Laurent Series and z-Transform
- Geometric Series
Causality A

20180707

2 formulas of z

① $\frac{3}{2}\frac{-1}{(z-0.5)(z-2)} = \left(\frac{1}{z-0.5} - \frac{1}{z-2}\right)$

$\updownarrow z^{-1}$

② $\frac{3}{2}\frac{-z^2}{(z-2)(z-0.5)} = \left(\frac{0.5z}{(z-0.5)} - \frac{2z}{(z-2)}\right)$

$$\boxed{\frac{3}{2}\,\frac{-1}{(z-0.5)(z-2)} \quad \xleftrightarrow{\;z^{-1}\;} \quad \frac{3}{2}\,\frac{-z^{2}}{(z-2)(z-0.5)}}$$

$$\frac{3}{2}\,\frac{-1}{(z-0.5)(z-2)} = \frac{3}{2}\,\frac{2}{3}\left(\frac{1}{z-0.5}-\frac{1}{z-2}\right)$$

$$= \left(\frac{1}{z-0.5}-\frac{1}{z-2}\right)$$

$$\Big\downarrow z^{-1}$$

$$\frac{3}{2}\,\frac{-1}{(z^{-1}-0.5)(z^{-1}-2)} = \frac{3}{2}\,\frac{2}{3}\left(\frac{1}{z^{-1}-0.5}-\frac{1}{z^{-1}-2}\right)$$

$$= \left(\frac{2}{2z^{-1}-1}-\frac{0.5}{0.5z^{-1}-1}\right)$$

$$= \left(\frac{2z}{2-z}-\frac{0.5z}{0.5-z}\right)$$

$$= \left(\frac{-2z}{z-2}+\frac{0.5z}{z-0.5}\right)$$

$$= z\left(\frac{-2}{z-2}+\frac{0.5}{z-0.5}\right)$$

$$= z\left(\frac{-\frac{3}{2}z}{(z-2)(z-0.5)}\right)$$

$$= \frac{3}{2}\,\frac{-z^{2}}{(z-2)(z-0.5)}$$

$$\frac{3}{2}\,\frac{-z^{2}}{(z-2)(z-0.5)} = \frac{3}{2}\,\frac{2}{3}\left(\frac{0.5z}{(z-0.5)}-\frac{2z}{(z-2)}\right)$$

$f(z), g(z)$: causal form of $\boxed{\text{Laurent}}$ series

$$\frac{\text{nominator polynomial of } z}{\text{denominator polynomial of } z}$$

$f(z^{-1}), g(z^{-1})$: anti-causal form of $\boxed{\text{Laurent}}$ series

$$\frac{\text{nominator polynomial of } z^{-1}}{\text{denominator polynomial of } z^{-1}}$$

$X(z), Y(z)$: causal form of $\boxed{z\text{- Trans}}$

$$\frac{\text{nominator polynomial of } z^{-1}}{\text{denominator polynomial of } z^{-1}}$$

$X(z^{-1}), Y(z^{-1})$: anti-causal form of $\boxed{z\text{- Trans}}$

$$\frac{\text{nominator polynomial of } z}{\text{denominator polynomial of } z}$$

2 formulas

$$\boxed{\frac{1}{z - p}} \qquad \boxed{\frac{1}{z^{-1} - p}}$$

2 representations each

$$\boxed{\frac{1}{z - p}} \qquad \boxed{-\frac{p^{-1}}{1 - p^{-1} z}} \triangleq f(z) = X(z^{-1})$$

$$\| \qquad \|$$

$$\boxed{\frac{z^{-1}}{1 - p z^{-1}}} \triangleq Y(z) = g(z^{-1})$$

$$\boxed{\frac{1}{z^{-1} - p}} \qquad \boxed{\frac{z}{1 - p z}} \triangleq g(z) = Y(z^{-1})$$

$$\| \qquad \|$$

$$\boxed{-\frac{p^{-1}}{1 - p^{-1} z^{-1}}} \triangleq X(z) = f(z^{-1})$$

z polynomials $\qquad f(z) \qquad g(z) \qquad$ z polynomials

$$\begin{array}{cc} f(z) & g(z) \\ \| & \| \\ Y(z) & X(z) \end{array}$$

z^{-1} polynomials $\qquad Y(z) \qquad X(z) \qquad$ z^{-1} polynomials

z polynomials

z^{-1} polynomials

$$\begin{array}{ccc} X(z^{-1}) & = & f(z) \\ \| & & \| \\ g(z^{-1}) & = & Y(z) \end{array} \qquad \begin{array}{ccc} g(z) & = & Y(z^{-1}) \\ \| & & \| \\ X(z) & = & f(z^{-1}) \end{array}$$

z polynomials

z^{-1} polynomials

$\boxed{\text{Laurent}}$ $f(z), g(z)$: causal, $f(z^{-1}), g(z^{-1})$: anti-causal

$\boxed{\text{z- Trans}}$ $X(z), Y(z)$: causal, $X(z^{-1}), Y(z^{-1})$: anti-causal

$$
\begin{tikzcd}
\boxed{\dfrac{1}{z-p}} \arrow[r, shift left, "\frac{p^{-1}}{p^{-1}}"] \arrow[d, shift right, "\frac{z^{-1}}{z^{-1}}"'] & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z}} \arrow[l, shift left] \\
\boxed{\dfrac{z^{-1}}{1-pz^{-1}}} \arrow[u, shift right] &
\end{tikzcd}
$$

$= f(z)$ $\quad \leftrightarrow \quad -(p^{-1})^{n+1}$

$X(z^{-1})$

$\|$

$g(z^{-1})$

$Y(z) \quad \leftrightarrow \quad (p)^{n-1}$

$$
\begin{tikzcd}
\boxed{\dfrac{1}{z^{-1}-p}} \arrow[r, shift left, "\frac{p^{-1}}{p^{-1}}"] \arrow[d, shift right, "\frac{z^{-1}}{z^{-1}}"'] & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}} \arrow[l, shift left] \\
\boxed{\dfrac{z}{1-pz}} \arrow[u, shift right] &
\end{tikzcd}
$$

