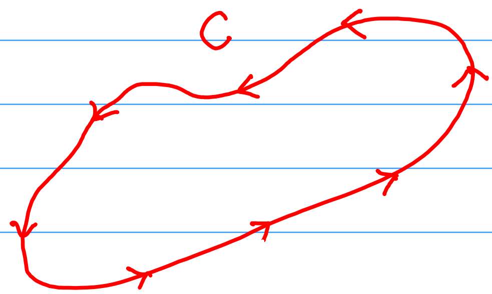
Vector Calculus (H.1)
Green's Theorem

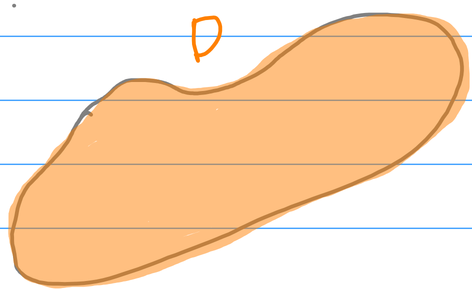
20151021

Green's Theorem.

simple closed curve
positively oriented
piecewise smooth C, the region enclosed by C D

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$





$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

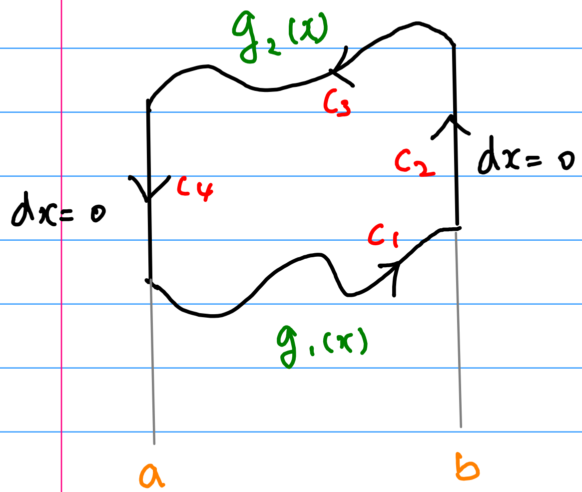
$$\int \frac{\partial P}{\partial y}\,dy \Rightarrow P$$

$$\int \frac{\partial Q}{\partial x}\,dx \Rightarrow Q$$

$$\boxed{\int f'(x)\,dx \Rightarrow f(x)} \qquad \int_a^b f'(x)\,dx = f(b) - f(a)$$

Fundamenta theorem

Type 1 regions and contours



double integral

$$\boxed{\iint_D \left(\frac{\partial P}{\partial y}\right) dA}$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\, dy\, dx$$

$$= \int_a^b P(x, g_2(x)) - P(x, g_1(x))\, dx$$

line integral

$$\boxed{\int_C P\, dx} = \int_{C_1 + C_2 + C_3 + C_4} P\, dx$$

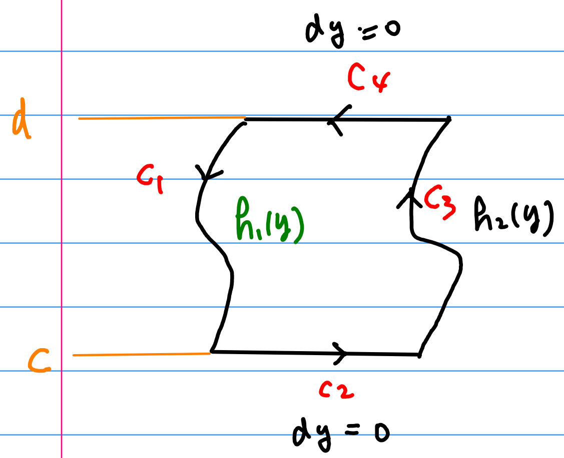
$$= \int_{C_1} P\, dx - \int_{-C_3} P\, dx$$

$$\boxed{\begin{array}{l} C_2, C_4: \ dx = 0 \\ \int_{C_2} P\, dx = \int_{C_4} P\, dx = 0 \end{array}}$$

$$= \int_a^b P(x, g_1(x))\, dx - \int_a^b P(x, g_2(x))\, dx$$

$$\underbrace{\boxed{\int_C P\, dx}}_{\text{line integral}} = - \underbrace{\boxed{\iint_D \left(\frac{\partial P}{\partial y}\right) dA}}_{\text{double integral}}$$

Type 2 regions and contours



double integral

$$\boxed{\iint_D \left(\frac{\partial Q}{\partial x}\right) dA}$$

$$= \int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial Q}{\partial y}\, dx\, dy$$

$$= \int_c^d Q(h_2(y), y) - Q(h_1(y), y)\, dy$$

line integral

$$\boxed{\int_C Q\, d} = \int_{C_1 + C_2 + C_3 + C_4} Q\, dy$$

$$= -\int_{-C_1} Q\, dy + \int_{C_3} Q\, dy$$

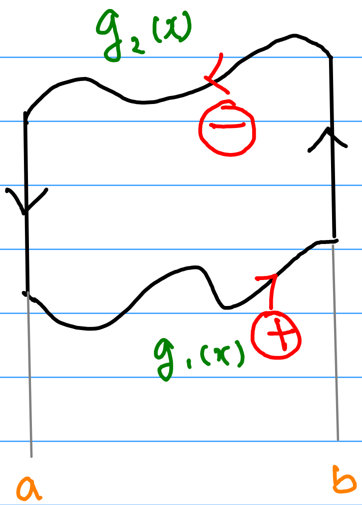
C_2, C_4: $dy = 0$

$$\int_{C_2} Q\, dy = \int_{C_4} Q\, dy = 0$$

$$= -\int_c^d Q(h_1(y), y)\, dy + \int_c^d Q(h_2(y), y)\, dy$$

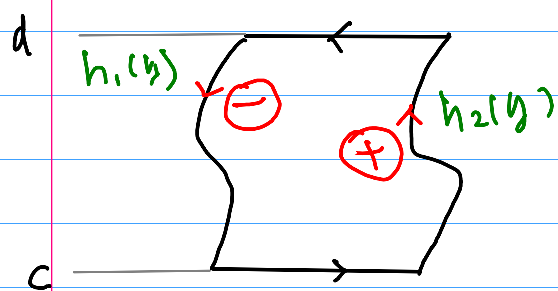
$$\boxed{\int_C Q\, dy} = + \boxed{\iint_D \left(\frac{\partial Q}{\partial x}\right) dA}$$

line integral double integral



$$\boxed{\int_C P\,dx} = - \boxed{\iint_D \left(\frac{\partial P}{\partial y}\right) dA}$$

