

Laurent Series and z-Transform

- Geometric Series

Permutations B

20240527 Mon

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Unshifted Sequence

(1) $a^n u(n)$

(2) $a^{-n} u(n)$

(3) $a^n u(-n)$

(4) $a^{-n} u(-n)$

(5) $a^n u(-n-1)$

(6) $a^{-n} u(-n-1)$

(7) $a^n u(n-1)$

(8) $a^{-n} u(n-1)$

Shifted Sequence

$a^{n+i} u(n)$ (1')

$a^{-n-i} u(n)$ (2')

$a^{n-i} u(-n)$ (3')

$a^{-n+i} u(-n)$ (4')

$a^{n+i} u(-n-1)$ (5')

$a^{-n-i} u(-n-1)$ (6')

$a^{n-i} u(n-1)$ (7')

$a^{-n+i} u(n-1)$ (8')

Inter-permutations over unshifted sequence and shifted sequence

Intra-permutations over unshifted sequence

Intra-permutations over shifted sequence

Unshifted Sequence**Shifted Sequence****Shifted Sequence**

(1) $a^n u(n)$

$a^{n+i} u(n)$ (1')

$a^{-n-i} u(n)$ (1'')

(2) $a^{-n} u(n)$

$a^{-n-i} u(n)$ (2')

$a^{n+i} u(n)$ (2'')

(3) $a^n u(-n)$

$a^{n-i} u(-n)$ (3')

$a^{-n+i} u(-n)$ (3'')

(4) $a^{-n} u(-n)$

$a^{-n+i} u(-n)$ (4')

$a^{n-i} u(-n)$ (4'')

(5) $a^n u(-n-1)$

$a^{n+i} u(-n-1)$ (5')

$a^{-n-i} u(-n-1)$ (5'')

(6) $a^{-n} u(-n-1)$

$a^{-n-i} u(-n-1)$ (6')

$a^{n+i} u(-n-1)$ (6'')

(7) $a^n u(n-1)$

$a^{n-i} u(n-1)$ (7')

$a^{-n+i} u(n-1)$ (7'')

(8) $a^{-n} u(n-1)$

$a^{-n+i} u(n-1)$ (8')

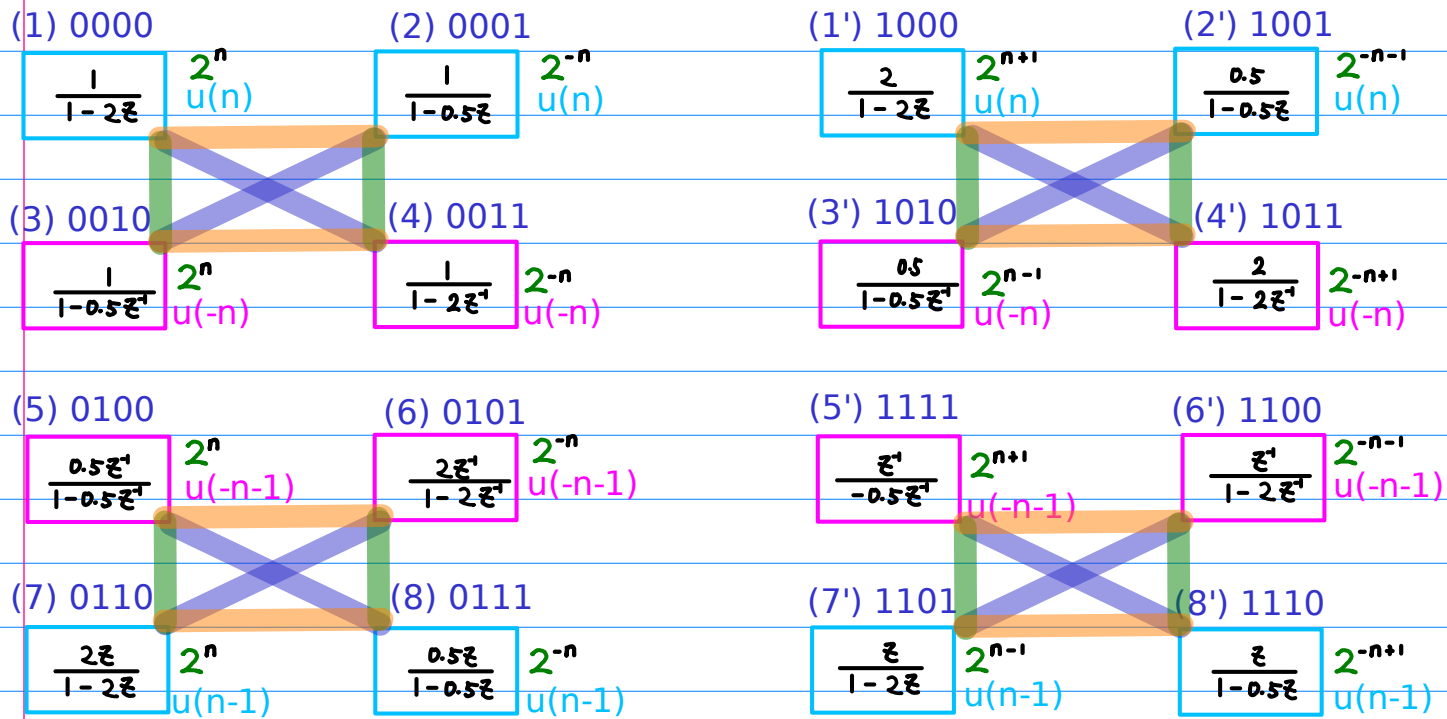
$a^{n-i} u(n-1)$ (8'')

many possible permutations are possible
but consider these two first

Inter-permutation

(x) --> (x')

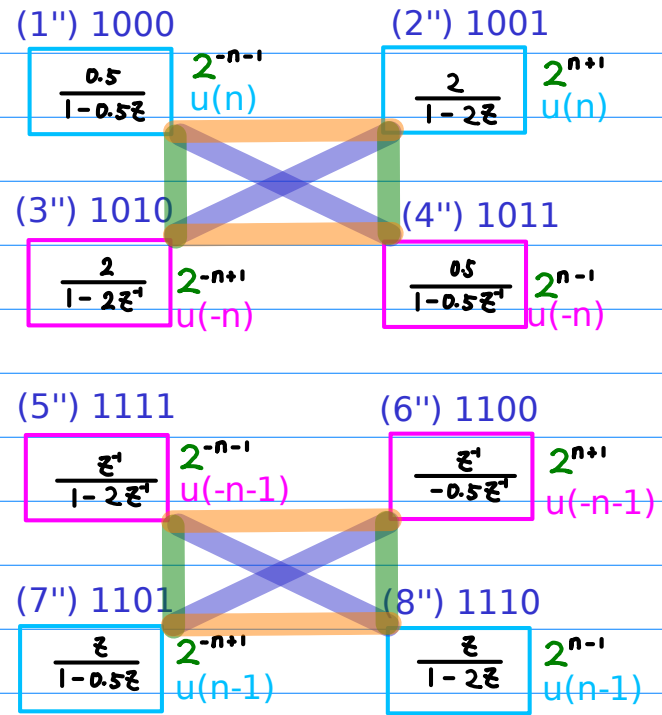
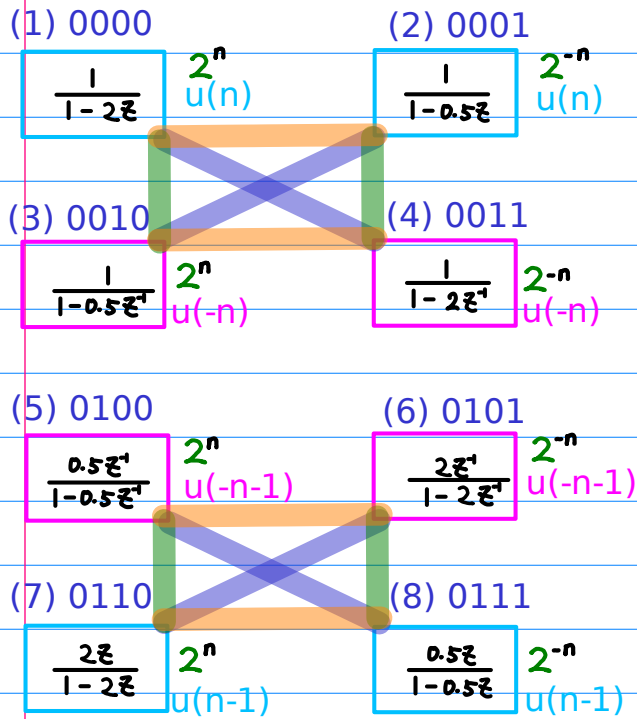
(1)~(8) --> (1')~(8')



(1)	0 0 0 0	→	1 0 0 0	(1')	n	→	$n+1$	shl(n)
(2)	0 0 0 1	→	1 0 0 1	(2')	n	→	$n+1$	shl(n)
(3)	0 0 1 0	→	1 0 1 0	(3')	n	→	$n-1$	shr(n)
(4)	0 0 1 1	→	1 0 1 1	(4')	n	→	$n-1$	shr(n)
(5)	0 1 0 0	→	1 1 0 0	(5')	n	→	$n+1$	shl(n)
(6)	0 1 0 1	→	1 1 0 1	(6')	n	→	$n+1$	shl(n)
(7)	0 1 1 0	→	1 1 1 0	(7')	n	→	$n-1$	shr(n)
(8)	0 1 1 1	→	1 1 1 1	(7')	n	→	$n-1$	shr(n)

Inter-permutation

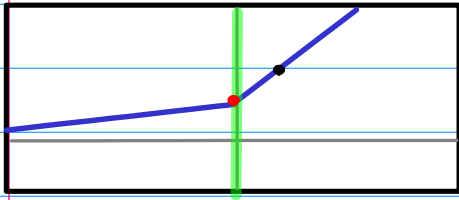
$(x) \quad \rightarrow \quad (x'')$
 $(1) \sim (8) \quad \rightarrow \quad (1'') \sim (8'')$



$\text{shl}(n) = n + 1$
 $\text{shr}(n) = n - 1$

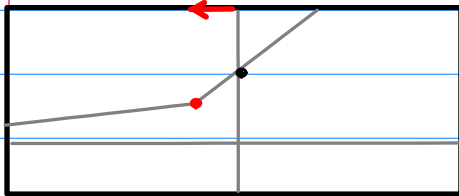
(1) -- (1')
(5) -- (5')

2^n



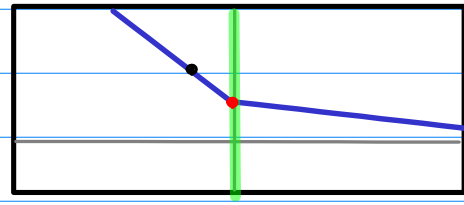
shift left
 $n \leftarrow n+1$

2^{n+1}



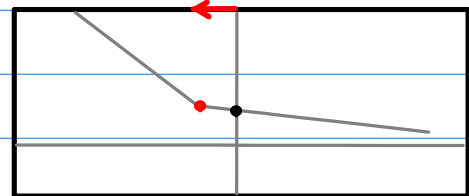
(2) -- (2')
(6) -- (6')

2^{-n}



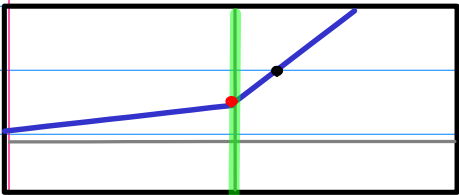
shift left
 $n \leftarrow n+1$

$2^{-(n+1)}$



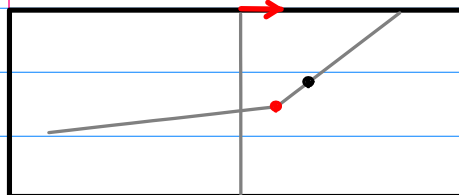
(3) -- (3')
(7) -- (7')

