2y$$



$$\frac{\partial f}{\partial x}(1, 1)\vec{i} + \frac{\partial f}{\partial y}(1, 1)\vec{j} = 3\vec{i} + 3\vec{j}$$

$$f(1, -1) = 1+1+1=3$$

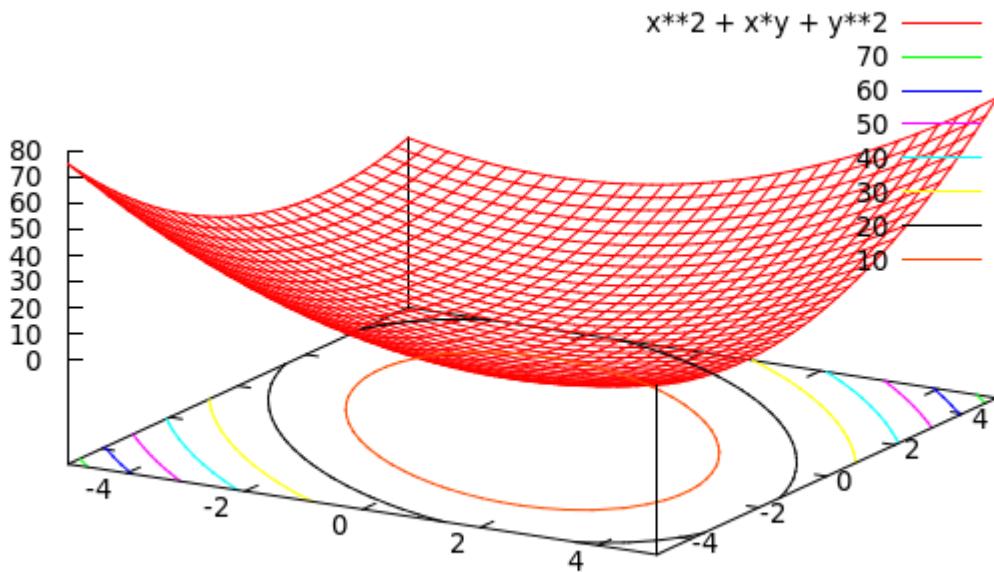
$$y=1 \Rightarrow \frac{\partial f}{\partial x} = 2x - 1$$



$$x=-1 \Rightarrow \frac{\partial f}{\partial y} = -1+2y$$

$$\frac{\partial f}{\partial x}(1, -1)\vec{i} + \frac{\partial f}{\partial y}(1, -1)\vec{j} = -3\vec{i} - 3\vec{j}$$