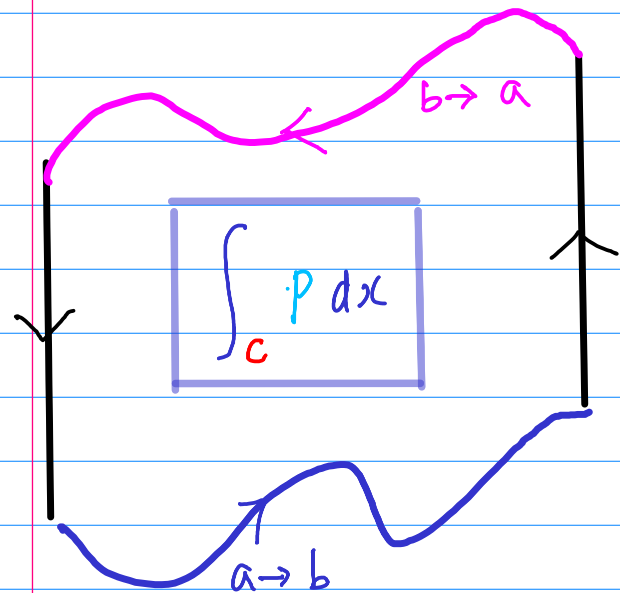
$$+g_1 - g_2$$

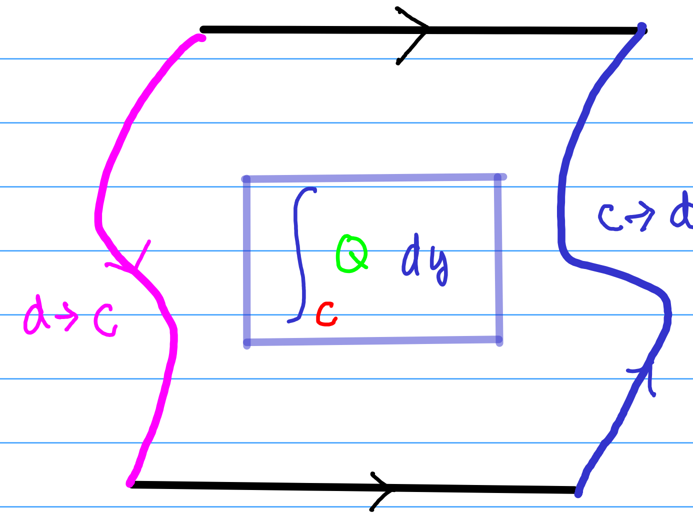


$$\boxed{\int_C Q\,dy} = + \boxed{\iint_D \left(\frac{\partial Q}{\partial x}\right) dA}$$

$$+h_2 - h_1$$

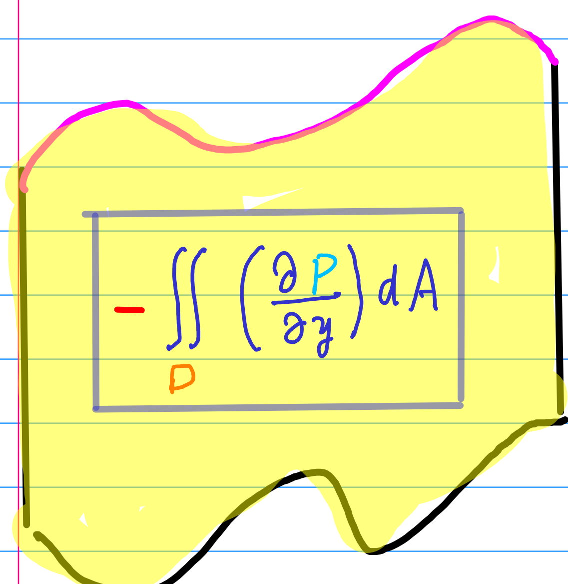
$$\boxed{\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

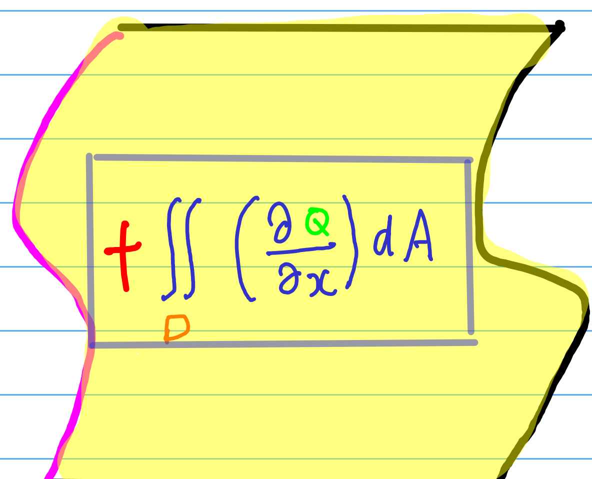




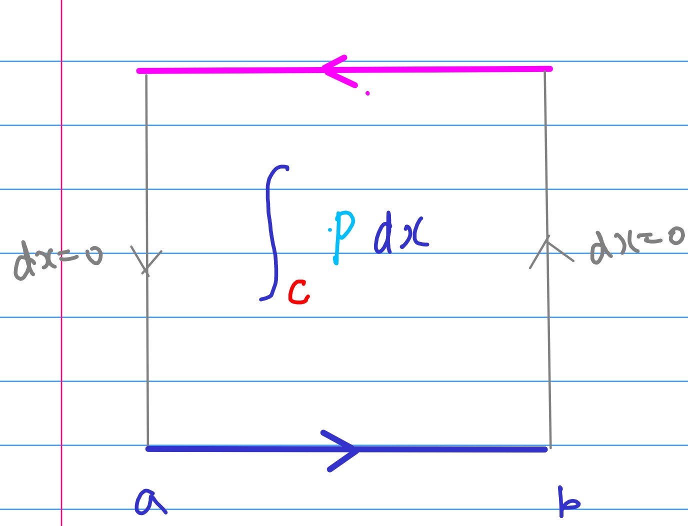
$$\int_a^b \overbrace{\boxed{\int \left(\frac{\partial P}{\partial y}\right) dy}}^{P} \; dx \qquad\qquad \int_c^d \overbrace{\boxed{\int \left(\frac{\partial Q}{\partial x}\right) dx}}^{Q} \; dy$$