2^n



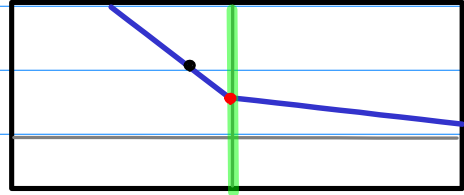
shift right
 $n \leftarrow n-1$

2^{n-1}



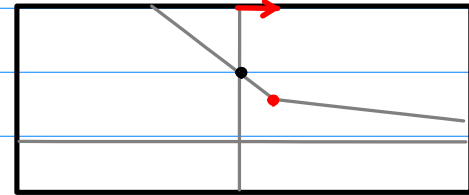
(4) -- (4')
(8) -- (8')

2^{-n}



shift right
 $n \leftarrow n-1$

$2^{-(n-1)}$

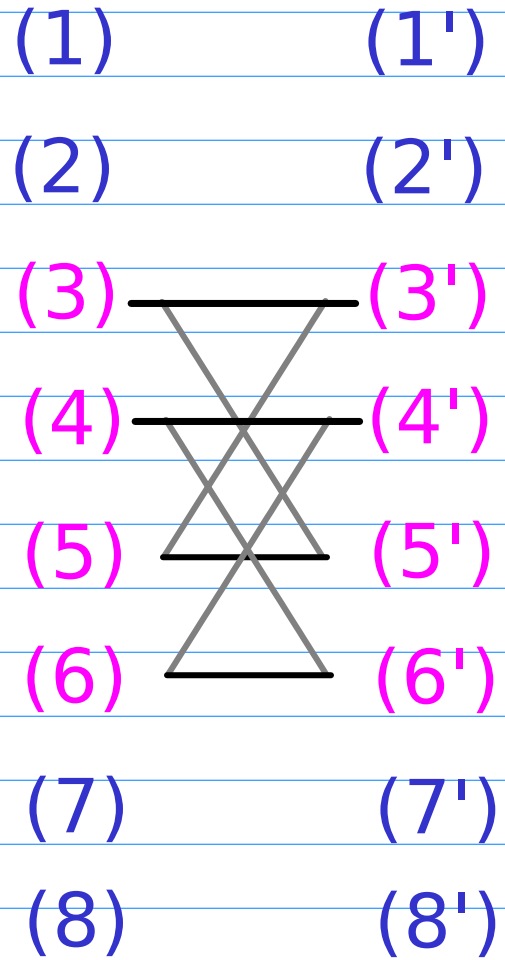
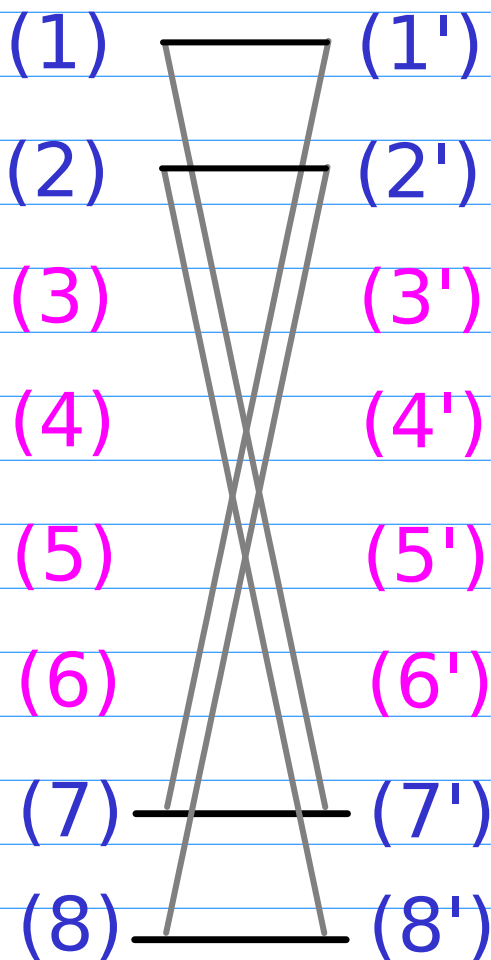
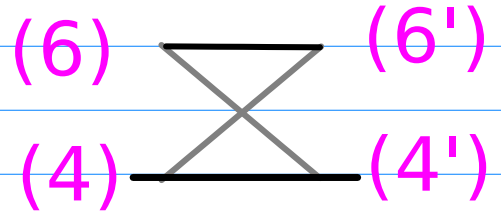
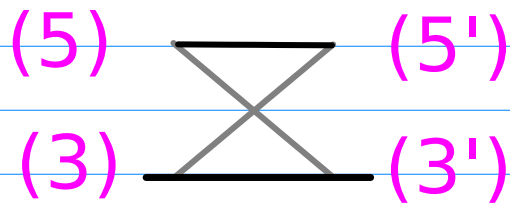
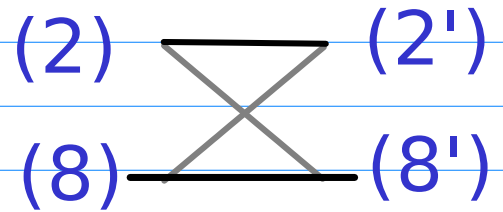
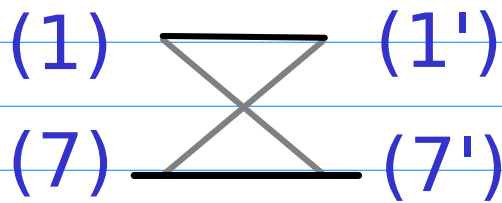


(1) - (1')
(7) - (7')
(5) - (5')
(3) - (3')

(2) - (2')
(8) - (8')
(6) - (6')
(4) - (4')

(1) - (1')
(2) - (2')
(3) - (3')
(4) - (4')

(5) - (5')
(6) - (6')
(7) - (7')
(8) - (8')



Butterfly Relations

$(x) \dashrightarrow (x')$
 $(1) \sim (8) \dashrightarrow (1') \sim (8')$

(*) unit starting

(1) $\xrightarrow{*a \leftarrow (SL, id)}$ (1') SL

$\xrightarrow{/z \leftarrow (SL, SL)}$

$\xrightarrow{*z \Rightarrow (SR, SR)}$

(7) $\xrightarrow{/a \rightarrow (SR, id)}$ (7') SR

C.R. starting

(*) C.R. starting

(5) $\xrightarrow{*a \leftarrow (SL, id)}$ (5') SL

$\xrightarrow{/z \leftarrow (SL, SL)}$

$\xrightarrow{*z \Rightarrow (SR, SR)}$

(3) $\xrightarrow{/a \rightarrow (SR, id)}$ (3') SR

unit starting

(*) unit starting

(2) $\xrightarrow{/a \leftarrow (SL, id)}$ (2') SL

$\xrightarrow{/z \leftarrow (SL, SL)}$

$\xrightarrow{*z \Rightarrow (SR, SR)}$

(8) $\xrightarrow{*a \rightarrow (SR, id)}$ (8') SR

C.R. starting

(*) C.R. starting

(6) $\xrightarrow{/a \leftarrow (SL, id)}$ (6') SL

$\xrightarrow{/z \leftarrow (SL, SL)}$

$\xrightarrow{*z \Rightarrow (SR, SR)}$

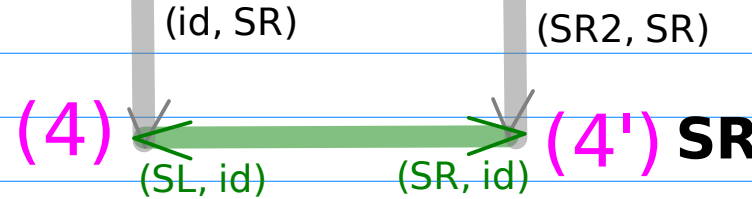
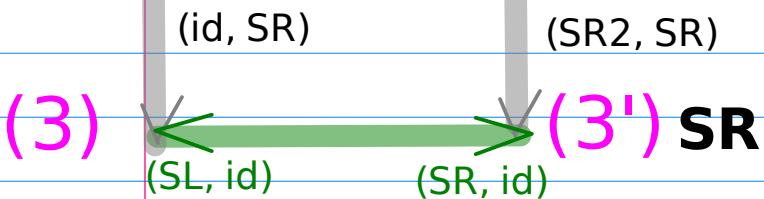
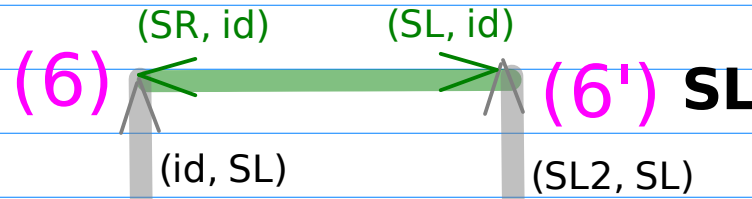
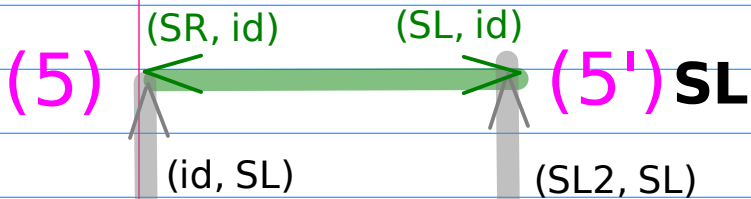
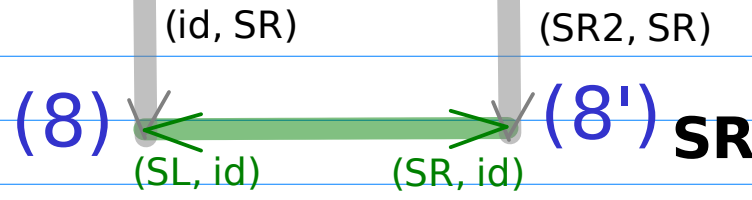
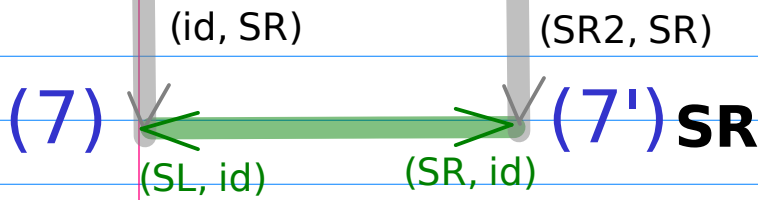
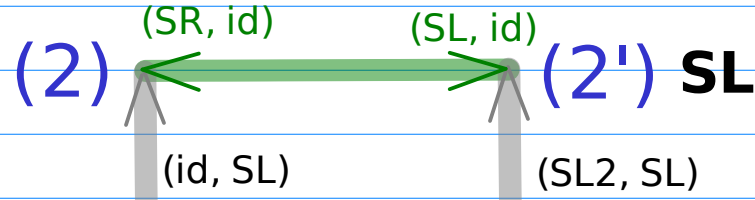
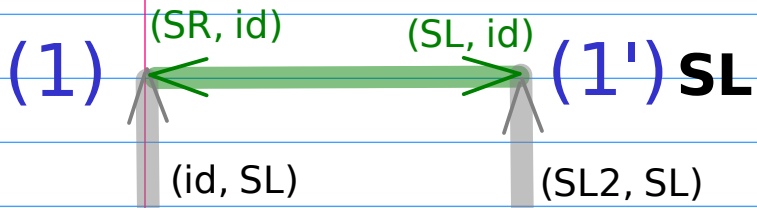
(4) $\xrightarrow{*a \rightarrow (SR, id)}$ (4') SR

unit starting

(Exp Shift, Range Shift)