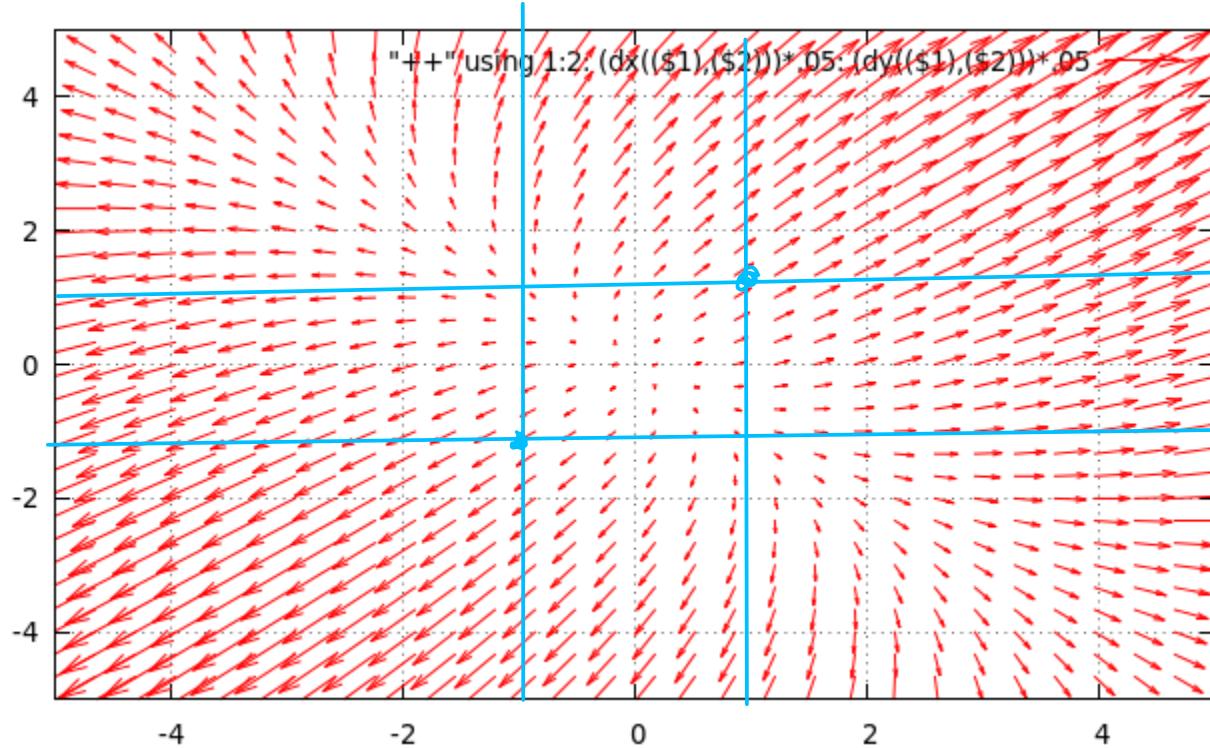
## \* Contour graphs in gnuplot



$$f(x, y) = x^2 + xy + y^2$$

```
gnuplot> set xrange [-5:5]
gnuplot> set yrange [-5:5]
gnuplot> set sample 30
gnuplot> set isosample 31
gnuplot> set contour base
gnuplot> set cntrparam levels 10
gnuplot> splot x**2 + x*y + y**2
```

# \* Gradient field plot in gnuplot



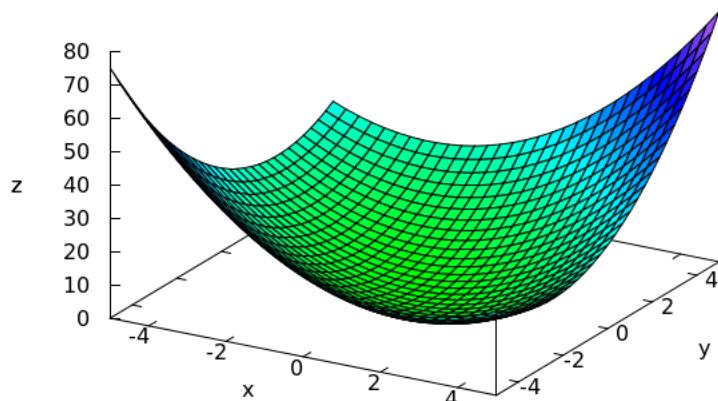
$$\nabla f(x,y)$$

$$f(x,y) = x^2 + xy + y^2$$

```
gnuplot> set xrange [-5:5]
gnuplot> set yrange [-5:5]
gnuplot> set sample 30
gnuplot> set isosample 31
gnuplot> dx(x,y) = 2*x + y
gnuplot> dy(x,y) = x + 2*y
gnuplot> plot "++" using 1:2: (dx((\$1),(\$2))*.05): (dy((\$1),(\$2))*.05) w vec
```

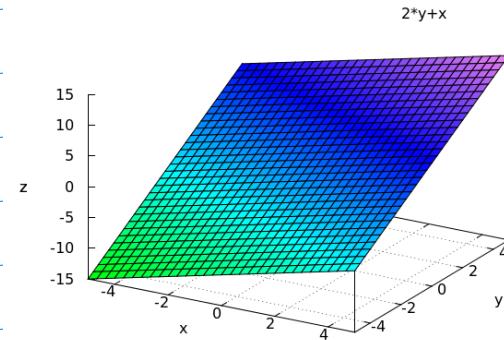
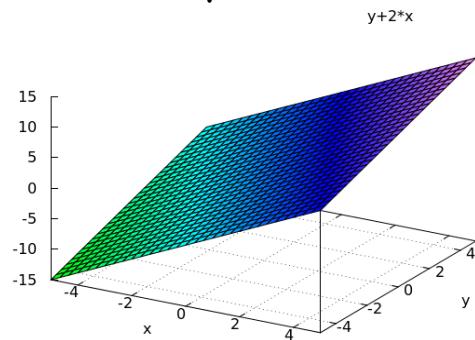
$$f(x, y) = x^2 + xy + y^2$$