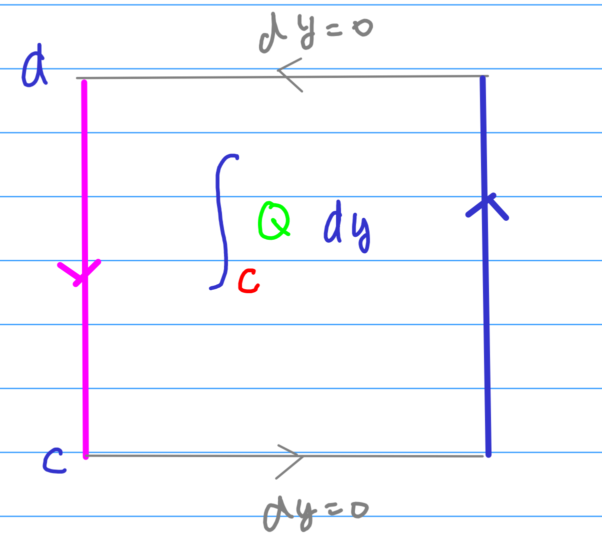
$$\int_a^b -\left(\text{TOP} - \text{BOTTOM}\right) dx \qquad\qquad \int_c^d \left(\text{RIGHT} - \text{LEFT}\right) dy$$



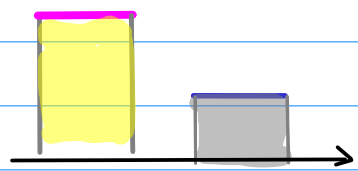


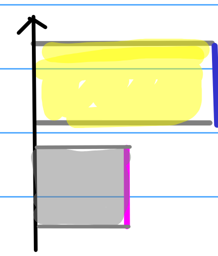
Rectangular regions and contours





$$\int_a^b -(\text{TOP} - \text{BOTTOM})\,dx \qquad \int_c^d (\text{RIGHT} - \text{LEFT})\,dy$$





$$\int_a^b \overset{\nearrow P}{\boxed{\int \left(\frac{\partial P}{\partial y}\right) dy}}\, dx \qquad \int_c^d \overset{\nearrow Q}{\boxed{\int \left(\frac{\partial Q}{\partial x}\right) dx}}\, dy$$

$$\boxed{-\iint_D \left(\frac{\partial P}{\partial y}\right) dA} \qquad \boxed{+\iint_D \left(\frac{\partial Q}{\partial x}\right) dA}$$

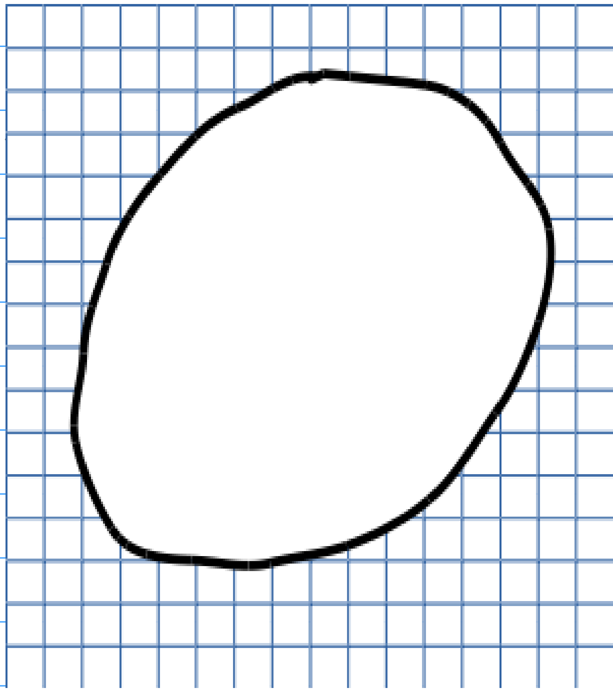
General Contour

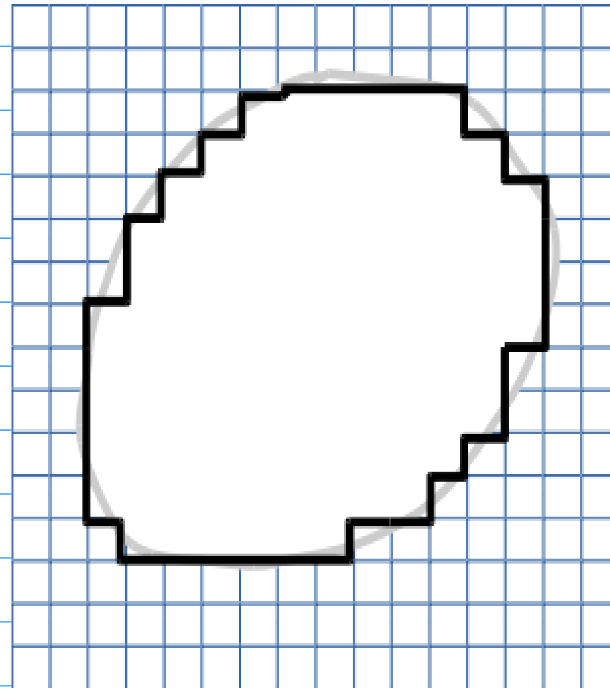
$$\int_C P\,dx \quad + \quad \int_C Q\,dy$$

$$\equiv \quad -\iint_D \left(\frac{\partial P}{\partial y}\right) dA \quad + \quad \iint_D \left(\frac{\partial Q}{\partial x}\right) dA$$