Butterfly Relations

$(x) \quad \dashrightarrow \quad (x')$
 $(1) \sim (8) \quad \dashrightarrow \quad (1') \sim (8')$



(Exp Shift, Range Shift)

Decomposition of Exp and Rng Shifts (1)

$$(id, SR) + (SR, id) = (SR, SR)$$

(*) unit starting

(1)

(1') SL

(id, SR)

*z ⇒ (SR, SR)

C.R. starting

(7)

/a ⇒ (SR, id)

(7') SR

(*) unit starting

(2)

(2') SL

(id, SR)

*z ⇒ (SR, SR)

C.R. starting

(8)

*a ⇒ (SR, id)

(8') SR

(*) C.R. starting

(5)

(5') SL

(id, SR)

*z ⇒ (SR, SR)

unit starting

(3)

/a ⇒ (SR, id)

(3') SR

(*) C.R. starting

(6)

(6') SL

(id, SR)

*z ⇒ (SR, SR)

unit starting

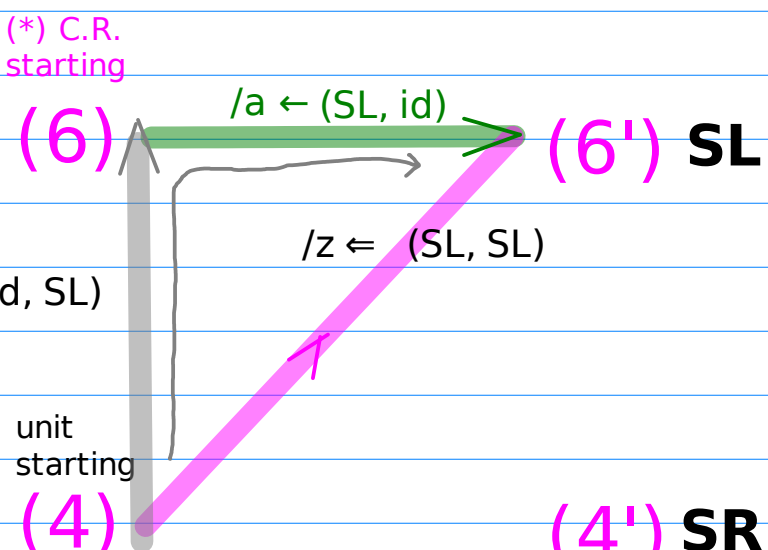
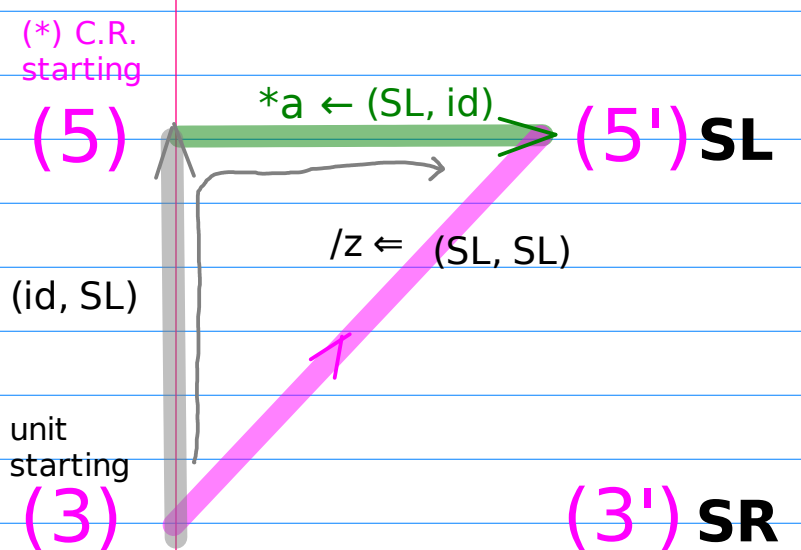
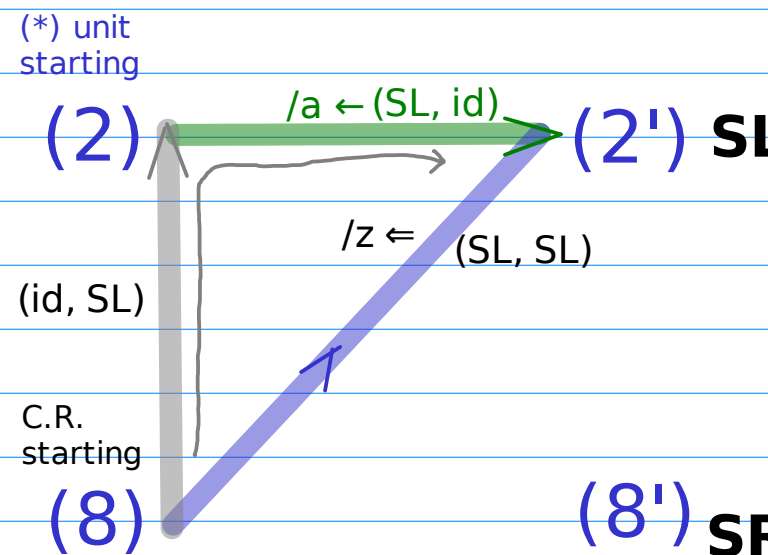
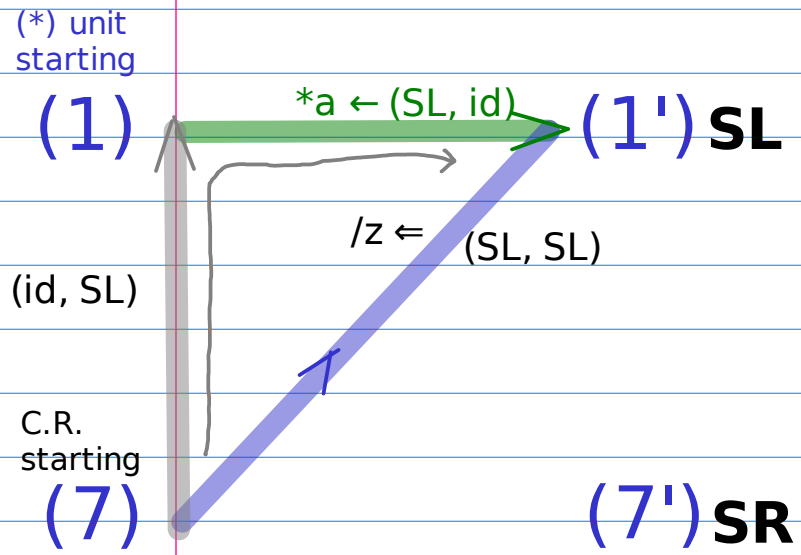
(4)

*a ⇒ (SR, id)

(4') SR

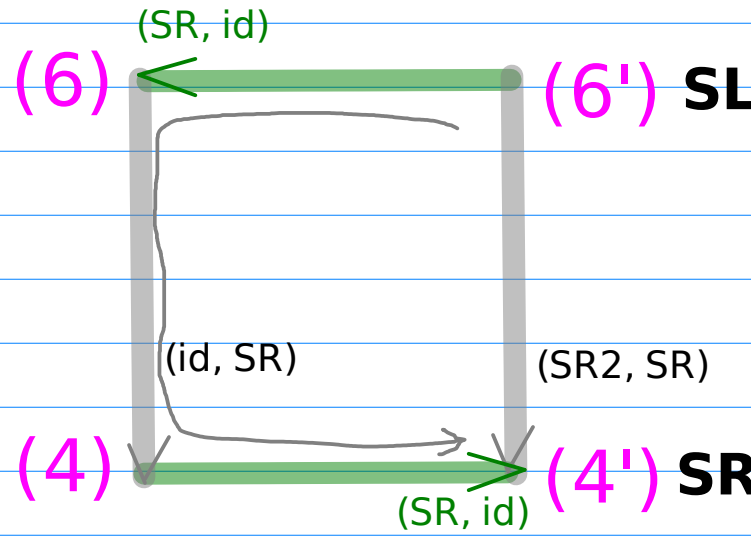
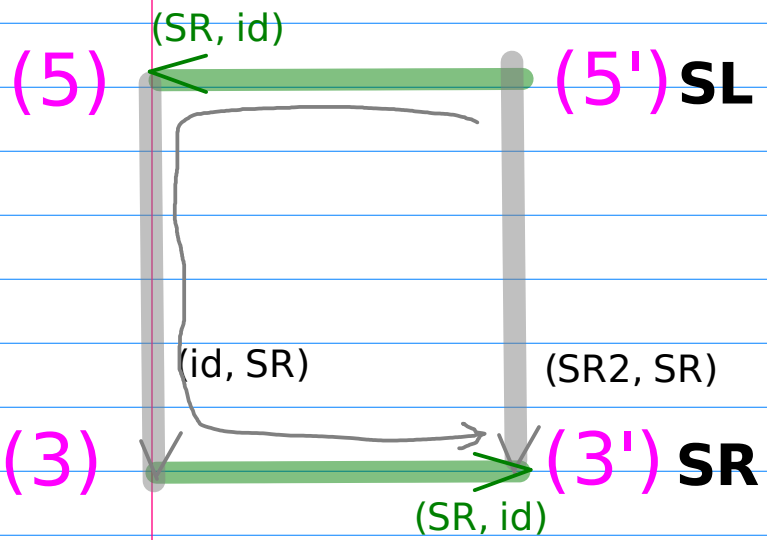
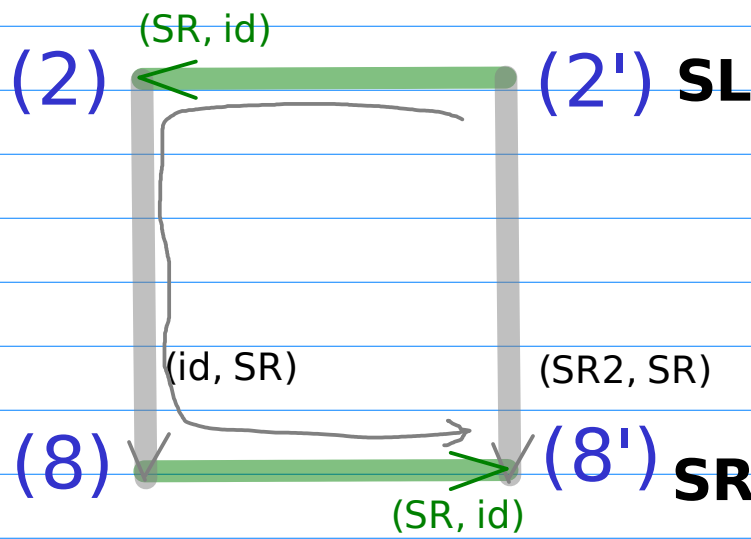
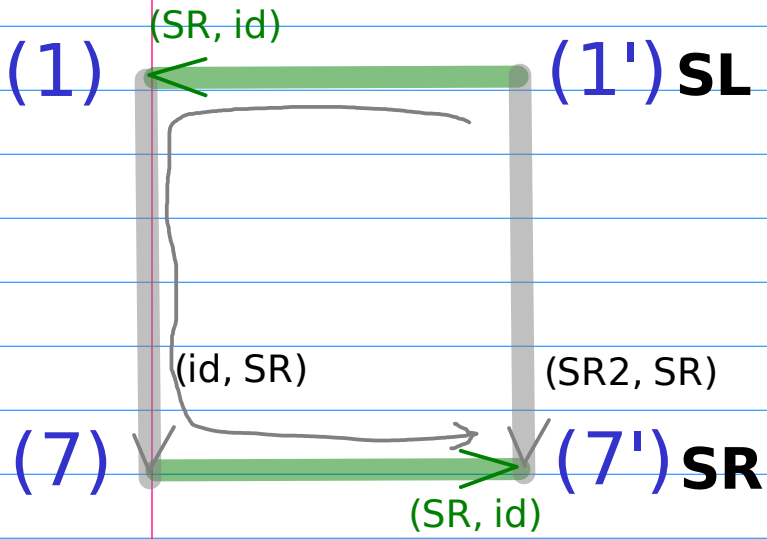
Decomposition of Exp and Rng Shifts (2)