$$y^2 + x \cdot y + x^2$$

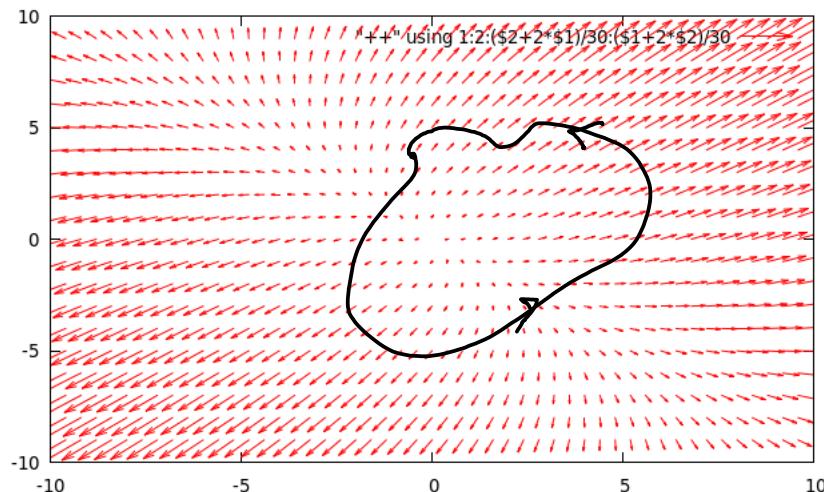


$$P(x, y) = \frac{\partial f}{\partial x}(x, y) = y + 2x$$

$$Q(x, y) = \frac{\partial f}{\partial y}(x, y) = 2y + x$$



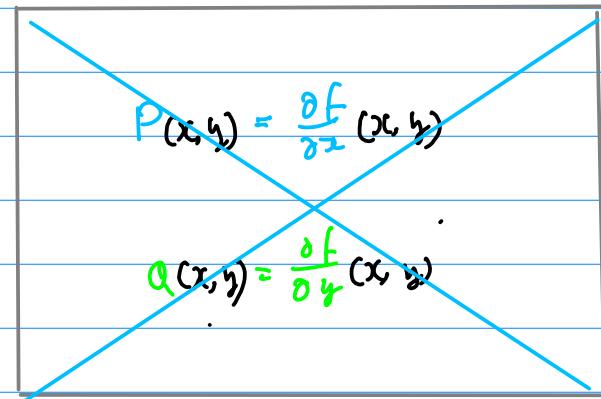
$$\nabla f(x, y) = P(x, y) \vec{i} + Q(x, y) \vec{j} = (y + 2x) \vec{i} + (2y + x) \vec{j}$$



$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

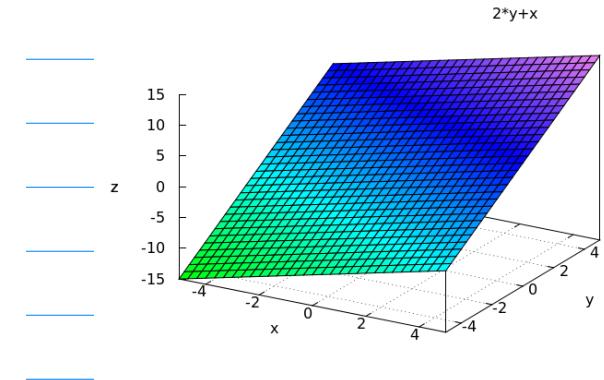
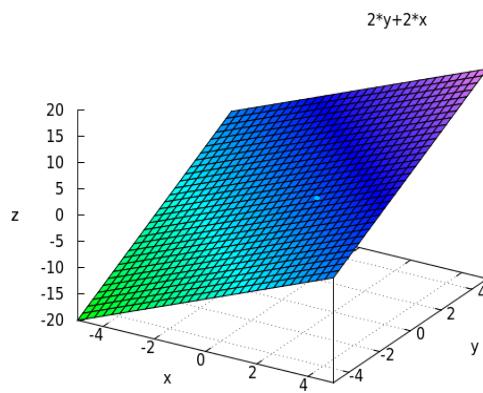
conservative