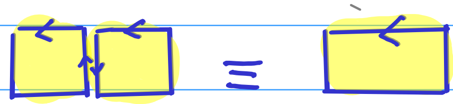
$$\boxed{\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

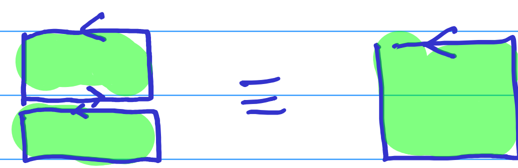
Approximation (Digitization)



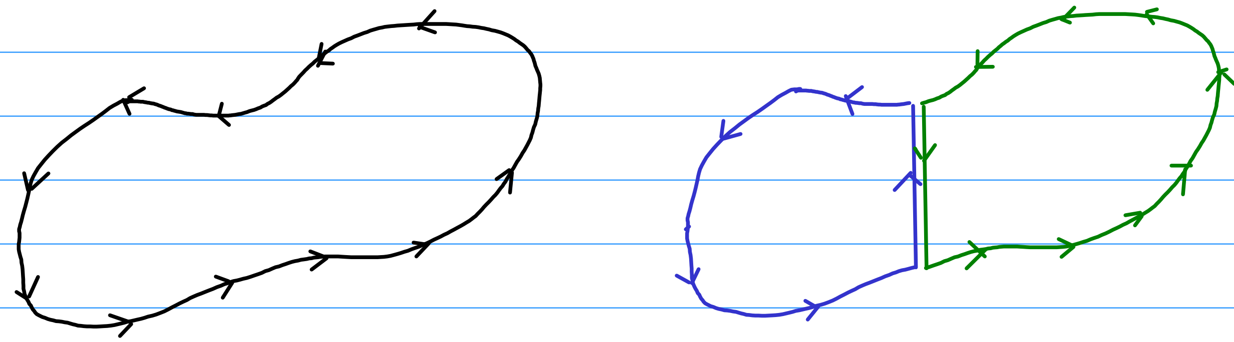


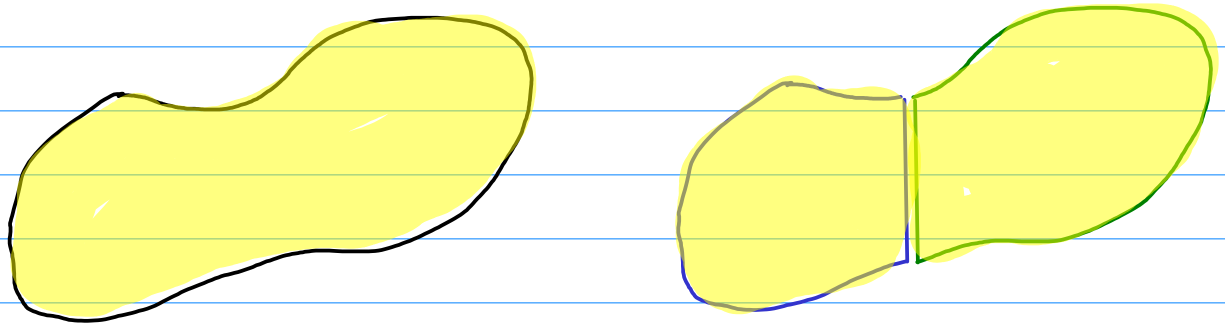
approximated by a collection of ~~rectangles~~



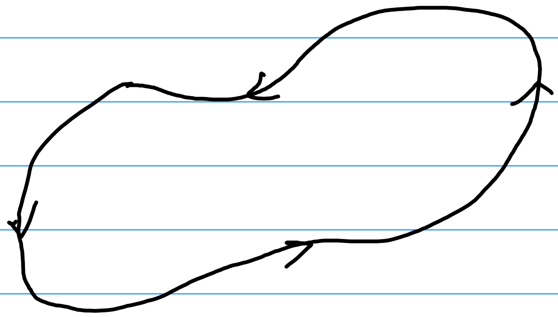
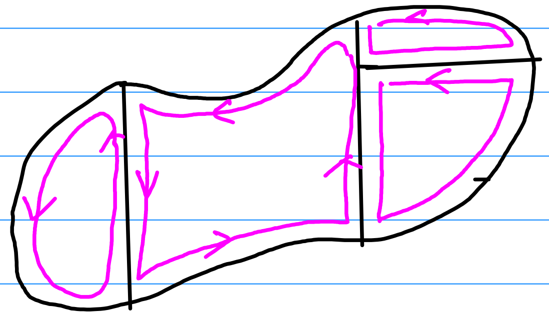
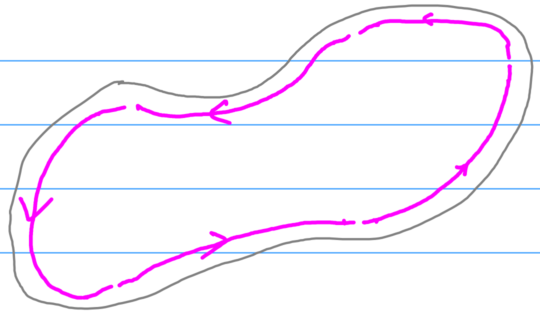
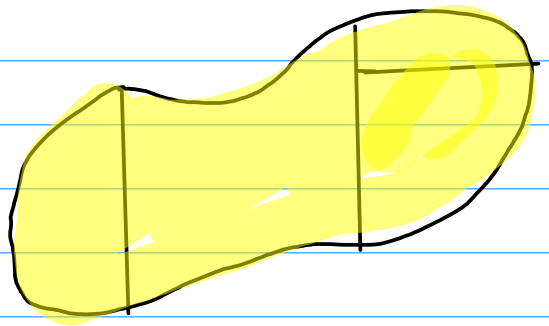


http://ocw.mit.edu/courses/mathematics/18-02sc-multivariable-calculus-fall-2010/
3.-double-integrals-and-line-integrals-in-the-plane/part-c-greens-theorem/
session-67-proof-of-greens-theorem/MIT18_02SC_notes_67.pdf