$= f(z^{-1})$

$X(z) \quad \leftrightarrow \quad -(p^{-1})^{n+1}$

$\|$

$g(z) \quad \leftrightarrow \quad (p)^{n-1}$

$Y(z^{-1})$

2 formulas of z
2 representations

$f(z),\ g(z)$	$X(z),\ Y(z)$
$f(z^{-1}),\ g(z^{-1})$	$X(z^{-1}),\ Y(z^{-1})$

* Simple Pole Forms

$$p \qquad \boxed{\frac{1}{z-p}} \xleftrightarrow{z^{-1}} \boxed{\frac{1}{z^{-1}-p}} \qquad p^{-1}$$

$$\frac{z^{-1}}{z^{-1}} \cdot \qquad \cdot \frac{z}{z} \qquad\qquad\qquad \frac{z}{z} \cdot \qquad \cdot \frac{z^{-1}}{z^{-1}}$$

$$p \qquad \boxed{\frac{z^{-1}}{1-pz^{-1}}} \xleftrightarrow{z^{-1}} \boxed{\frac{z}{1-pz}} \qquad p^{-1}$$

* Geometric Series Forms

$n \geq 0$ $\qquad |z| < p$ $\qquad\qquad |z| > p^{-1}$ $\qquad n \geq 0$

$$\boxed{-(p^{-1})^{n+1}\ \text{causal } f(z)} \quad \boxed{-\frac{p^{-1}}{1-p^{-1}z}} \xleftrightarrow{z^{-1}} \boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} \quad \boxed{\text{causal } X(z)\ -(p^{-1})^{n+1}}$$

$$\cdot \frac{p}{p}\,\frac{z^{-1}}{z^{-1}} \qquad\qquad\qquad \frac{p}{p}\,\frac{z}{z} \cdot$$

$$\boxed{\text{causal } X(z)\ (p)^{n-1}} \quad \boxed{\frac{z^{-1}}{1-pz^{-1}}} \xleftrightarrow{z^{-1}} \boxed{\frac{z}{1-pz}} \quad \boxed{(p)^{n-1}\ \text{causal } f(z)}$$

$n \geq 1$ $\qquad |z| > p$ $\qquad\qquad |z| < p^{-1}$ $\qquad n \geq 1$

Laurent Series

$f(z) \ (|z| < \rho) \leftrightarrow a_n \ (n \geq 0) \quad -(\rho^{-1}, \rho^{-2}, \rho^{-3}, \cdots)$

$f(z^{-1}) \ (|z| > \rho^{-1}) \leftrightarrow a_{-n} \ (n < 1) \quad -(\rho^{-1}, \rho^{-2}, \rho^{-3}, \cdots)$

$f(z) \ (|z| > \rho) \leftrightarrow -a_n \ (n < 0) \quad (\rho^{0}, \rho^{1}, \rho^{2}, \cdots)$

$f(z^{-1}) \ (|z| < \rho^{-1}) \leftrightarrow -a_{-n} \ (n \geq 1) \quad (\rho^{0}, \rho^{1}, \rho^{2}, \cdots)$

$$f(z) \ (|z| < p) \longleftrightarrow a_n \ (n \geq 0) \qquad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$$

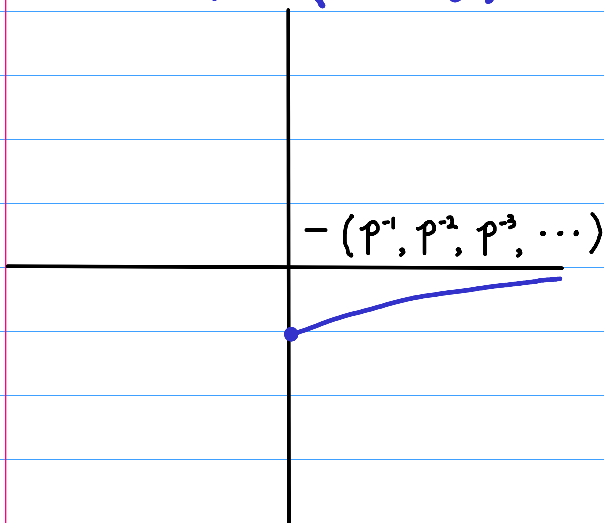
$$f(z^{-1}) \ (|z| > p^{-1}) \longleftrightarrow a_{-n} \ (n < 1) \qquad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$$

$$f(z) \ (|z| > p) \longleftrightarrow -a_n \ (n < 0) \qquad (p^{0}, p^{1}, p^{2}, \cdots)$$

$$f(z^{-1}) \ (|z| < p^{-1}) \longleftrightarrow -a_{-n} \ (n \geq 1) \qquad (p^{0}, p^{1}, p^{2}, \cdots)$$