$f(x, y) \cdot \times$  no potential function exists

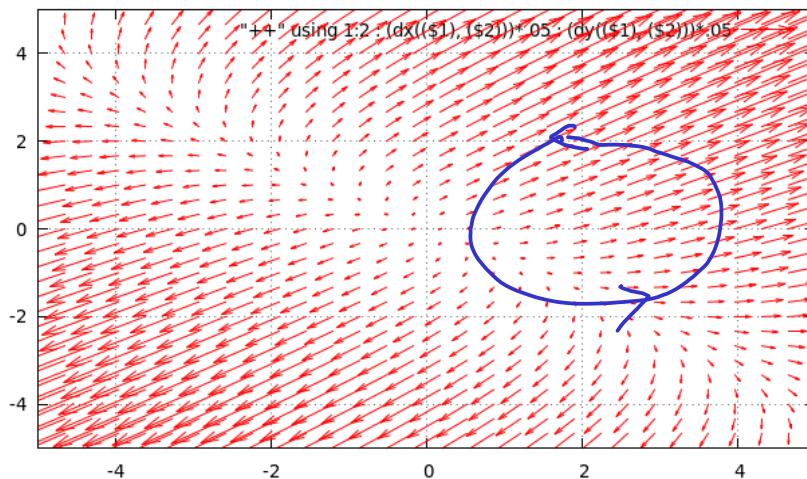


$$P(x, y) = 2y + 2x$$

$$Q(x, y) = 2y + x$$



$$P(x, y) \vec{i} + Q(x, y) \vec{j} = (2y + 2x) \vec{i} + (2y + x) \vec{j}$$



$$\oint_C \vec{F} \cdot d\vec{r} \neq 0$$

non-conservative

# \* Exact Differential

$$P(x, y) dx + Q(x, y) dy : \text{a differential}$$

if this is a total differential of a certain function  $f$   
then it is exact differential