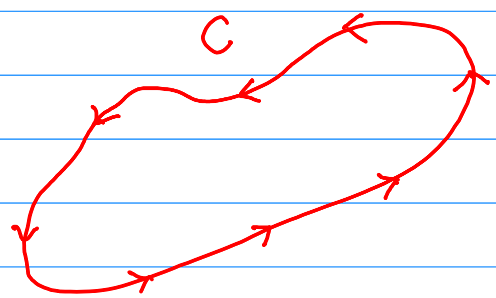


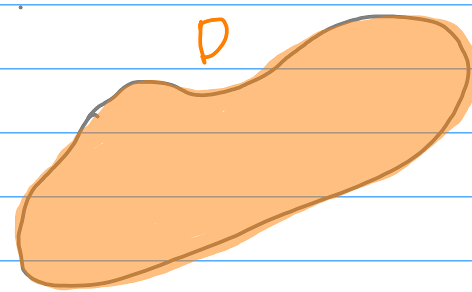


$$\boxed{\int_{C} P\,dx + Q\,dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$





If P, Q are the x, y component of a gradient vector field then there exists a potential function f

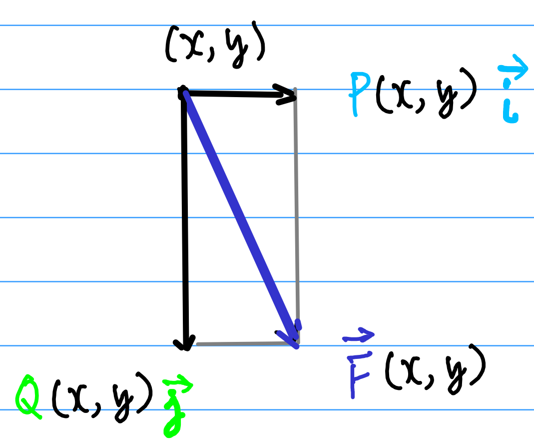
$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

$$= P\,\vec{i} + Q\,\vec{j}$$

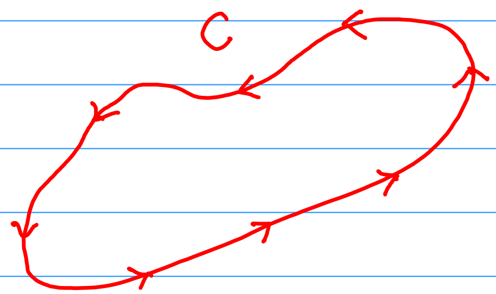
$$= \vec{F}$$

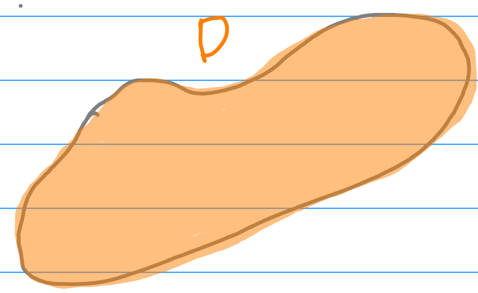
$$f(x,y) = \int P(x,y)\,dx$$

$$f(x,y) = \int Q(x,y)\,dy$$



$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$





P, Q can be x, y component of a **gradient** vector field of $f(x.y)$ $\Longleftrightarrow$

$$P = \frac{\partial f}{\partial x} \qquad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x}$$

$$Q = \frac{\partial f}{\partial y} \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}$$

conservative field

Vector Field $\vec{F}(x, y)$

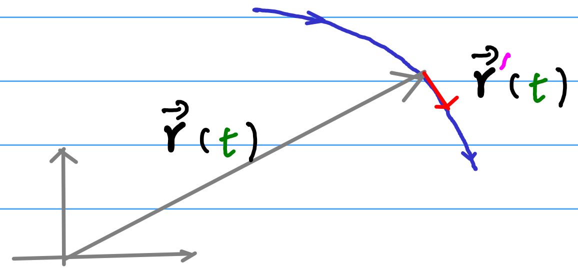
$$\vec{F}(x, y) = P(x, y)\,\vec{i} + Q(x, y)\,\vec{j}$$

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$$

$$\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t))$$
$$= P(x, y)\,\vec{i} + Q(x, y)\,\vec{j}$$

$$\int \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot \vec{T}\, d\vec{s}$$
$$= \int \vec{F} \cdot \boxed{\frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}}\, \boxed{d\vec{s}}$$
$$= \int \vec{F} \cdot \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}\, \boxed{\|\vec{r}'(t)\|\, dt}$$
$$= \int \vec{F} \cdot \vec{r}'(t)\, dt$$
$$= \int \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

$$\boxed{\vec{T} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}}$$



Vector Field $\vec{F}(x, y)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F} \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b (P\,\vec{i} + Q\,\vec{j}) \cdot (x'\,\vec{i} + y'\,\vec{j})\, dt$$

$$= \int_a^b P\, x'\, dt + \int_a^b Q\, y'\, dt$$

$$= \int_C P\, dx + \int_C Q\, dy$$

P, Q can be x, y component of a gradient vector field

$\Rightarrow$ conservative field

any closed contour C

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_C P\, dx + Q\, dy = \int_C \nabla f \cdot d\vec{r} = 0}$$

$$\boxed{\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t)) = P(x, y)\,\vec{i} + Q(x, y)\,\vec{j}}$$

Example

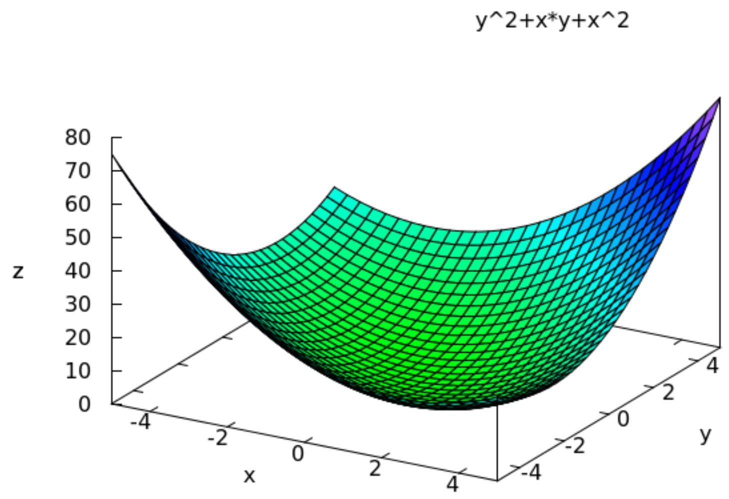
$$f(x,y) = x^2 + xy + y^2$$

$$\frac{\partial f}{\partial x} = 2x + y \qquad \frac{\partial f}{\partial y} = x + 2y$$

$$\nabla f = \underbrace{(2x+y)}_{P}\vec{i} + \underbrace{(x+2y)}_{Q}\vec{j}$$

$$\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x} = 1$$: Conservative Vector Field