$(id, SL) + (SL, id) = (SL, SL)$



Decomposition of Exp and Rng Shifts (3)

$$(SR, id) + (id, SR) + (SR, id) = (SR2, SR)$$



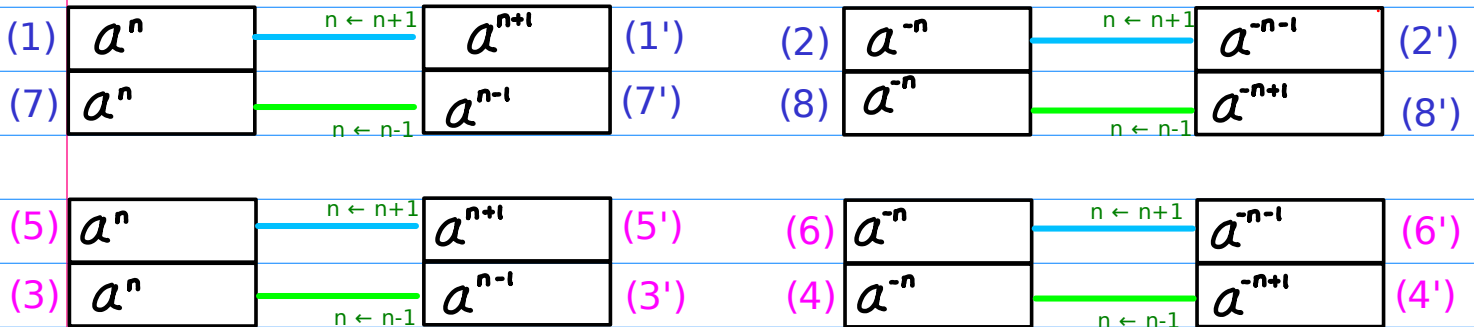
$$(id, SR) + (SR, id) = (SR, SR)$$

$$(id, SL) + (SL, id) = (SL, SL)$$

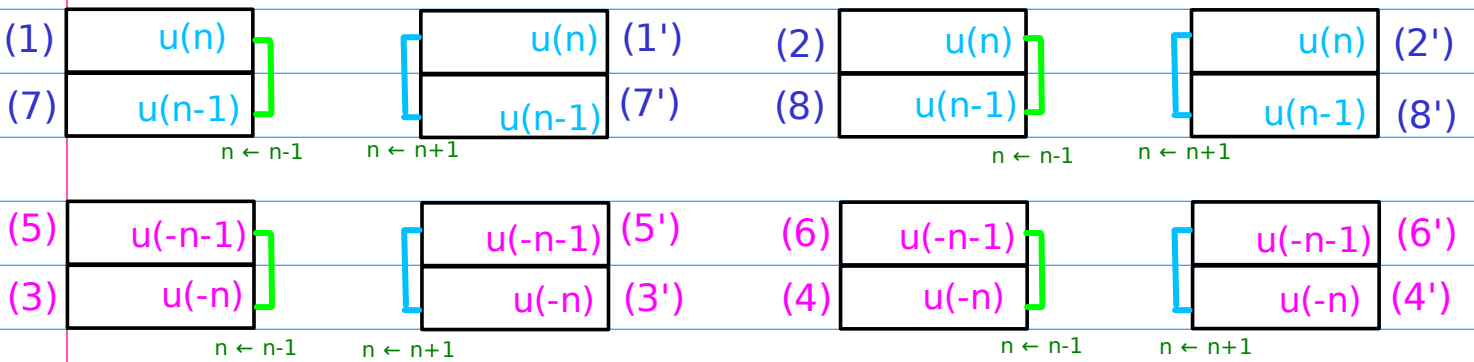
$$(x) \quad \dashrightarrow \quad (x')$$

$$(1) \sim (8) \quad \dashrightarrow \quad (1') \sim (8')$$

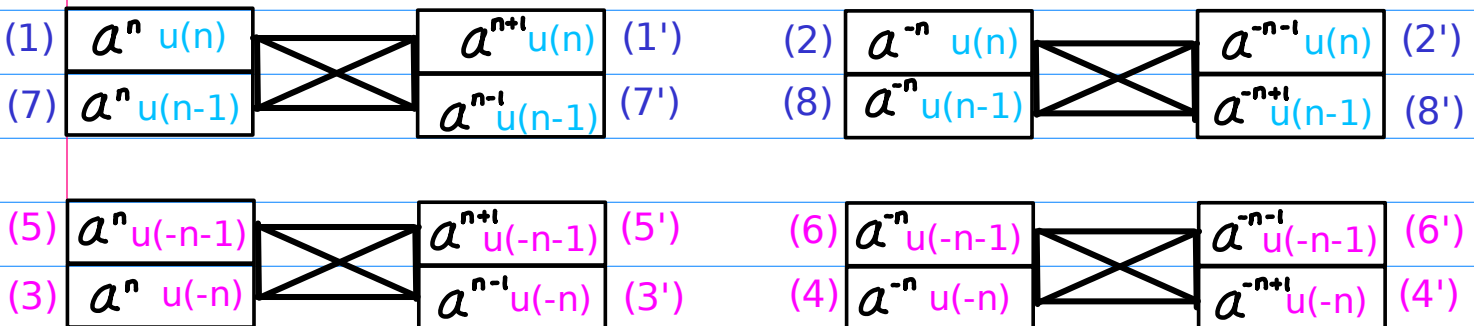
Exponent Shifts



Range Shifts



Exponent & Range Permutations



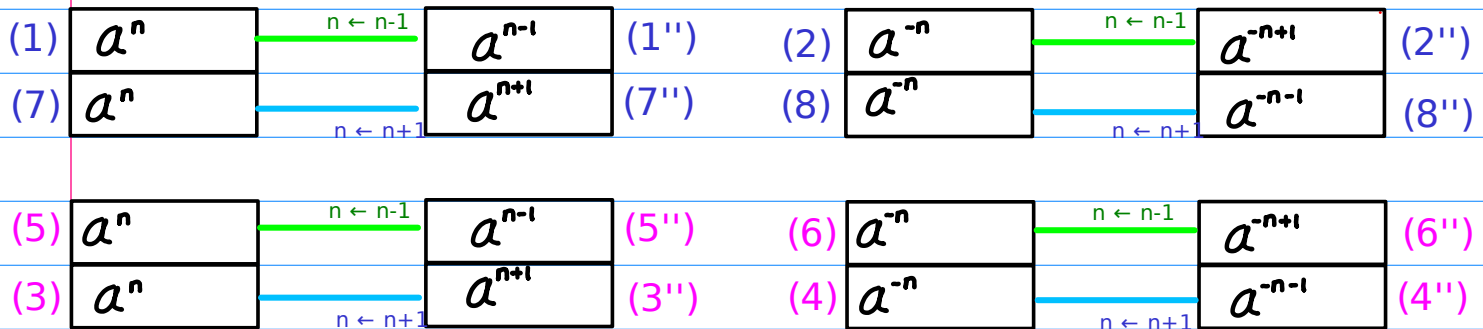
$$(id, SR) + (SL, id) = (SL, SR)$$

$$(id, SL) + (SR, id) = (SR, SL)$$

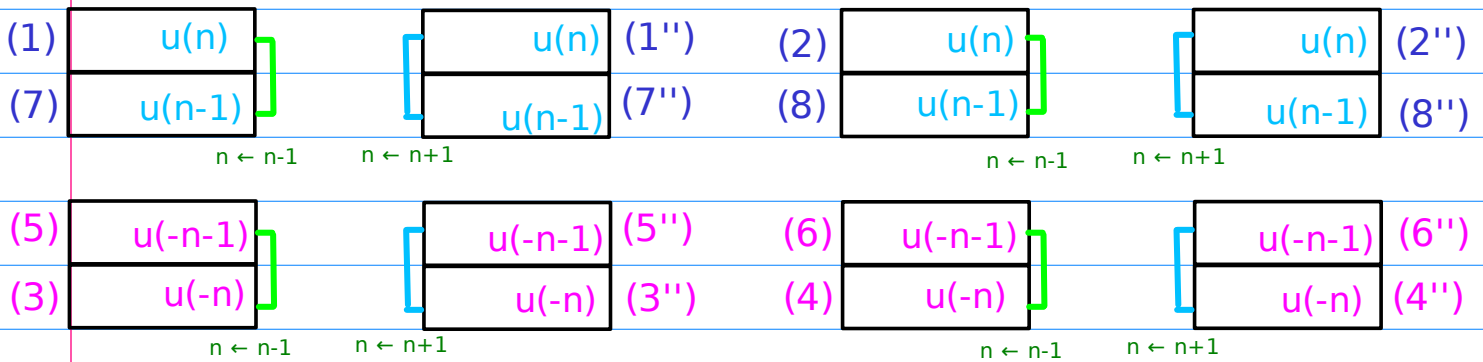
$$(x) \quad \dashrightarrow \quad (x'')$$

$$(1) \sim (8) \quad \dashrightarrow \quad (1'') \sim (8'')$$

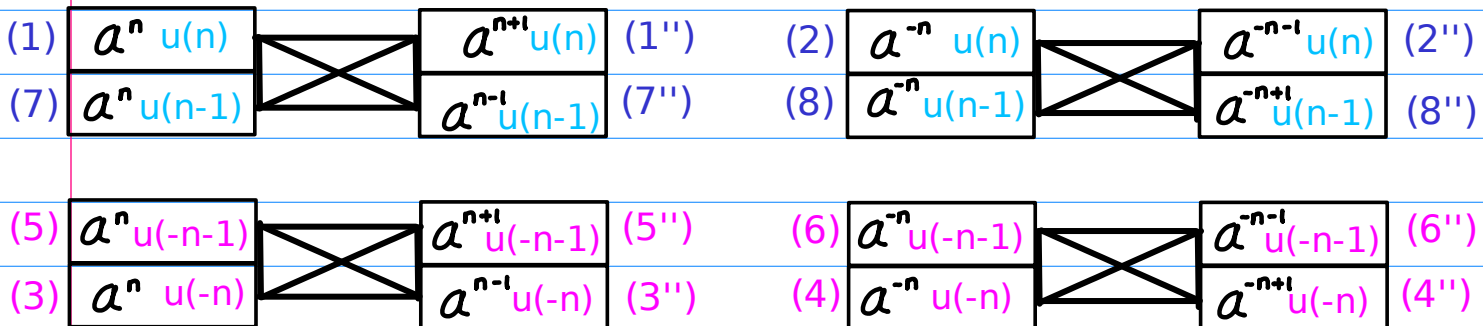
Exponent Shifts



Range Shifts



Exponent & Range Permutations



$$\begin{array}{l} (1) \ a^n u(n) \\ (7) \ a^n u(n-1) \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{l} a^{n+i} u(n) \quad (1') \\ a^{n-i} u(n-1) \quad (7') \end{array}$$

$$\begin{array}{l} (2) \ a^{-n} u(n) \\ (8) \ a^{-n} u(n-1) \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{l} a^{-n-i} u(n) \quad (2') \\ a^{-n+i} u(n-1) \quad (8') \end{array}$$

$$\begin{array}{l} (5) \ a^n u(-n-1) \\ (3) \ a^n u(-n) \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{l} a^{n+i} u(-n-1) \quad (5') \\ a^{n-i} u(-n) \quad (3') \end{array}$$

$$\begin{array}{l} (6) \ a^{-n} u(-n-1) \\ (4) \ a^{-n} u(-n) \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{l} a^{-n-i} u(-n-1) \quad (6') \\ a^{-n+i} u(-n) \quad (4') \end{array}$$