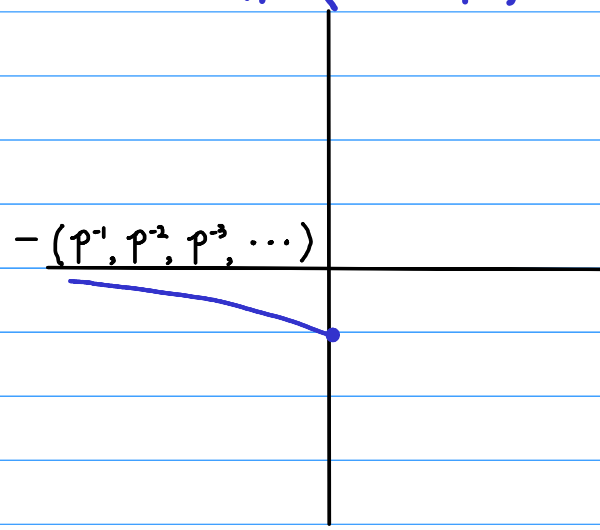
$f(z)$ $(|z| < p)$

a_n $(n \geq 0)$



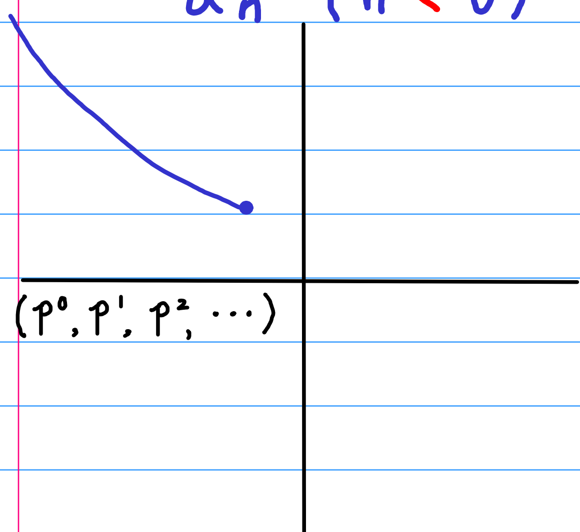
$f(z^{-1})$ $(|z| > p^{-1})$

a_{-n} $(n < 1)$



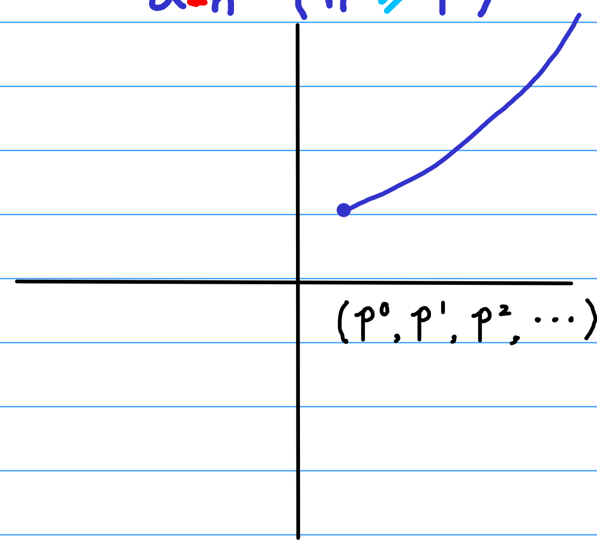
$f(z)$ $(|z| > p)$

$-a_n$ $(n < 0)$



$f(z^{-1})$ $(|z| < p^{-1})$

$-a_{-n}$ $(n \geq 1)$



Ⓐ $f(z)$ for $\boxed{|z|<p}$, $g(z)$ for $\boxed{|z|<p^{-1}}$ Laurent S

Geometric Series Forms

$|z|<p$

p $\boxed{f(z)} = \boxed{-\dfrac{p^{-1}}{1-p^{-1}z}}$ $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}$

$\boxed{\dfrac{z^{-1}}{1-pz^{-1}}}$ $\boxed{\dfrac{z}{1-pz}} = \boxed{g(z)}$ p^{-1}

$|z|<p^{-1}$

Ⓑ $f(z^{-1})$ for $\boxed{|z|>p^{-1}}$, $g(z^{-1})$ for $\boxed{|z|>p}$

Geometric Series Forms

$|z|<p$ $|z|>p^{-1}$

$f(z) = \boxed{-\dfrac{p^{-1}}{1-p^{-1}z}} \xleftrightarrow{z^{-1}} \boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}} = \boxed{f(z^{-1})}$

$\boxed{g(z^{-1})} = \boxed{\dfrac{z^{-1}}{1-pz^{-1}}} \xleftrightarrow{z^{-1}} \boxed{\dfrac{z}{1-pz}} = g(z)$

$|z|>p$ $|z|<p^{-1}$

Laurent Series $\boxed{a_n} \leftrightarrow f(z)$

symmetric ranges

$$a_n \ (n \geq 0) \xleftrightarrow{z^{-1}} a_{-n} \ (n < 1)$$

symmetric ranges

$$b_{-n} \ (n < 0) \xleftrightarrow{z^{-1}} b_n \ (n \geq 1)$$

$$a_n \ (n \geq 0)$$

complementary ranges

$$-a_n \ (n < 0)$$

$$-b_n \ (n < 1)$$

complementary ranges

$$b_n \ (n \geq 1)$$

$$a_n \ (n \geq 0) \xleftrightarrow{z^{-1}} a_{-n} \ (n < 1)$$

$$-a_n \ (n < 0) \xleftrightarrow{z^{-1}} -a_{-n} \ (n \geq 1)$$

Laurent Series $a_n \leftrightarrow \boxed{f(z)}$

ROC's with reciprocal poles

$$\boxed{f(z)\ (|z|<p)} \xleftrightarrow{\ z^{-1}\ } \boxed{f(z^{-1})\ (|z^{-1}|<p)}$$

ROC's with reciprocal poles

$$\boxed{g(z^{-1})\ (|z^{-1}|<p^{-1})} \xleftrightarrow{\ z^{-1}\ } \boxed{g(z)\ (|z|<p^{-1})}$$

$$\begin{tikzcd}
\boxed{f(z)\ (|z|<p)} \arrow[d, equal, "\text{complementary ROC}"] & \boxed{f(z^{-1})\ (|z|>p^{-1})} \arrow[d, equal, "\text{complementary ROC}"] \\
\boxed{g(z^{-1})\ (|z|>p)} & \boxed{g(z)\ (|z|<p^{-1})}
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{f(z)\ (|z|<p)} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, equal] & \boxed{f(z^{-1})\ (|z|>p^{-1})} \arrow[d, equal] \\
\boxed{f(z)\ (|z|>p)} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{f(z^{-1})\ (|z|<p^{-1})}
\end{tikzcd}$$

Laurent Series $\quad a_n \leftrightarrow f(z) \quad b_n \leftrightarrow g(z)$

Geometric Series Forms

$$\underset{|z|<p}{f(z) = \boxed{-\frac{p^{-1}}{1-p^{-1}z}}} \xleftrightarrow{z^{-1}} \underset{|z|>p^{-1}}{\boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} = f(z^{-1})}$$

$$\cdot\frac{p}{p}\frac{z^{-1}}{z^{-1}} \qquad\qquad \cdot\frac{p}{p}\frac{z}{z}$$

$$\underset{|z|>p}{g(z^{-1}) = \boxed{\frac{z^{-1}}{1-pz^{-1}}}} \xleftrightarrow{z^{-1}} \underset{|z|<p^{-1}}{\boxed{\frac{z}{1-pz}} = g(z)}$$

	$f(z)$ $(\lvert z\rvert<p)$ a_n $(n \geq 0)$	$\xleftrightarrow{z^{-1}}$	$f(z^{-1})$ $(\lvert z^{-1}\rvert<p)$ a_{-n} $(n<1)$
			$g(z)$ $(\lvert z\rvert>p^{-1})$ $-b_n$ $(n<1)$
$f(z)$ $(\lvert z\rvert>p)$ $-a_n$ $(n<0)$	$g(z^{-1})$ $(\lvert z^{-1}\rvert<p^{-1})$ b_{-n} $(n<0)$	$\xleftrightarrow{z^{-1}}$	$g(z)$ $(\lvert z\rvert<p^{-1})$ b_n $(n \geq 1)$