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \qquad \oint_C \vec{F} \cdot d\vec{r} = 0$$



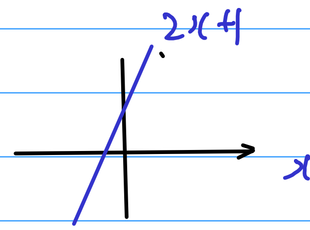
$F(x,y) = x^2 + xy + y^2$

$\frac{\partial f}{\partial x} = 2x + y$

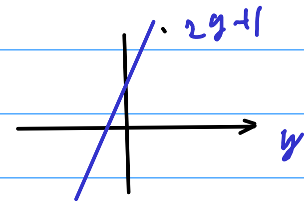
$\frac{\partial f}{\partial y} = x + 2y$

$f(1,1) = 1+1+1 = 3$

$y = 1 \Rightarrow \frac{\partial f}{\partial x} = 2x + 1$



$x = 1 \Rightarrow \frac{\partial f}{\partial y} = 1 + 2y$

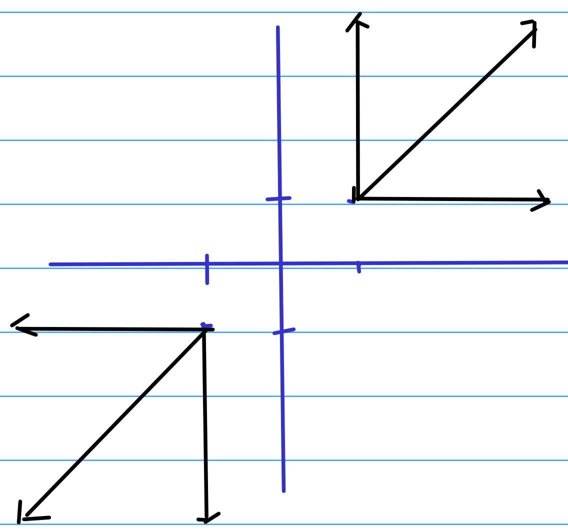


$\frac{\partial f}{\partial x}(1,1)\vec{i} + \frac{\partial f}{\partial y}(1,1)\vec{j} = 3\vec{i} + 3\vec{j}$

$f(-1,-1) = 1+1+1 = 3$

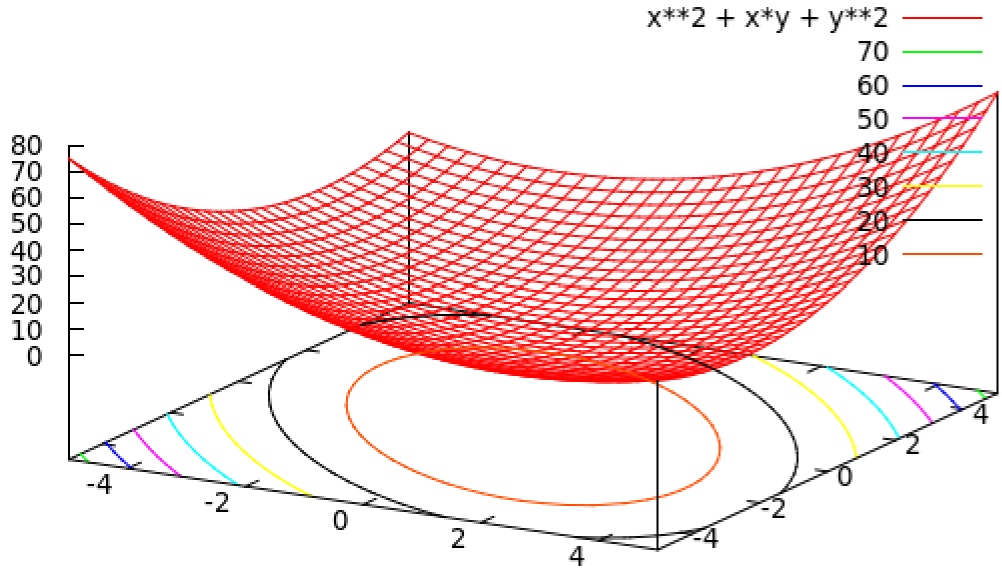
$y = -1 \Rightarrow \frac{\partial f}{\partial x} = 2x - 1$

$x = -1 \Rightarrow \frac{\partial f}{\partial y} = -1 + 2y$



$\frac{\partial f}{\partial x}(-1,-1)\vec{i} + \frac{\partial f}{\partial y}(-1,-1)\vec{j} = -3\vec{i} - 3\vec{j}$