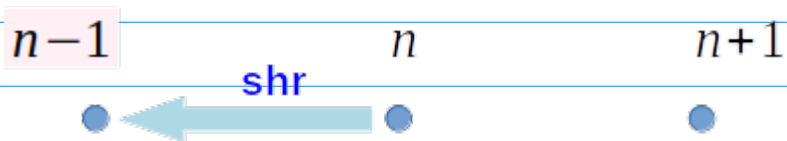
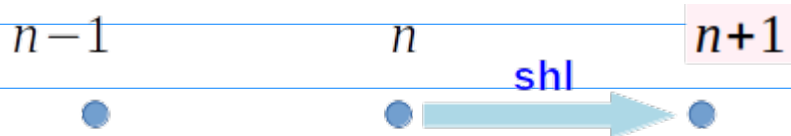
$$\begin{array}{l} (1) \ a^n u(n) \\ (7) \ a^n u(n-1) \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{l} a^{n-i} u(n) \quad (1'') \\ a^{n+i} u(n-1) \quad (7'') \end{array}$$

$$\begin{array}{l} (2) \ a^{-n} u(n) \\ (8) \ a^{-n} u(n-1) \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{l} a^{-n+i} u(n) \quad (2'') \\ a^{-n-i} u(n-1) \quad (8'') \end{array}$$

$$\begin{array}{l} (5) \ a^n u(-n-1) \\ (3) \ a^n u(-n) \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{l} a^{n-i} u(-n-1) \quad (5'') \\ a^{n+i} u(-n) \quad (3'') \end{array}$$

$$\begin{array}{l} (6) \ a^{-n} u(-n-1) \\ (4) \ a^{-n} u(-n) \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{l} a^{-n+i} u(-n-1) \quad (6'') \\ a^{-n-i} u(-n) \quad (4'') \end{array}$$

$R(n)$			$SHR(n)$
$u(n)$	$n \leftarrow n-1$	shr	$u(n-1)$
$u(n-1)$	$n \leftarrow n+1$	shl	$u(n)$
$u(-n)$	$n \leftarrow n+1$	shl	$u(-n-1)$
$u(-n-1)$	$n \leftarrow n-1$	shr	$u(-n)$



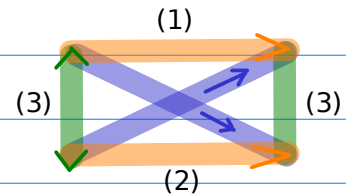
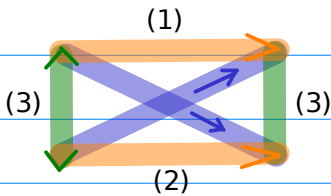
Exponent Permutations

$$\begin{array}{l} (1) \\ (7) \end{array} \begin{array}{|c|} \hline a^n u(n) \\ \hline a^n u(n-1) \\ \hline \end{array} \begin{array}{|c|} \hline \diagdown \quad \diagup \\ \hline \diagup \quad \diagdown \\ \hline \end{array} \begin{array}{|c|} \hline a^{n+1} u(n) \\ \hline a^{n-1} u(n-1) \\ \hline \end{array} \begin{array}{l} (1') \\ (7') \end{array}$$

$$\begin{array}{l} (2) \\ (8) \end{array} \begin{array}{|c|} \hline a^{-n} u(n) \\ \hline a^{-n} u(n-1) \\ \hline \end{array} \begin{array}{|c|} \hline \diagdown \quad \diagup \\ \hline \diagup \quad \diagdown \\ \hline \end{array} \begin{array}{|c|} \hline a^{-n-1} u(n) \\ \hline a^{-n+1} u(n-1) \\ \hline \end{array} \begin{array}{l} (2') \\ (8') \end{array}$$

$$\begin{array}{l} (5) \\ (3) \end{array} \begin{array}{|c|} \hline a^n u(-n-1) \\ \hline a^n u(-n) \\ \hline \end{array} \begin{array}{|c|} \hline \diagdown \quad \diagup \\ \hline \diagup \quad \diagdown \\ \hline \end{array} \begin{array}{|c|} \hline a^{n+1} u(-n-1) \\ \hline a^{n-1} u(-n) \\ \hline \end{array} \begin{array}{l} (5') \\ (3') \end{array}$$

$$\begin{array}{l} (6) \\ (4) \end{array} \begin{array}{|c|} \hline a^{-n} u(-n-1) \\ \hline a^{-n} u(-n) \\ \hline \end{array} \begin{array}{|c|} \hline \diagdown \quad \diagup \\ \hline \diagup \quad \diagdown \\ \hline \end{array} \begin{array}{|c|} \hline a^{-n-1} u(-n-1) \\ \hline a^{-n+1} u(-n) \\ \hline \end{array} \begin{array}{l} (6') \\ (4') \end{array}$$



(1) shift left exponent

(2) shift right exponent

(3) shift right/left range

(1) shift left exponent

(2) shift right exponent

(3) shift right/left range

(SR, id) shift right exponent

(id, SR) shift right range

(SR, SR)

(SR, id) shift right exponent

(id, SR) shift right range

(SR, SR)

(SL, id) shift left exponent

(id, SL) shift left range

(SL, SL)

(SL, id) shift left exponent

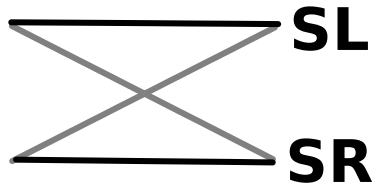
(id, SL) shift left range

(SL, SL)

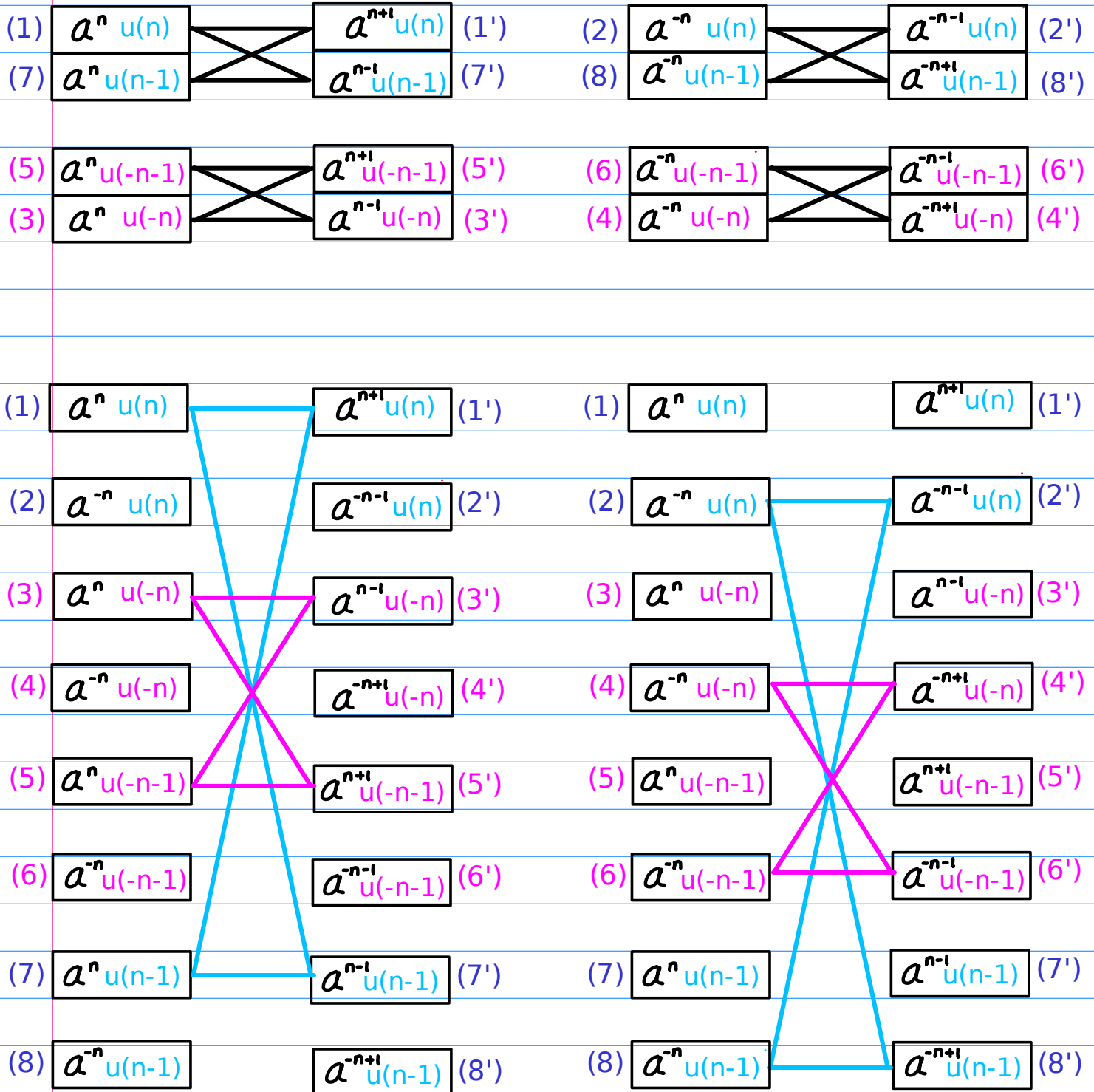
$$\begin{aligned} (id, SR) + (SR, id) &= (SR, SR) \\ (id, SL) + (SL, id) &= (SR, SL) \end{aligned}$$

$$\begin{aligned} (id, SR) + (SR, id) &= (SR, SR) \\ (id, SL) + (SL, id) &= (SR, SL) \end{aligned}$$

Unshifted Sequence



Shifted Sequence





Intra-Permutations over (1) ~ (8)

(1) $a^n u(n)$	$a^{n+i} u(n)$ (1')
(2) $a^{-n} u(n)$	$a^{-n-i} u(n)$ (2')
(3) $a^n u(-n)$	$a^{-i} u(-n)$ (3')
(4) $a^{-n} u(-n)$	$a^{-n+i} u(-n)$ (4')
(5) $a^n u(-n-1)$	$a^{n+i} u(-n-1)$ (5')
(6) $a^{-n} u(-n-1)$	$a^{-n-i} u(-n-1)$ (6')
(7) $a^n u(n-1)$	$a^{-i} u(n-1)$ (7')
(8) $a^{-n} u(n-1)$	$a^{-n+i} u(n-1)$ (8')

A. Flipping

Base Inverting
Range Flipping

B. Range Shifting = Range Flipping + Range Complementing

Range Flipping
Range Complementing

C. Complementary Inverting

Base Inverting
Range Complementing

a^n \times $R(n)$

a^n	a^{-n}	\times	$u(n)$	$u(-n-1)$
		\times	$u(n-1)$	$u(-n)$

(1)	$a^n u(n)$	$a^{-n} u(n)$	(1)
(7)	$a^n u(n-1)$	$a^{-n} u(n-1)$	(7)
(5)	$a^n u(-n-1)$	$a^{-n} u(-n-1)$	(5)
(3)	$a^n u(-n)$	$a^{-n} u(-n)$	(3)