Laurent Series using only $\boxed{a_n} \leftrightarrow \boxed{f(z)}$

$$
\begin{tikzcd}
 & \boxed{\begin{array}{ll} f(z) & (|z|<p) \\ a_n & (n \geq 0) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, equal] & \boxed{\begin{array}{ll} f(z^{-1}) & (|z^{-1}|<p) \\ a_{-n} & (n<1) \end{array}} \arrow[d, equal] & \boxed{\begin{array}{ll} g(z) & (|z|>p^{-1}) \\ -b_n & (n<1) \end{array}} \\
\boxed{\begin{array}{ll} f(z) & (|z|>p) \\ -a_n & (n<0) \end{array}} & \boxed{\begin{array}{ll} g(z^{-1}) & (|z^{-1}|<p^{-1}) \\ b_{-n} & (n<0) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{ll} g(z) & (|z|<p^{-1}) \\ b_n & (n \geq 1) \end{array}} &
\end{tikzcd}
$$

$$a_{-n} = -b_n \qquad\qquad -a_{-n} = b_n$$

$$
\begin{tikzcd}
\boxed{\begin{array}{ll} f(z) & (|z|<p) \\ a_n & (n \geq 0) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, equal] & \boxed{\begin{array}{ll} f(z^{-1}) & (|z|>p^{-1}) \\ a_{-n} & (n<1) \end{array}} \arrow[d, equal] \\
\boxed{\begin{array}{ll} f(z) & (|z|>p) \\ -a_n & (n<0) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{ll} f(z^{-1}) & (|z|<p^{-1}) \\ -a_{-n} & (n \geq 1) \end{array}}
\end{tikzcd}
$$

Z - Transform

$X(z^{-1})\ (|z| < p) \leftrightarrow x_{-n}\ (n < 1)\quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

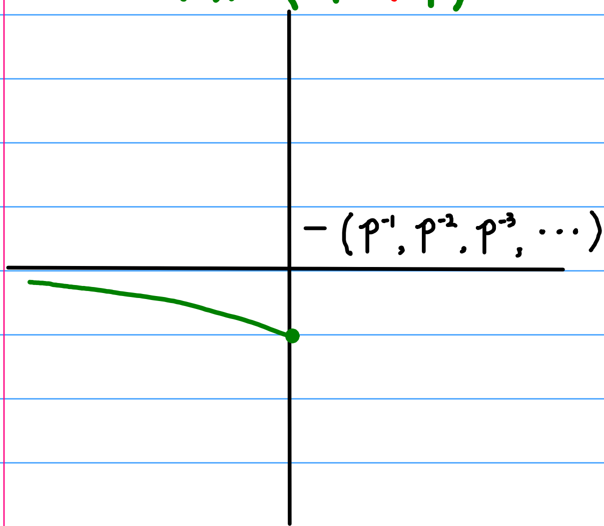
$X(z)\ (|z| > p^{-1}) \leftrightarrow x_n\ (n \geq 0)\quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$X(z^{-1})\ (|z| > p) \leftrightarrow -x_{-n}\ (n \geq 1)\quad (p^0, p^1, p^2, \cdots)$

$X(z)\ (|z| < p^{-1}) \leftrightarrow -x_n\ (n < 0)\quad (p^0, p^1, p^2, \cdots)$