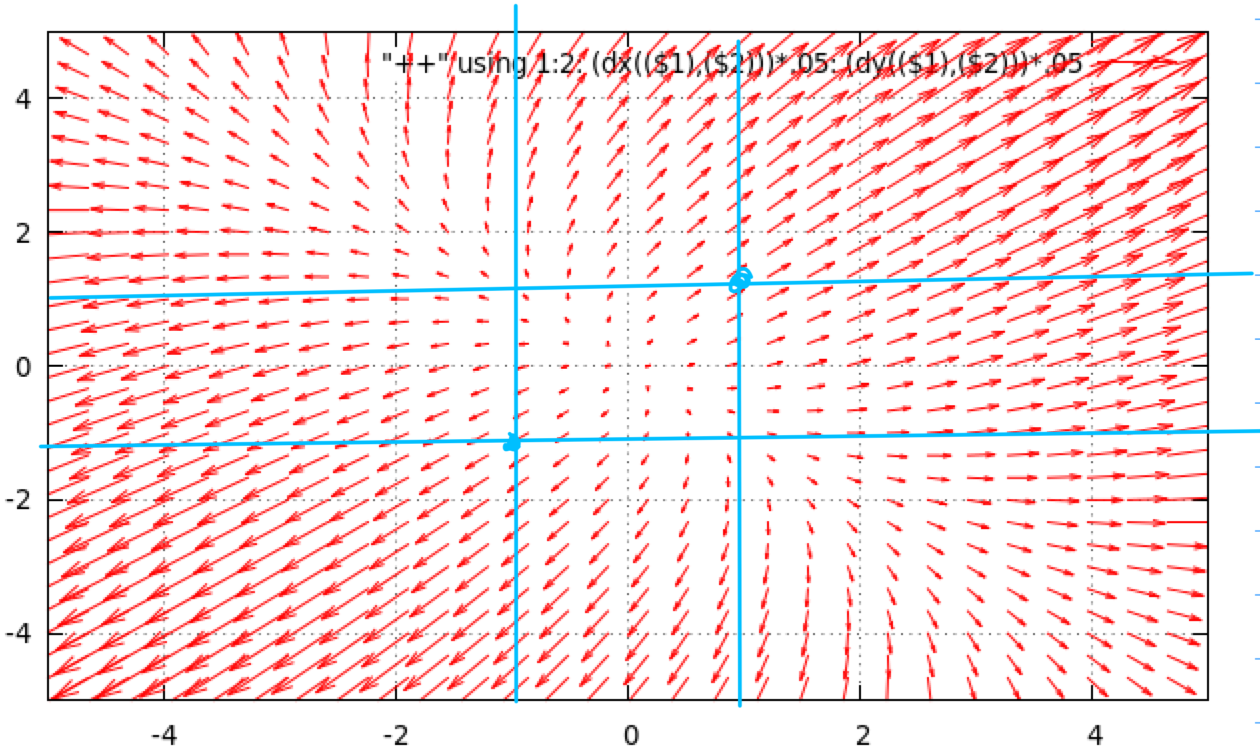
* Contour graphs in gnuplot



$$f(x,y) = x^2 + xy + y^2$$

```
gnuplot> set xrange [-5:5]
gnuplot> set yrange [-5:5]
gnuplot> set sample 30
gnuplot> set isosample 31
gnuplot> set contour base
gnuplot> set cntrparam levels 10
gnuplot> splot x**2 + x*y + y**2
```

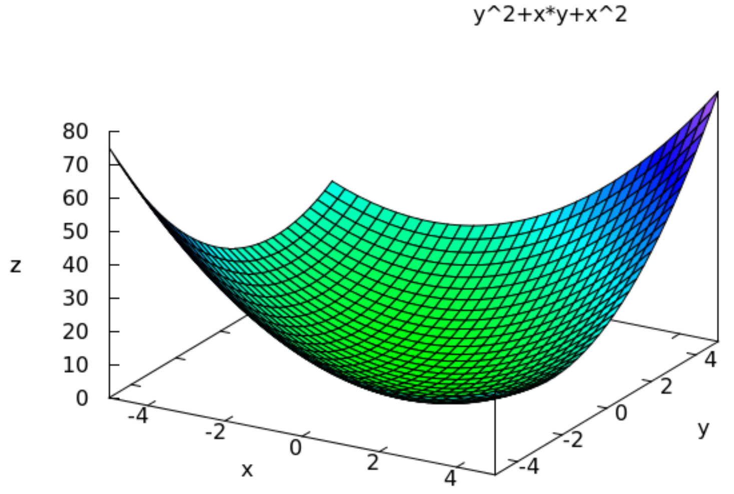
* Gradient field plot in gnuplot



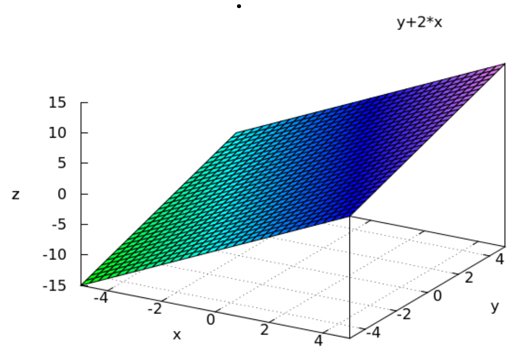
$\nabla f(x,y)$ $\qquad$ $f(x,y) = x^2 + xy + y^2$

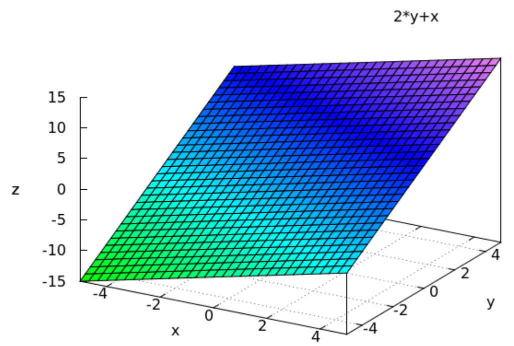
```
gnuplot> set xrange [-5:5]
gnuplot> set yrange [-5:5]
gnuplot> set sample 30
gnuplot> set isosample 31
gnuplot> dx(x,y) = 2*x + y
gnuplot> dy(x,y) = x + 2*y
gnuplot> plot "++" using 1:2: (dx(($1),($2)))*.05: (dy(($1),($2)))*.05 w vec
```