(1)	$a^n u(n)$	$a^{-n} u(n)$	(2)
(3)	$a^n u(n-1)$	$a^{-n} u(n-1)$	(4)
(5)	$a^n u(-n-1)$	$a^{-n} u(-n-1)$	(6)
(7)	$a^n u(-n)$	$a^{-n} u(-n)$	(8)

Intra-Permutations over (1') ~ (8')

$$(1) \quad a^n u(n)$$

$$(2) \quad a^{-n} u(n)$$

$$(3) \quad a^n u(-n)$$

$$(4) \quad a^{-n} u(-n)$$

$$(5) \quad a^n u(-n-1)$$

$$(6) \quad a^{-n} u(-n-1)$$

$$(7) \quad a^n u(n-1)$$

$$(8) \quad a^{-n} u(n-1)$$

$$a^{n+i} u(n) \quad (1')$$

$$a^{-n-i} u(n) \quad (2')$$

$$a^{n-i} u(-n) \quad (3')$$

$$a^{-n+i} u(-n) \quad (4')$$

$$a^{n+i} u(-n-1) \quad (5')$$

$$a^{-n-i} u(-n-1) \quad (6')$$

$$a^{n-i} u(n-1) \quad (7')$$

$$a^{-n+i} u(n-1) \quad (8')$$

D. Flipping2

Base Inverting

Shifted Range Flipping = Exponent Shifting2 + Range Flipping

E. Shifting2 = Exponent Shifting2 + Range Shifting

Shifted Range Flipping = Exponent Shifting2 + Range Flipping

Range Complementing

F. Complementary Inverting

Base Inverting

Range Complementing

a^n \times $R(n)$

a^{n+1}	a^{-n-1}
a^{n-1}	a^{-n+1}

 \times

$u(n)$	$u(-n-1)$
$u(n-1)$	$u(-n)$

(1') $a^{n+1}u(n)$	$a^{-n-1}u(n)$	(2')
(7') $a^{n-1}u(n-1)$	$a^{-n+1}u(n-1)$	(8')
(5') $a^{n+1}u(-n-1)$	$a^{-n-1}u(-n-1)$	(6')
(3') $a^{n-1}u(-n)$	$a^{-n+1}u(-n)$	(4')

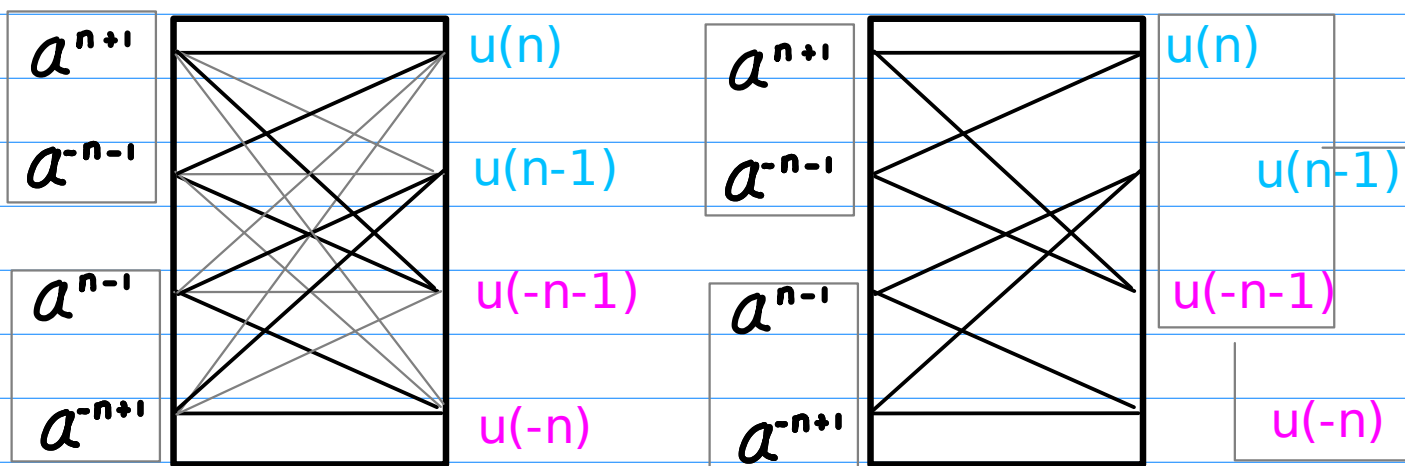
(1'') $a^{n-1}u(n)$	$a^{-n+1}u(n)$	(2'')
(7'') $a^{n+1}u(n-1)$	$a^{-n-1}u(n-1)$	(8'')
(5'') $a^{n-1}u(-n-1)$	$a^{-n+1}u(-n-1)$	(6'')
(3'') $a^{n+1}u(-n)$	$a^{-n-1}u(-n)$	(4'')

(1') $a^{n+1}u(n)$	$a^{-n-1}u(n)$	(2')
(3') $a^{n-1}u(-n)$	$a^{-n+1}u(-n)$	(4')
(5') $a^{n+1}u(-n-1)$	$a^{-n-1}u(-n-1)$	(6')
(7') $a^{n-1}u(n-1)$	$a^{-n+1}u(n-1)$	(8')

(1'') $a^{n-1}u(n)$	$a^{-n+1}u(n)$	(2'')
(3'') $a^{n+1}u(-n)$	$a^{-n-1}u(-n)$	(4'')
(5'') $a^{n-1}u(-n-1)$	$a^{-n+1}u(-n-1)$	(6'')
(7'') $a^{n+1}u(n-1)$	$a^{-n-1}u(n-1)$	(8'')

$$a^n \times R(n)$$

a^{n+1}	a^{-n-1}	\times	$u(n)$	$u(-n-1)$
a^{n-1}	a^{-n+1}	\times	$u(n-1)$	$u(-n)$



$n \leftarrow n+2$ or $n \leftarrow n-2$

(1') $a^{n+1}u(n)$	$a^{-n-1}u(n)$	(2')
(3') $a^{n-1}u(-n)$	$a^{-n+1}u(-n)$	(4')
(5') $a^{n+1}u(-n-1)$	$a^{-n-1}u(-n-1)$	(6')
(7') $a^{n-1}u(n-1)$	$a^{-n+1}u(n-1)$	(8')

(1'') $a^{n-1}u(n)$	$a^{-n+1}u(n)$	(2'')
(3'') $a^{n+1}u(-n)$	$a^{-n-1}u(-n)$	(4'')
(5'') $a^{n-1}u(-n-1)$	$a^{-n+1}u(-n-1)$	(6'')
(7'') $a^{n+1}u(n-1)$	$a^{-n-1}u(n-1)$	(8'')



(1')	a^{n+1}	$u(n)$	a^{-n-1}	$u(n)$	(2') (1'')	a^{n-1}	$u(n)$	a^{-n+1}	$u(n)$	(3')
(3')	a^{n-1}	$u(-n)$	a^{-n+1}	$u(-n)$	(4') (3'')	a^{n+1}	$u(-n)$	a^{-n-1}	$u(-n)$	(5')
(5')	a^{n+1}	$u(-n-1)$	a^{-n-1}	$u(-n-1)$	(6') (5'')	a^{n-1}	$u(-n-1)$	a^{-n+1}	$u(-n-1)$	(7')
(7')	a^{n-1}	$u(n-1)$	a^{-n+1}	$u(n-1)$	(8') (7'')	a^{n+1}	$u(n-1)$	a^{-n-1}	$u(n-1)$	(9')

permutation over (1) ~ (8)

A. Flipping

Base Inverting
Range Flipping

B. Range Shifting = Range Flipping + Range Complementing

Range Flipping
Range Complementing

C. Complementary Inverting

Base Inverting
Range Complementing

permutation over (1') ~ (8')

D. Flipping₂

Base Inverting
Shifted Range Flipping = Exponent Shifting₂ + Range Flipping

E. Shifting₂ = Exponent Shifting₂ + Range Shifting

Shifted Range Flipping = Exponent Shifting₂ + Range Flipping
Range Complementing

F. Complementary Inverting

Base Inverting
Range Complementing

Range Shifting = Range Flipping + Range Complementing

Shifted Range Flipping = Exponent Shifting₂ + Range Flipping

Shifting₂ = Shifted Range Flipping + Range Complementing

= Exponent Shifting₂ + Range (Flipping+Complementing)

= Exponent Shifting₂ + Range Shifting

permutation over (1) ~ (8)

A. Flipping	(1) - (4)	(5) - (8)
Base Inverting	(2) - (3)	(6) - (7)
Range Flipping	(3) - (2)	(7) - (6)
	(4) - (1)	(8) - (5)
B. Range Shifting	(1) - (7)	(5) - (3)
Range Flipping	(2) - (8)	(6) - (4)
Range Complementing	(3) - (5)	(7) - (1)
	(4) - (6)	(8) - (2)
C. Complementary Inverting	(1) - (6)	(5) - (2)
Base Inverting	(6) - (1)	(6) - (1)
Range Complementing	(2) - (5)	(7) - (4)
	(5) - (2)	(8) - (3)

permutation over (1') ~ (8')

D. Flipping2	(1') - (4')	(5') - (8')
Base Inverting	(2') - (3')	(6') - (7')
Shifted Range Flipping	(3') - (2')	(7') - (6')
	(4') - (1')	(8') - (5')
E. Shifting2	(1') - (7')	(5') - (3')
Shifted Range Flipping	(2') - (8')	(6') - (4')
Range Complementing	(3') - (5')	(7') - (1')
	(4') - (6')	(8') - (2')
F. Complementary Inverting	(1') - (6')	(5') - (2')
Base Inverting	(6') - (1')	(6') - (1')
Range Complementing	(2') - (5')	(7') - (4')
	(5') - (2')	(8') - (3')



permutation over (1) ~ (8)

	A	B	C
Base Inverting	X		X
Range Flipping	X	X	
Range Complementing		X	X

permutation over (1') ~ (8')

	D	E	F
Base Inverting	X		X
Shifted Range Flipping	X	X	
Range Complementing		X	X

Range Shifting = Range Flipping + Range Complementing
Shifted Range Flipping = Exponent Shifting2 + Range Flipping

Shifting2 = Shifted Range Flipping + Range Complementing
= Exponent Shifting2 + Range (Flipping+Complementing)
= Exponent Shifting2 + Range Shifting



Over (1) ~ (8)

Base Inverting

$$a^n \longleftrightarrow a^{-n}$$

Range Flipping

$$R(n) \longleftrightarrow R(-n)$$

Range Complementing

$$R(n) \longleftrightarrow \overline{R(n)}$$

Over (1') ~ (8')

Base Inverting

$$a^n \longleftrightarrow a^{-n}$$

Shifted Range Flipping

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

Range Complementing

$$R(n) \longleftrightarrow \overline{R(n)}$$

A.I Flipping
Base Inverting
Range Flipping

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow R(-n)$$

$$a^n R(n) \longleftrightarrow a^{-n} R(-n)$$

D.I Flipping2
Base Inverting
Shifted Range Flipping

$$a^n \longleftrightarrow a^{-n}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

$$a^n R(n) \longleftrightarrow a^{-sh2(n)} R(-n)$$

B.I Range Shifting
Range Flipping
Range Complementing

$$R(n) \longleftrightarrow R(-n)$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$R(n) \longleftrightarrow \overline{R(-n)}$$