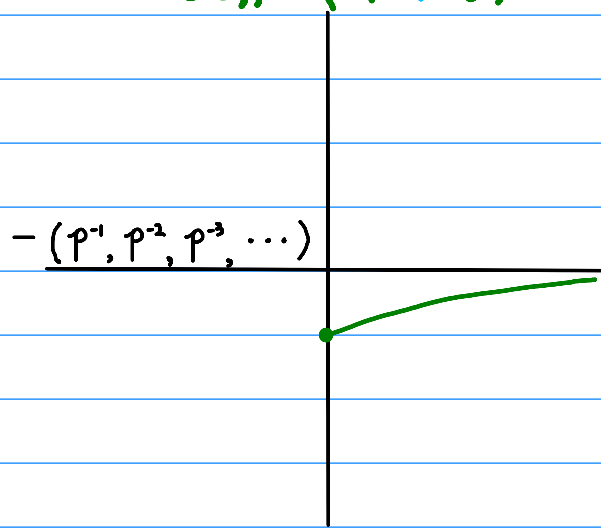
$X(z^{-1})$ $(|z| < p)$

x_{-n} $(n < 1)$



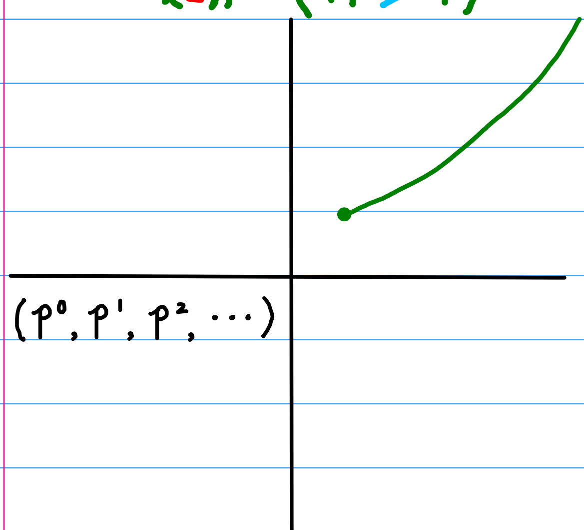
$X(z)$ $(|z| > p^{-1})$

x_n $(n \geq 0)$



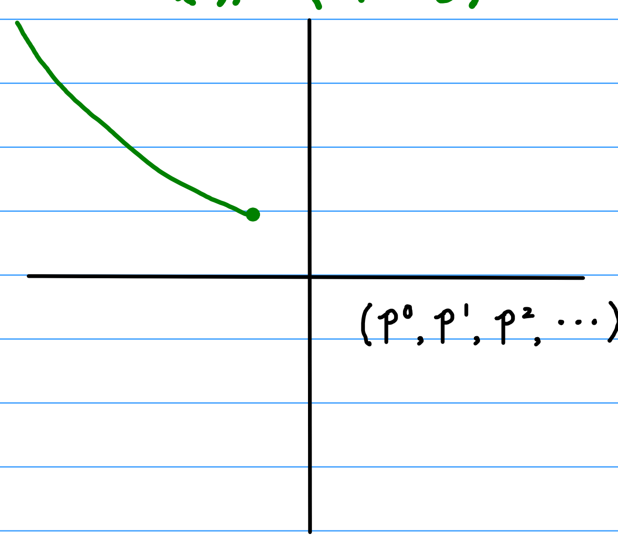
$X(z^{-1})$ $(|z| > p)$

$-x_{-n}$ $(n \geq 1)$



$X(z)$ $(|z| < p^{-1})$

$-x_n$ $(n < 0)$

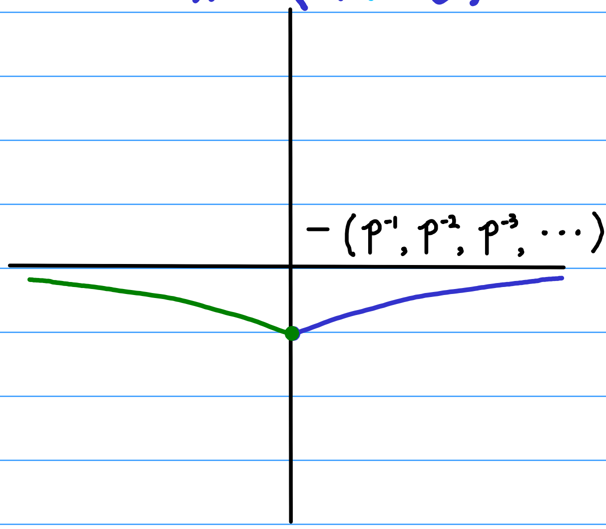


$X(z^{-1})$ $(|z| < p)$

x_{-n} $(n < 1)$

$f(z)$ $(|z| < p)$

a_n $(n \geq 0)$

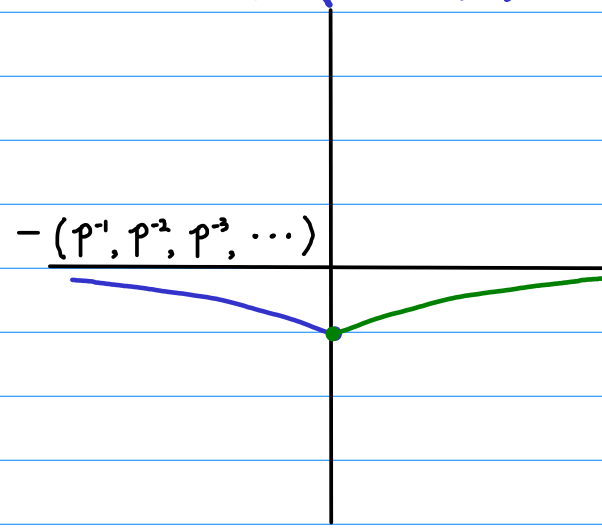


$X(z)$ $(|z| > p^{-1})$

x_n $(n \geq 0)$

$f(z^{-1})$ $(|z| > p^{-1})$

a_{-n} $(n < 1)$

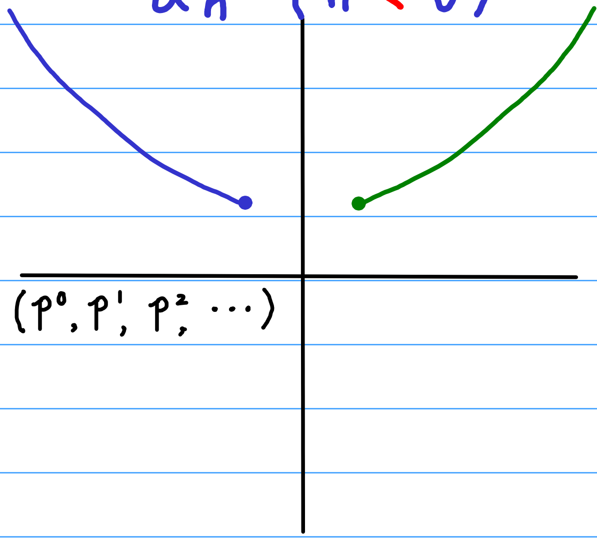


$X(z^{-1})$ $(|z| > p)$

$-x_{-n}$ $(n \geq 1)$

$f(z)$ $(|z| > p)$

$-a_n$ $(n < 0)$

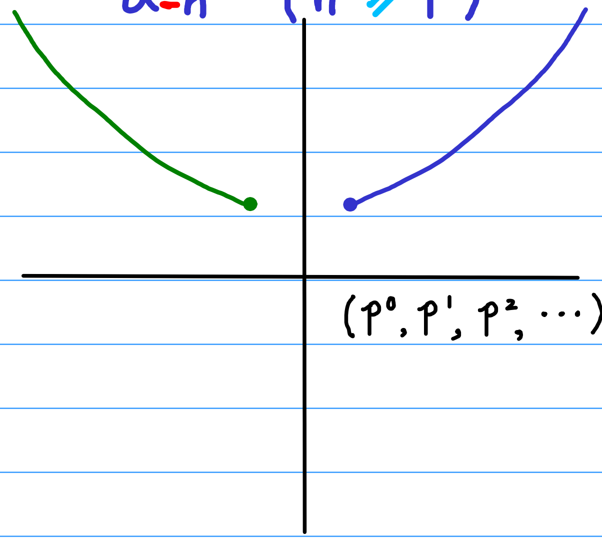


$X(z)$ $(|z| < p^{-1})$

$-x_n$ $(n < 0)$

$f(z^{-1})$ $(|z| < p^{-1})$

$-a_{-n}$ $(n \geq 1)$



Ⓐ $X(z)$ for $\boxed{|z| > p^{-1}}$, $Y(z)$ for $\boxed{|z| > p}$ z-Transform

Geometric Series Forms

$$\boxed{-\frac{p^{-1}}{1 - p^{-1} z}} \qquad \overset{|z| > p^{-1}}{\boxed{-\frac{p^{-1}}{1 - p^{-1} z^{-1}}}} = \boxed{X(z)} \quad p^{-1}$$

$$p \quad \boxed{Y(z)} = \overset{|z| > p}{\boxed{\frac{z^{-1}}{1 - p z^{-1}}}} \qquad \boxed{\frac{z}{1 - p z}}$$