$$f(x,y) = x^2 + xy + y^2$$

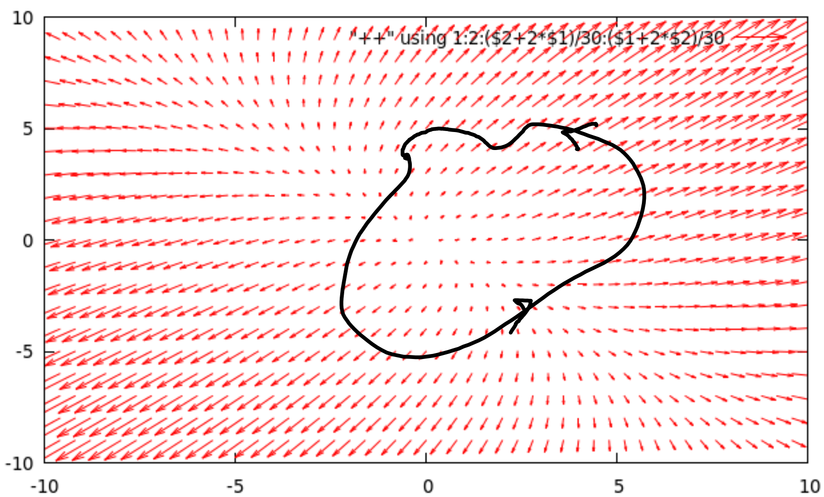


$$P(x,y) = \frac{\partial f}{\partial x}(x,y) = y + 2x \qquad Q(x,y) = \frac{\partial f}{\partial y}(x,y) = 2y + x$$





$$\nabla f(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j} = (y+2x)\,\vec{i} + (2y+x)\,\vec{j}$$



$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

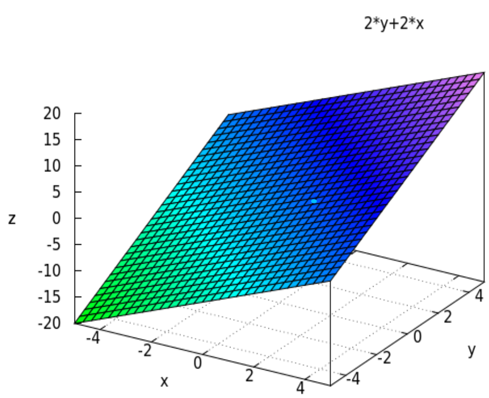
conservative

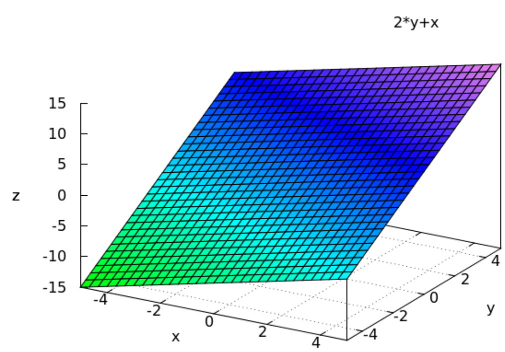
$f(x, y)$ · ✗ no potential function exists

$$\cancel{P(x,y) = \frac{\partial f}{\partial x}(x,y)}$$

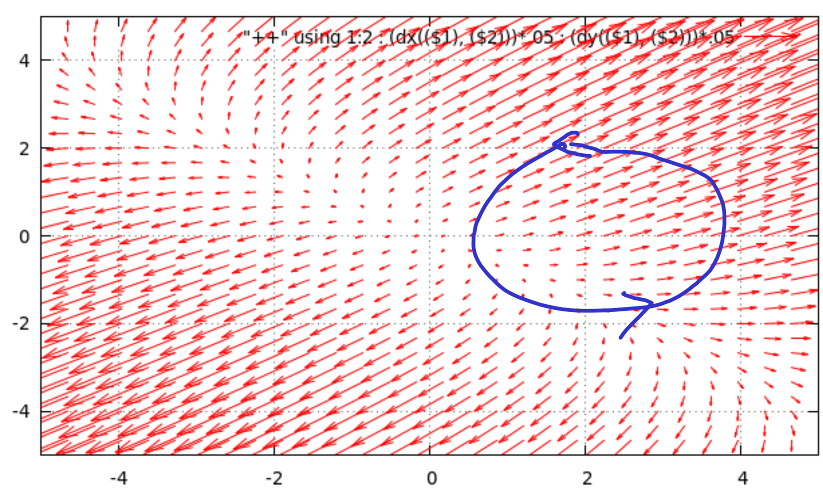
$$\cancel{Q(x,y) = \frac{\partial f}{\partial y}(x,y)}$$

$P(x,y) = 2y + 2x$ $\qquad$ $Q(x,y) = 2y + x$





$$P(x,y)\,\vec{i} + Q(x,y)\,\vec{j} = (2y+2x)\,\vec{i} + (2y+x)\,\vec{j}$$



$$\oint_C \vec{F} \cdot d\vec{r} \neq 0$$

~~conservative~~

* Exact Differential

$\boxed{P(x,y)\,dx + Q(x,y)\,dy}$: a differential

if this is a total differential of a certain function f
then it is exact differential

$$\boxed{\Rightarrow \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = dz}$$

$z = f(x,y)$

$$P(x,y) = \frac{\partial f}{\partial x} \qquad Q(x,y) = \frac{\partial f}{\partial y}$$

the necessary and sufficient condition

$\boxed{P(x,y)\,dx + Q(x,y)\,dy}$: an exact differential

$$\Leftrightarrow \boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

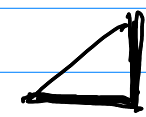
$$\oint_C xy\,dx = \int_{C_1} xy\,dx + \int_{C_2} xy\,dx + \int_{C_3} xy\,dx$$

$y=0$ $\qquad$ $y=1$ $\qquad$ $y=2x$

$$= \int_0^1 0\,dx + \int_1^1 x\,dx + \int_1^0 2x^2\,dx$$

$$= \left[\frac{2}{3}x^3\right]_1^0 = -\frac{2}{3}$$

$y=2x$



$$\oint_C x^2y^3\,dx = \int_{C_1} x^2y^3\,dy + \int_{C_2} x^2y^3\,dy + \int_{C_3} x^2y^3\,dy$$

$x=[0,1]$ $\qquad$ $x=1$ $\qquad$ $x=\frac{y}{2}$

$$= \int_0^0 x^2y^3\,dy + \int_0^2 y^3\,dy + \int_2^0 \frac{y^5}{4}\,dy$$

$$= \left[\frac{1}{4}y^4\right]_0^2 + \left[\frac{1}{24}y^6\right]_2^0$$

$$= 4 - \cancel{\frac{64}{24}}\ \cancel{\frac{16}{6}}\ \frac{8}{3} = \frac{12-8}{3} = +\frac{4}{3}$$

$\frac{2}{3}$