E.I Shifting2
Shifted Range Flipping
Range Complementing

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

C.I Complementary Inverting
Base Inverting
Range Complementing

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

F.I Complementary Inverting
Base Inverting
Range Complementing

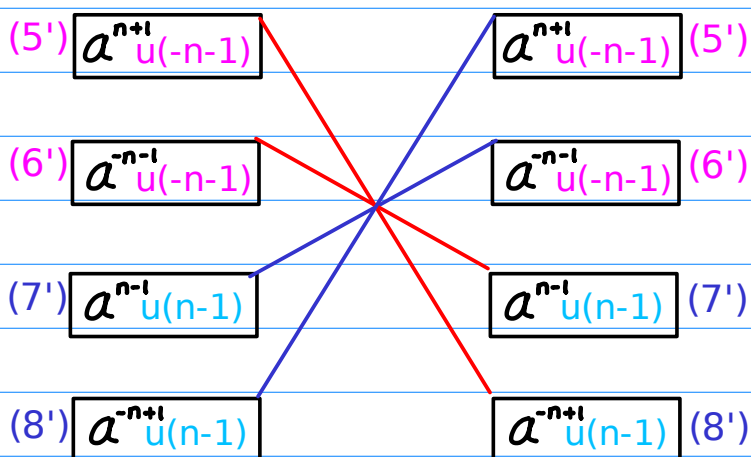
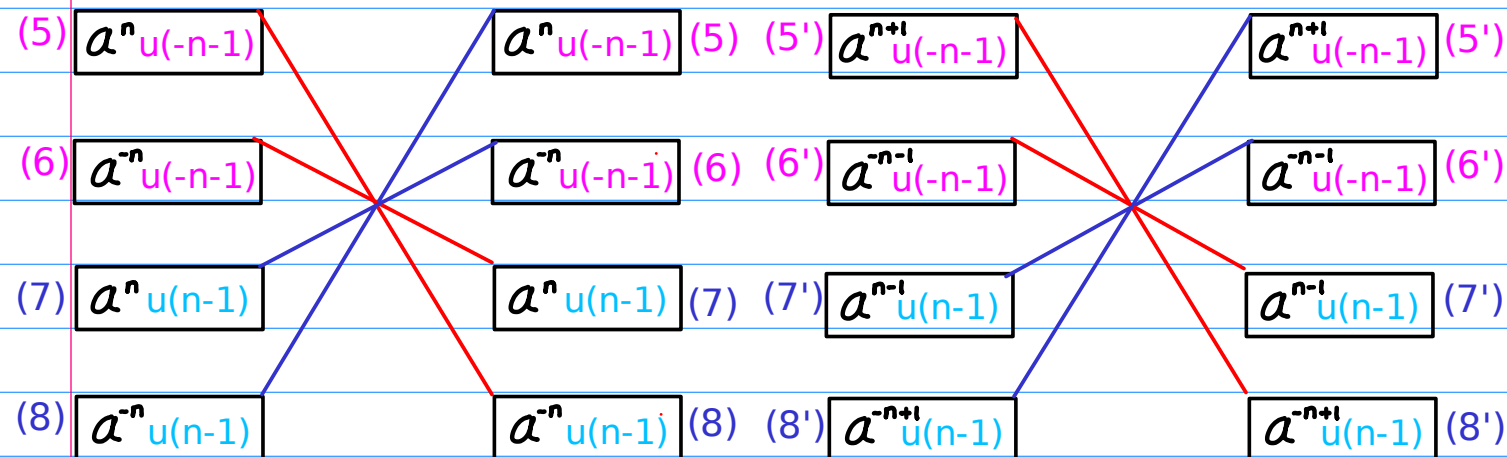
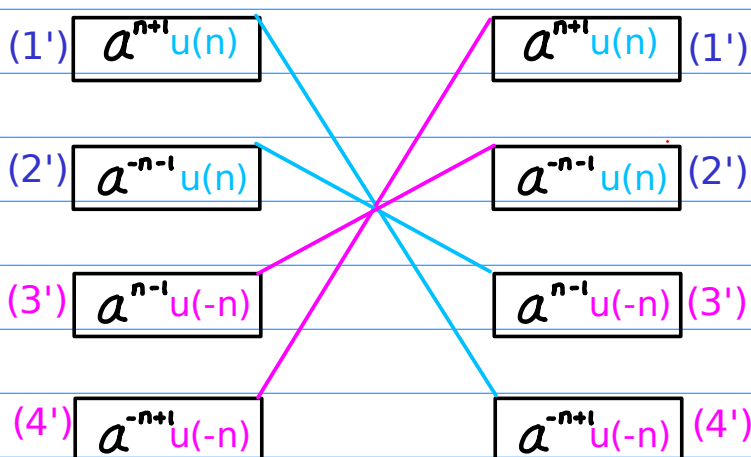
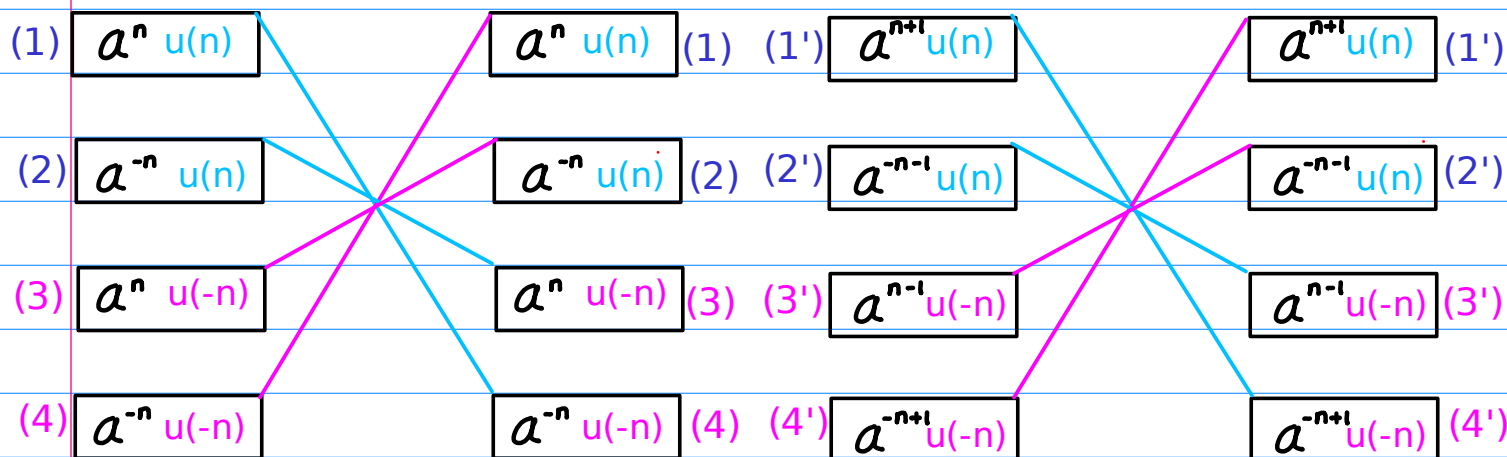
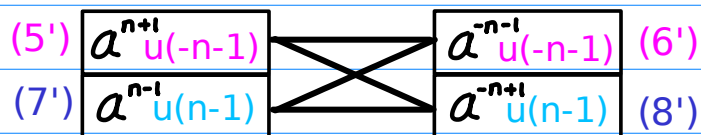
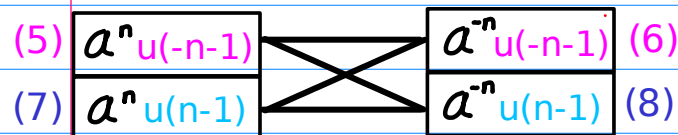
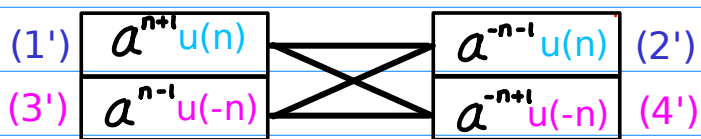
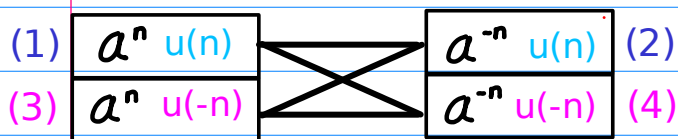
$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

A.I Flipping
Base Inverting
Range Flipping

D.I Flipping2
Base Inverting
Shifted Range Flipping

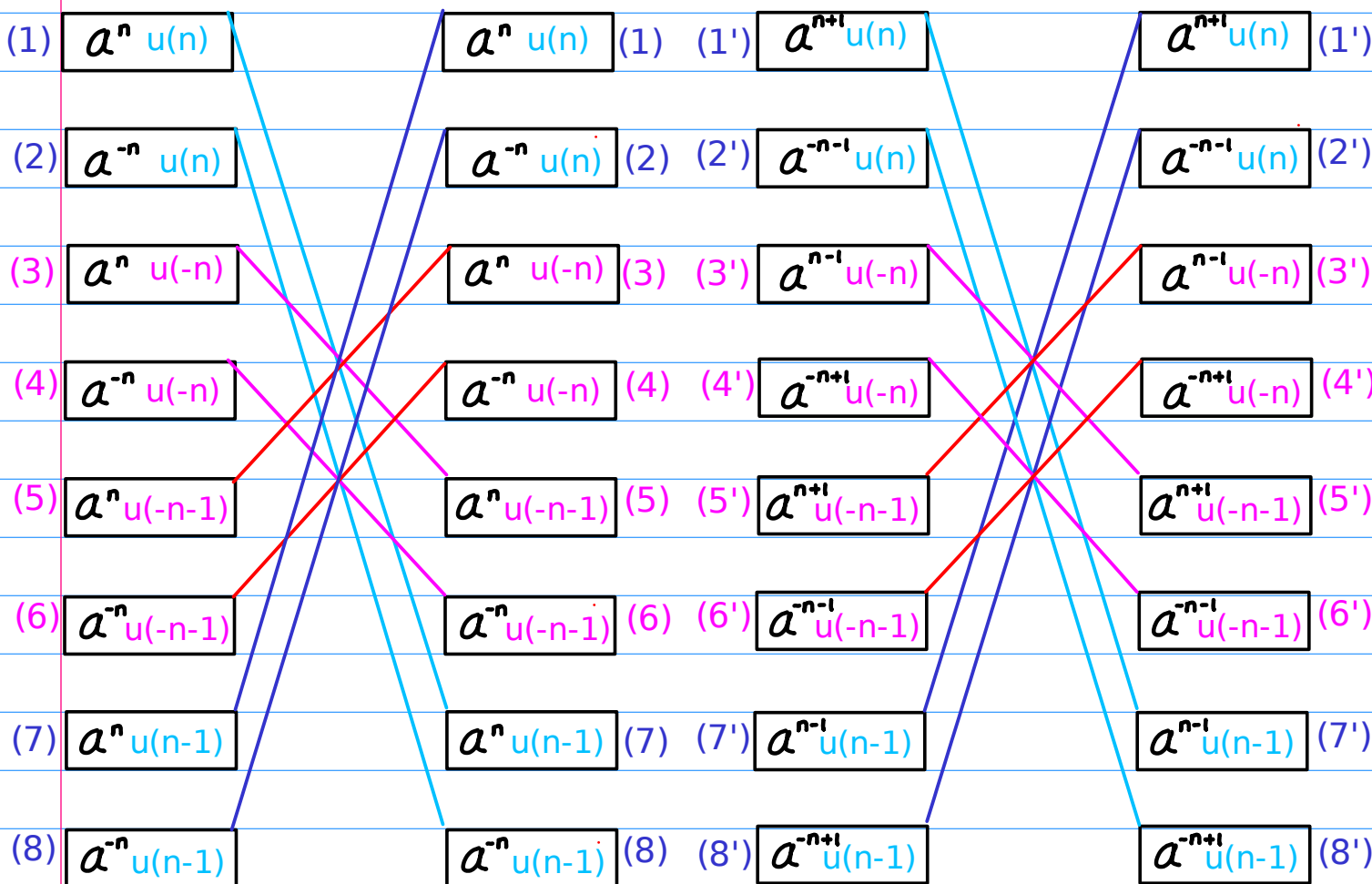
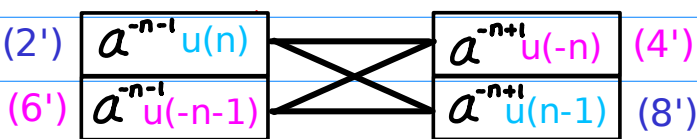
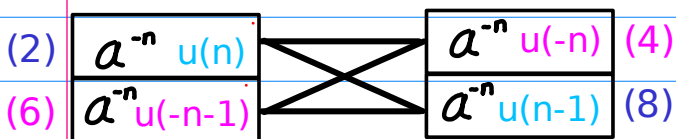
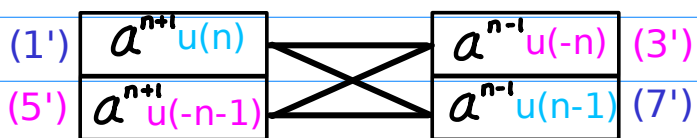
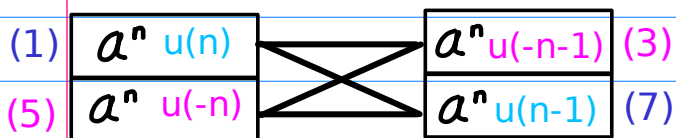