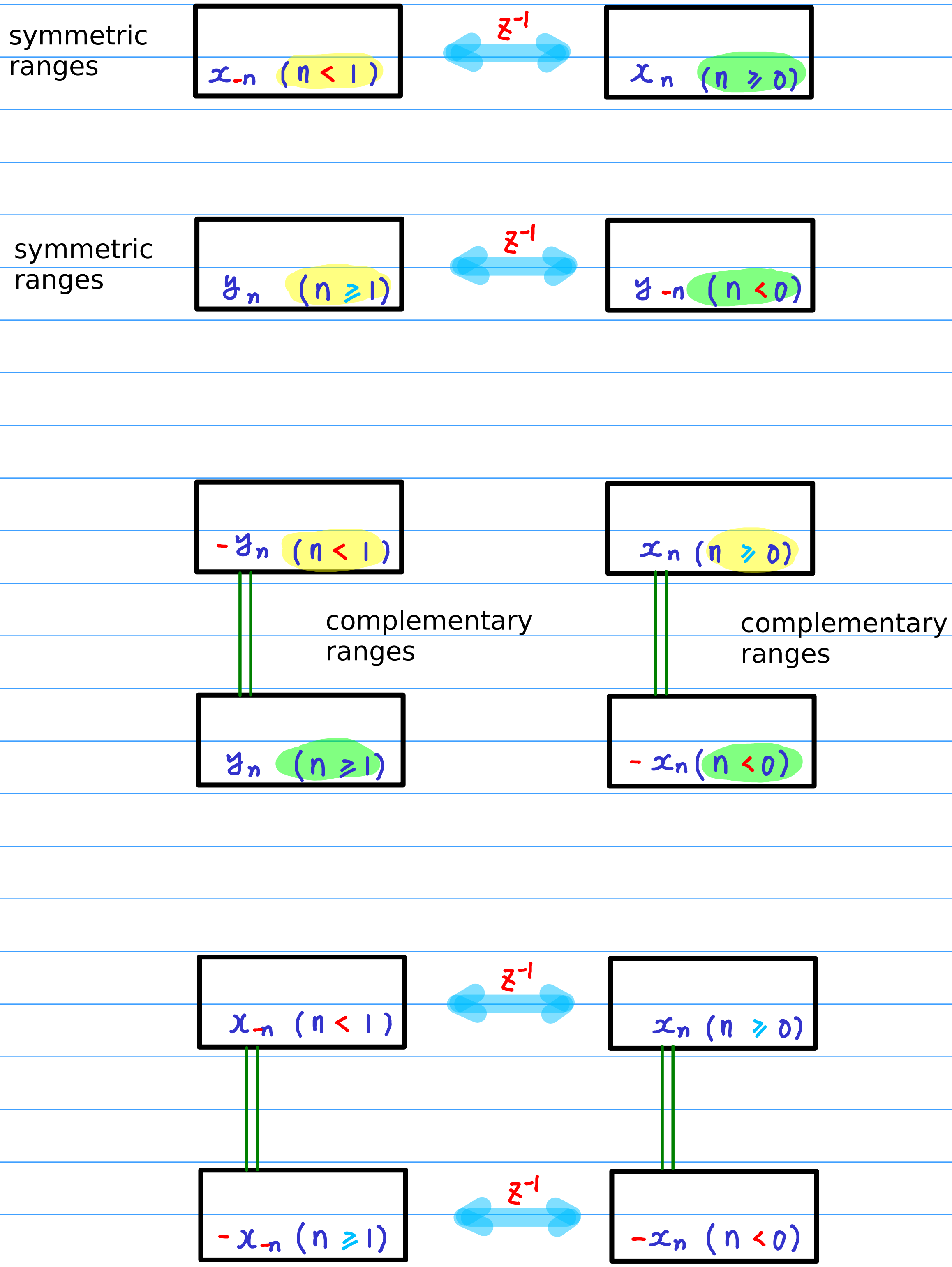
Ⓑ $X(z^{-1})$ for $\boxed{|z| < p}$, $Y(z^{-1})$ for $\boxed{|z| < p^{-1}}$

Geometric Series Forms

$$\boxed{X(z^{-1})} = \overset{|z| < p}{\boxed{-\frac{p^{-1}}{1 - p^{-1} z}}} \xleftrightarrow{z^{-1}} \overset{|z| > p^{-1}}{\boxed{-\frac{p^{-1}}{1 - p^{-1} z^{-1}}}} = X(z)$$

$$Y(z) = \underset{|z| > p}{\boxed{\frac{z^{-1}}{1 - p z^{-1}}}} \xleftrightarrow{z^{-1}} \underset{|z| < p^{-1}}{\boxed{\frac{z}{1 - p z}}} = \boxed{Y(z^{-1})}$$

z - Transform $\boxed{x_n} \leftrightarrow X(z)$



z - Transform $x_n \leftrightarrow \boxed{X(z)}$

ROC's with reciprocal poles

$$\boxed{X(z^{-1})\ (|z^{-1}| > p^{-1})} \xleftrightarrow{z^{-1}} \boxed{X(z)\ (|z| > p^{-1})}$$

ROC's with reciprocal poles

$$\boxed{Y(z)\ (|z| > p)} \xleftrightarrow{z^{-1}} \boxed{Y(z^{-1})\ (|z^{-1}| > p)}$$

$$\boxed{Y(z)\ (|z| < p)} \qquad \boxed{X(z)\ (|z| > p^{-1})}$$

complementary ROC

complementary ROC

$$\boxed{Y(z)\ (|z| > p)} \qquad \boxed{X(z)\ (|z| < p)}$$

$$\boxed{X(z^{-1})\ (|z| < p)} \xleftrightarrow{z^{-1}} \boxed{X(z)\ (|z| > p^{-1})}$$

$$\boxed{X(z^{-1})\ (|z| > p)} \xleftrightarrow{z^{-1}} \boxed{X(z)\ (|z| < p)}$$

Z- Transform $X(z) \leftrightarrow x_n$ $Y(z) \leftrightarrow y_n$

Geometric Series Forms

$$
\begin{tikzcd}[column sep=huge, row sep=large]
X(z^{-1}) = \underset{|z| < p}{\boxed{-\dfrac{p^{-1}}{1 - p^{-1} z}}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, equal, "\cdot \frac{p}{p}\frac{z^{-1}}{z^{-1}}"] & \underset{|z| > p^{-1}}{\boxed{-\dfrac{p^{-1}}{1 - p^{-1} z^{-1}}}} = X(z) \arrow[d, equal, "\cdot \frac{p}{p}\frac{z}{z}"'] \\
Y(z) = \underset{|z| > p}{\boxed{\dfrac{z^{-1}}{1 - p z^{-1}}}} \arrow[r, leftrightarrow, "z^{-1}"] & \underset{|z| < p^{-1}}{\boxed{\dfrac{z}{1 - p z}}} = Y(z^{-1})
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=large, row sep=large]
\boxed{\begin{array}{c} Y(z)\ (|z| < p) \\ -y_n\ (n < 1) \end{array}} & \boxed{\begin{array}{c} X(z^{-1})\ (|z^{-1}| > p^{-1}) \\ x_{-n}\ (n < 1) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, equal] & \boxed{\begin{array}{c} X(z)\ (|z| > p^{-1}) \\ x_n\ (n \geq 0) \end{array}} \arrow[d, equal] & \\
 & \boxed{\begin{array}{c} Y(z)\ (|z| > p) \\ y_n\ (n \geq 1) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{c} Y(z^{-1})\ (|z^{-1}| > p) \\ y_{-n}\ (n < 0) \end{array}} & \boxed{\begin{array}{c} X(z)\ (|z| < p^{-1}) \\ -x_n\ (n < 0) \end{array}}
\end{tikzcd}
$$