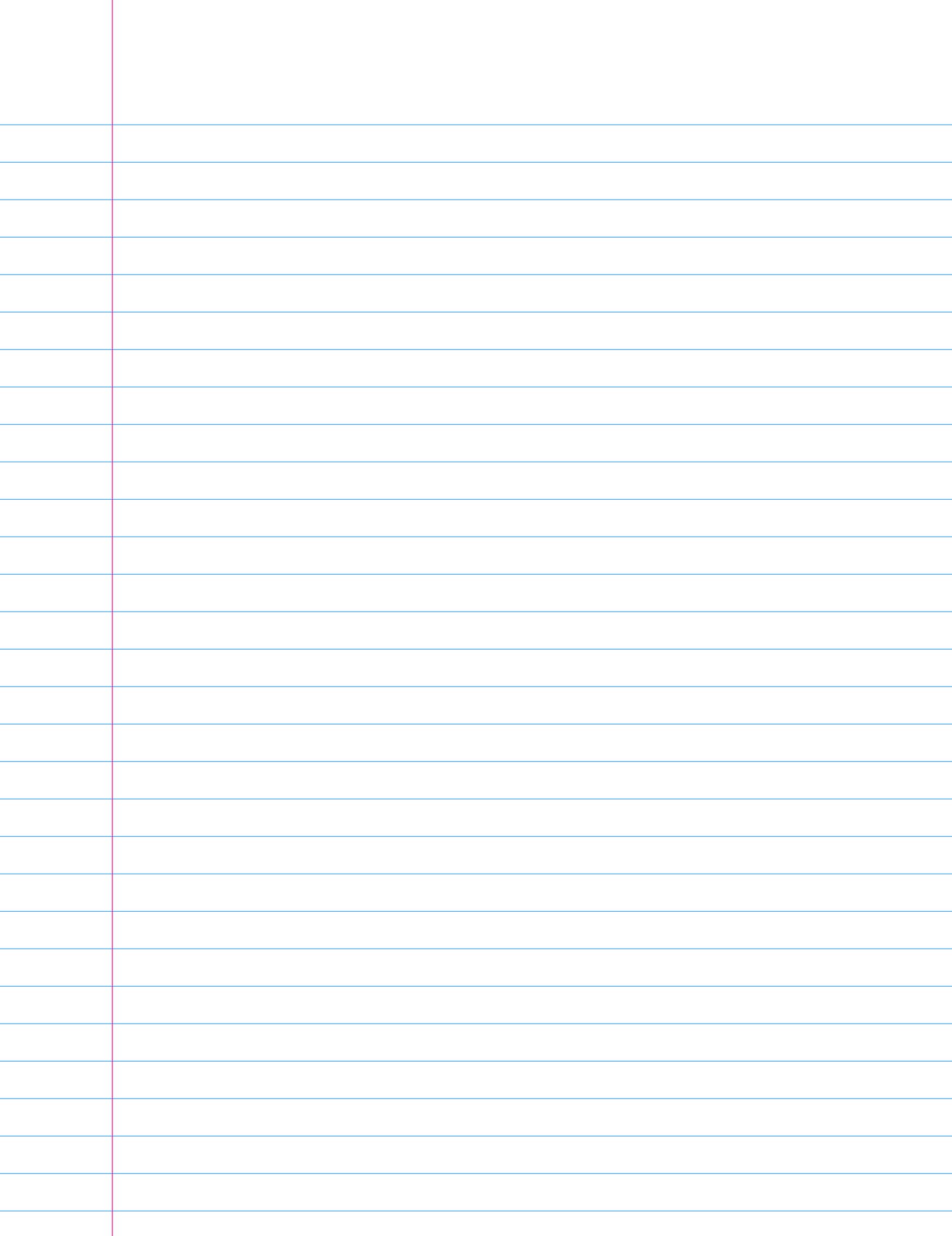
$$\Rightarrow \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = dz \quad z = f(x, y)$$

$$\begin{array}{ccc} P(x, y) & & Q(x, y) \\ \parallel & & \parallel \\ \frac{\partial F}{\partial x} & & \frac{\partial F}{\partial y} \end{array}$$

the necessary and sufficient condition

$$P(x, y) dx + Q(x, y) dy : \text{an exact differential}$$

$$\Leftrightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

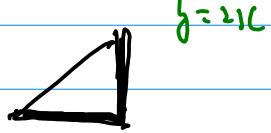


$$\oint_C xy \, dx = \int_{C_1} xy \, dx + \int_{C_2} xy \, dx + \int_{C_3} xy \, dx$$

$y=0$        $y=1$        $y=2x$

$$= \int_0^1 0 \, dx + \int_1^1 x \, dx + \int_1^0 2x^2 \, dx$$

$$= \left[ \frac{2}{3} x^3 \right]_1^0 = -\frac{2}{3}$$



$$\oint_C x^2 y^3 \, dx = \int_{C_1} x^2 y^3 \, dy + \int_{C_2} x^2 y^3 \, dy + \int_{C_3} x^2 y^3 \, dy$$

$x=[0, 1]$        $x=1$        $x=\frac{y}{2}$

$$= \int_0^0 x^2 y^3 \, dy + \int_0^2 y^3 \, dy + \int_2^0 \frac{y^5}{4} \, dy$$

$$= \left[ \frac{1}{4} y^4 \right]_0^2 + \left[ \frac{1}{24} y^6 \right]_2^0$$

$$= 4 - \frac{64}{24} \cancel{\frac{16}{3}} 8 = \frac{12-8}{3} = +\frac{4}{3}$$

(2/3)