z-Transform using only $\boxed{x_n} \leftrightarrow \boxed{X(z)}$

$$
\begin{tikzcd}
\boxed{\begin{array}{c} Y(z)\ (|z|<p) \\ -y_n\ (n<1) \end{array}} & \boxed{\begin{array}{c} X(z^{-1})\ (|z^{-1}|>p^{-1}) \\ x_{-n}\ (n<1) \end{array}} \arrow[d, dash, shift left] \arrow[d, dash, shift right] \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{c} X(z)\ (|z|>p^{-1}) \\ x_n\ (n\geq 0) \end{array}} \arrow[d, dash, shift left] \arrow[d, dash, shift right] & \\
& \boxed{\begin{array}{c} Y(z)\ (|z|>p) \\ y_n\ (n\geq 1) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{c} Y(z^{-1})\ (|z^{-1}|>p) \\ y_{-n}\ (n<0) \end{array}} & \boxed{\begin{array}{c} X(z)\ (|z|<p^{-1}) \\ -x_n\ (n<0) \end{array}}
\end{tikzcd}
$$

$$x_{-n} = -y_n \qquad -x_{-n} = y_n$$

$$
\begin{tikzcd}
\boxed{\begin{array}{c} X(z^{-1})\ (|z|<p) \\ x_{-n}\ (n<1) \end{array}} \arrow[d, dash, shift left] \arrow[d, dash, shift right] \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{c} X(z)\ (|z|>p^{-1}) \\ x_n\ (n\geq 0) \end{array}} \arrow[d, dash, shift left] \arrow[d, dash, shift right] \\
\boxed{\begin{array}{c} X(z^{-1})\ (|z|>p) \\ -x_{-n}\ (n\geq 1) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{c} X(z)\ (|z|<p^{-1}) \\ -x_n\ (n<0) \end{array}}
\end{tikzcd}
$$

Laurent Series and z-Transform

with the same ROC

	$f(z)$ $(\lvert z\rvert < p)$ a_n $(n \geq 0)$	$\xleftrightarrow{z^{-1}}$	$f(z^{-1})$ $(\lvert z^{-1}\rvert < p)$ a_{-n} $(n < 1)$	$g(z)$ $(\lvert z\rvert > p^{-1})$ $-b_n$ $(n < 1)$
	$\Vert$		$\Vert$	
$f(z)$ $(\lvert z\rvert > p)$ $-a_n$ $(n < 0)$	$g(z^{-1})$ $(\lvert z^{-1}\rvert < p^{-1})$ b_{-n} $(n < 0)$	$\xleftrightarrow{z^{-1}}$	$g(z)$ $(\lvert z\rvert < p^{-1})$ b_n $(n \geq 1)$	

$Y(z)$ $(\lvert z\rvert < p)$ $-y_n$ $(n < 1)$	$X(z^{-1})$ $(\lvert z^{-1}\rvert > p^{-1})$ x_{-n} $(n < 1)$	$\xleftrightarrow{z^{-1}}$	$X(z)$ $(\lvert z\rvert > p^{-1})$ x_n $(n \geq 0)$	
	$\Vert$		$\Vert$	
	$Y(z)$ $(\lvert z\rvert > p)$ y_n $(n \geq 1)$	$\xleftrightarrow{z^{-1}}$	$Y(z^{-1})$ $(\lvert z^{-1}\rvert > p)$ y_{-n} $(n < 0)$	$X(z)$ $(\lvert z\rvert < p^{-1})$ $-x_n$ $(n < 0)$

$\boxed{\text{Laurent Series}}$ and $\boxed{z\text{-Transform}}$ $\quad a_n, \ x_n$

with the same ROC

$$\begin{array}{ccc}
\boxed{\begin{array}{ll} f(z) & (|z|<p) \\ a_n & (n \geq 0) \end{array}} & \xleftrightarrow{z^{-1}} & \boxed{\begin{array}{ll} f(z^{-1}) & (|z|>p^{-1}) \\ a_{-n} & (n<1) \end{array}} \\
\| & & \| \\
\boxed{\begin{array}{ll} f(z) & (|z|>p) \\ -a_n & (n<0) \end{array}} & \xleftrightarrow{z^{-1}} & \boxed{\begin{array}{ll} f(z^{-1}) & (|z|<p^{-1}) \\ -a_{-n} & (n \geq 1) \end{array}}
\end{array}$$

$$a_n = x_{-n} \qquad\qquad a_{-n} = x_n$$

$$\begin{array}{ccc}
\boxed{\begin{array}{ll} X(z^{-1}) & (|z|<p) \\ x_{-n} & (n<1) \end{array}} & \xleftrightarrow{z^{-1}} & \boxed{\begin{array}{ll} X(z) & (|z|>p^{-1}) \\ x_n & (n \geq 0) \end{array}} \\
\| & & \| \\
\boxed{\begin{array}{ll} X(z^{-1}) & (|z|>p) \\ -x_{-n} & (n \geq 1) \end{array}} & \xleftrightarrow{z^{-1}} & \boxed{\begin{array}{ll} X(z) & (|z|<p^{-1}) \\ -x_n & (n<0) \end{array}}
\end{array}$$

Laurent Series and z-Transform Examples

with the same ROC

	$n \geq 0$	$\lvert z\rvert < p$		$\lvert z\rvert > p^{-1}$	$n < 1$	
a_n	causal f(z) $-(p^{-1})^{n+1}$	$-\dfrac{p^{-1}}{1-p^{-1}z}$	$\xleftrightarrow{z^{-1}}$	$-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}$	anti-causal f(z) $-(p^{-1})^{-n+1}$	a_{-n}
	$\cdot(-1)$	$= \cdot\dfrac{p}{p}\dfrac{z^{-1}}{z^{-1}}$		$= \cdot\dfrac{p}{p}\dfrac{z}{z}$	$\cdot(-1)$	
b_{-n}	$(p)^{-n-1}$ anti-causal f(z)	$\dfrac{z^{-1}}{1-pz^{-1}}$	$\xleftrightarrow{z^{-1}}$	$\dfrac{z}{1-pz}$	$(p)^{n-1}$ causal f(z)	b_n
	$n < 0$	$\lvert z\rvert > p$		$\lvert z\rvert < p^{-1}$	$n \geq 1$	

z-Transform

	$n < 1$	$\lvert z\rvert < p$		$\lvert z\rvert > p^{-1}$	$n \geq 0$	
x_{-n}	anti-causal X(z) $-(p^{-1})^{-n+1}$	$-\dfrac{p^{-1}}{1-p^{-1}z}$	$\xleftrightarrow{z^{-1}}$	$-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}$	causal X(z) $-(p^{-1})^{n+1}$	x_n
	$\cdot(-1)$	$= \cdot\dfrac{p}{p}\dfrac{z^{-1}}{z^{-1}}$		$= \cdot\dfrac{p}{p}\dfrac{z}{z}$	$\cdot(-1)$	
y_n	$(p)^{n-1}$ causal X(z)	$\dfrac{z^{-1}}{1-pz^{-1}}$	$\xleftrightarrow{z^{-1}}$	$\dfrac{z}{1-pz}$	$(p)^{-n-1}$ anti-causal X(z)	y_{-n}
	$n \geq 1$	$\lvert z\rvert > p$		$\lvert z\rvert < p^{-1}$	$n < 0$	

$\boxed{\text{Laurent Series}}$ and $\boxed{z\text{-Transform}}$

$$f(z)\ (|z|<p) \quad \leftrightarrow \quad a_n \quad (n \geq 0) \qquad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$$
$$X(z^{-1})\ (|z|<p) \quad \leftrightarrow \quad x_{-n} \quad (n < 1) \qquad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$$

$$f(z^{-1})\ (|z|>p^{-1}) \quad \leftrightarrow \quad a_{-n} \quad (n < 1) \qquad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$$
$$X(z)\ (|z|>p^{-1}) \quad \leftrightarrow \quad x_n \quad (n \geq 0) \qquad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$$

$$f(z)\ (|z|>p) \quad \leftrightarrow \quad -a_n \quad (n < 0) \qquad (p^{0}, p^{1}, p^{2}, \cdots)$$
$$X(z^{-1})\ (|z|>p) \quad \leftrightarrow \quad -x_{-n} \quad (n \geq 1) \qquad (p^{0}, p^{1}, p^{2}, \cdots)$$

$$f(z^{-1})\ (|z|<p^{-1}) \quad \leftrightarrow \quad -a_{-n} \quad (n \geq 1) \qquad (p^{0}, p^{1}, p^{2}, \cdots)$$
$$X(z)\ (|z|<p^{-1}) \quad \leftrightarrow \quad -x_n \quad (n < 0) \qquad (p^{0}, p^{1}, p^{2}, \cdots)$$

①

$f(z) \ (|z| < p) \longleftrightarrow a_n \ (n \geq 0)$

the same ROC

symmetric ranges

$X(z^{-1}) \ (|z| < p) \longleftrightarrow x_{-n} \ (n < 1)$

$|z| < p$ $\qquad$ $n \geq 0$

$f(z) \ \boxed{-\dfrac{p^{-1}}{1 - p^{-1}z}}$ $\qquad$ $\boxed{\text{causal } f(z) \quad -(p^{-1})^{n+1}}$ $\ a_n$

$|z| < p$ $\qquad$ $n < 1$

$X(z) \ \boxed{-\dfrac{p^{-1}}{1 - p^{-1}z}}$ $\qquad$ $\boxed{\text{anti-causal } X(z) \quad -(p^{-1})^{-n+1}}$ $\ x_{-n}$

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$f(z^{-1})$ $(|z| > p^{-1})$ $\leftrightarrow$ a_{-n} $(n < 1)$

the same ROC

symmetric ranges

$X(z)$ $(|z| > p^{-1})$ $\leftrightarrow$ x_n $(n \geq 0)$

$|z| > p^{-1}$

$f(z^{-1})$ $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}$

$n < 1$

anti-causal f(z)

$-(p^{-1})^{-n+1}$ a_{-n}

$|z| > p^{-1}$

$X(z)$ $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}$

$n \geq 0$

causal X(z)

$-(p^{-1})^{n+1}$ x_n

$n = 0, 1, 2, \cdots$ $\qquad -p^{-n-1} z^n$

$$\boxed{-\left(p^{-1} + p^{-2} z^{1} + p^{-3} z^{2} + \cdots\right) = \sum_{n=0}^{\infty} -\left(p^{-1}\right)^{n+1} z^n \quad (n \geq 0)}$$

$$\boxed{p^{0} z^{-1} + p^{1} z^{-2} + p^{2} z^{-3} + \cdots = \sum_{n=-1}^{-\infty} (p)^{-n-1} z^n \quad (n < 0)}$$

$n = -1, -2, -3, \cdots$ $\qquad p^{-n-1} z^n$

$n = 0, -1, -2, \cdots$ $\qquad -p^{n-1} z^n$

$$\boxed{-\left(p^{-1} + p^{-2} z^{-1} + p^{-3} z^{-2} + \cdots\right) = \sum_{n=0}^{-\infty} -(p)^{n-1} z^n \quad (n < 1)}$$

$$\boxed{p^{0} z^{1} + p^{1} z^{2} + p^{2} z^{3} + \cdots = \sum_{n=1}^{\infty} (p)^{n-1} z^n \quad (n \geq 1)}$$

$n = 1, 2, 3, \cdots$ $\qquad p^{n-1} z^n$

$n = 0, -1, -2, \cdots$ $\quad -p^{n-1} z^{-n}$

$$\boxed{-(p^{-1} + p^{-2} z^{1} + p^{-3} z^{2} + \cdots) = -\sum_{n=0}^{-\infty} (p^{-1})^{-n+1} z^{-n} \qquad n < 1}$$

$$\boxed{p^{0} z^{-1} + p^{1} z^{-2} + p^{2} z^{-3} + \cdots = \sum_{n=1}^{\infty} (p)^{n-1} z^{-n} \qquad n \geq 1}$$

$n = 1, 2, 3, \cdots$ $\quad p^{n-1} z^{-n}$

$n = 0, 1, 2, \cdots$ $\quad -p^{-n-1} z^{-n}$

$$\boxed{-(p^{-1} + p^{-2} z^{-1} + p^{-3} z^{-2} + \cdots) = \sum_{n=0}^{\infty} -(p^{-1})^{n+1} z^{-n} \qquad (n \geq 0)}$$

$$\boxed{p^{0} z^{1} + p^{1} z^{2} + p^{2} z^{3} + \cdots = \sum_{n=-1}^{-\infty} (p^{-1})^{n+1} z^{-n} \qquad (n < 0)}$$

$n = -1, -2, -3, \cdots$ $\quad p^{-n-1} z^{-n}$