(1) - (4) (5) - (8)
 (2) - (3) (6) - (7)
 (3) - (2) (7) - (6)
 (4) - (1) (8) - (5)

(1') - (4') (5') - (8')
 (2') - (3') (6') - (7')
 (3') - (2') (7') - (6')
 (4') - (1') (8') - (5')

**B.I Range Shifting
Range Flipping
Range Complementing**

**E.I Shifting2
Shifted Range Flipping
Range Complementing**

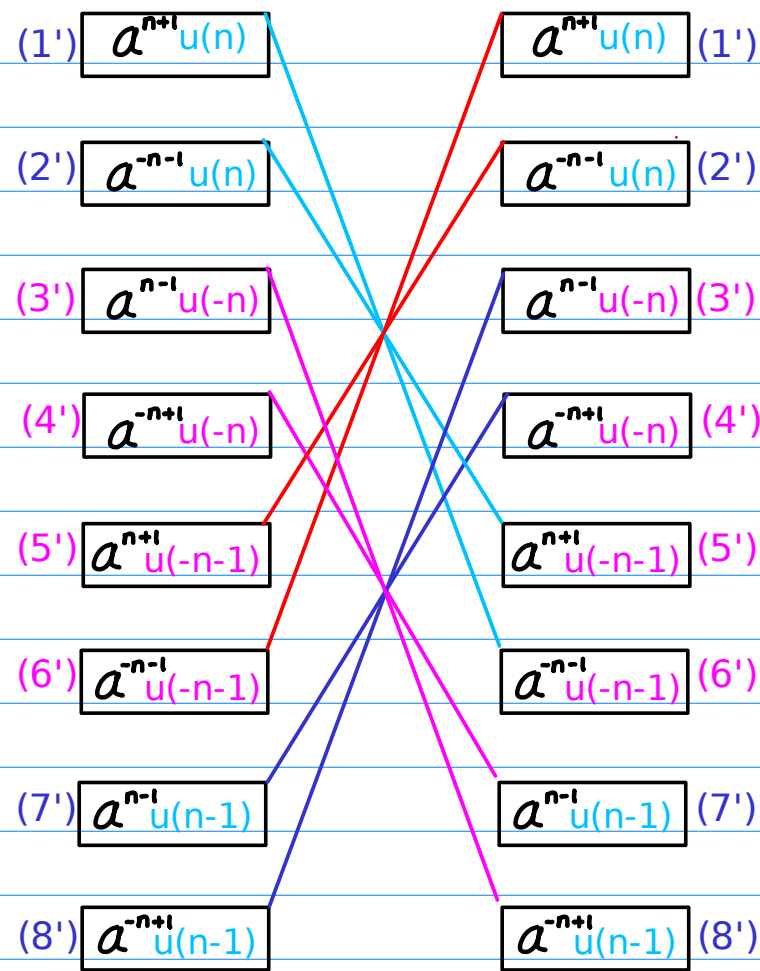
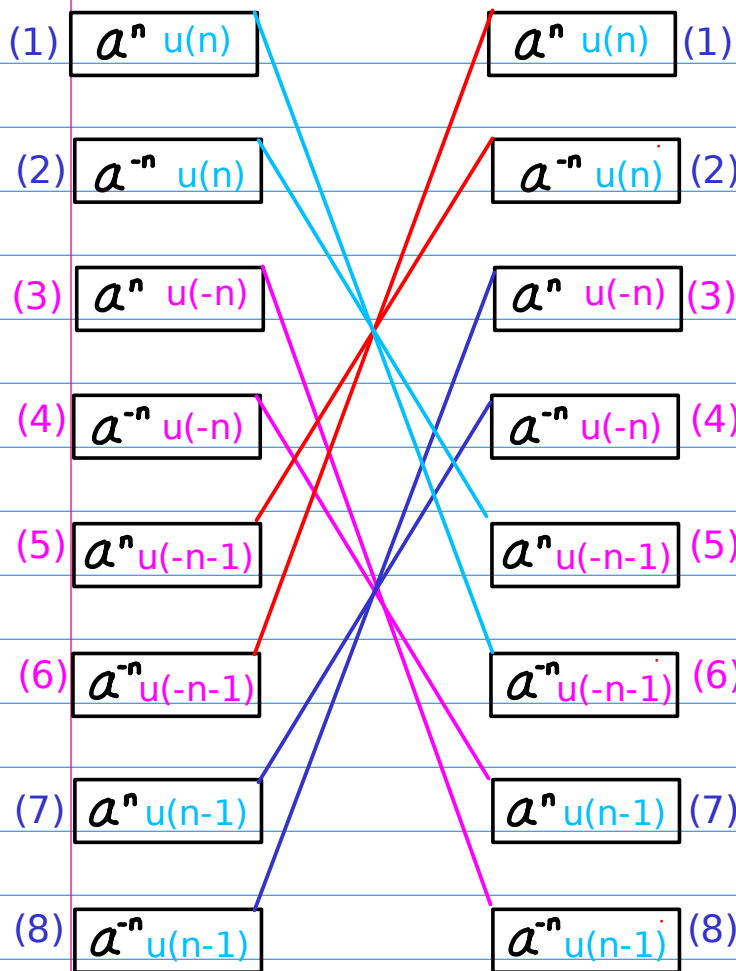
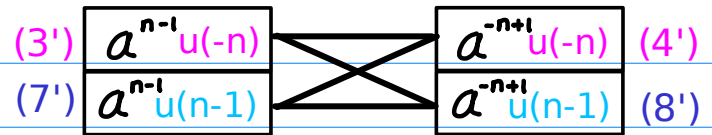
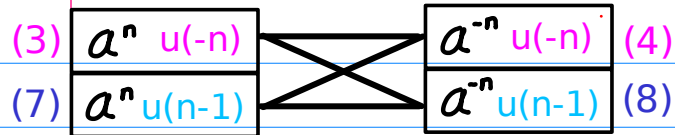
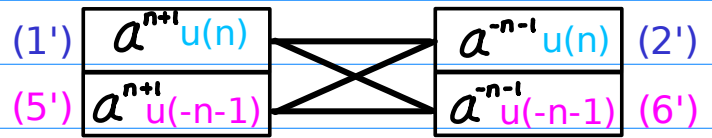
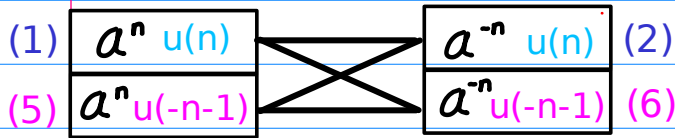


(1) - (7) (5) - (3)
 (2) - (8) (6) - (4)
 (3) - (5) (7) - (1)
 (4) - (6) (8) - (2)

(1') - (7') (5') - (3')
 (2') - (8') (6') - (4')
 (3') - (5') (7') - (1')
 (4') - (6') (8') - (2')

C.I Complementary Inverting Base Inverting Range Complementing

F.I Complementary Inverting Base Inverting Range Complementing



- (1) - (6)
- (6) - (1)
- (2) - (5)
- (5) - (2)
- (5) - (2)
- (3) - (4)
- (4) - (3)
- (7) - (8)
- (8) - (7)
- (8) - (3)

- (1') - (6')
- (6') - (1')
- (2') - (5')
- (5') - (2')
- (5') - (2')
- (3') - (4')
- (4') - (3')
- (7') - (8')
- (8') - (7')
- (8') - (3')

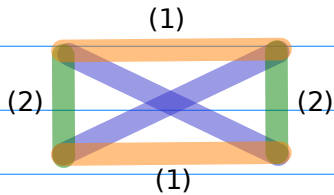


A.I Flipping

- (1) Base Inverting
- (2) Range Flipping

$$\begin{array}{|c|} \hline (1) \quad a^n u(n) \\ \hline (3) \quad a^n u(-n) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n} u(n) \quad (2) \\ \hline a^{-n} u(-n) \quad (4) \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (5) \quad a^n u(-n-1) \\ \hline (7) \quad a^n u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n} u(-n-1) \quad (6) \\ \hline a^{-n} u(n-1) \quad (8) \\ \hline \end{array}$$

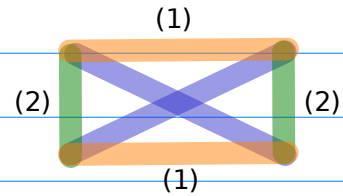


D.I Flipping2

- (1) Base Inverting
- (2) Shifted Range Flipping

$$\begin{array}{|c|} \hline (1') \quad a^{n+1} u(n) \\ \hline (3') \quad a^{n+1} u(-n) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n-1} u(n) \quad (2') \\ \hline a^{-n+1} u(-n) \quad (4') \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (5') \quad a^{n+1} u(-n-1) \\ \hline (7') \quad a^{n+1} u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n-1} u(-n-1) \quad (6') \\ \hline a^{-n+1} u(n-1) \quad (8') \\ \hline \end{array}$$



$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ R(n) & \xleftrightarrow{(2)} & R(-n) \\ a^n R(n) & \longleftrightarrow & a^{-n} R(-n) \end{array}$$

$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ a^n R(n) & \xleftrightarrow{(2)} & a^{sh2(n)} R(-n) \\ a^n R(n) & \longleftrightarrow & a^{-sh2(n)} R(-n) \end{array}$$

b^n	b^{-n}
a^n	a^{-n}
a^{-n}	a^n

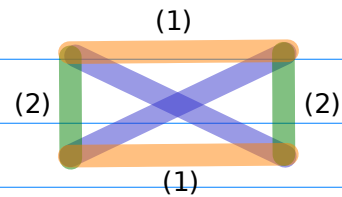
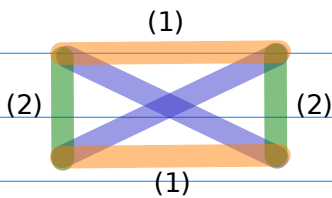
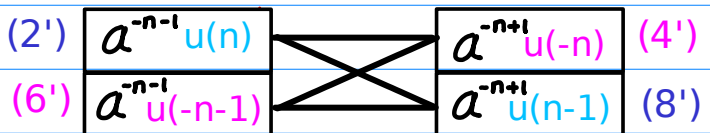
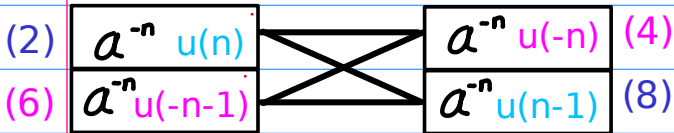
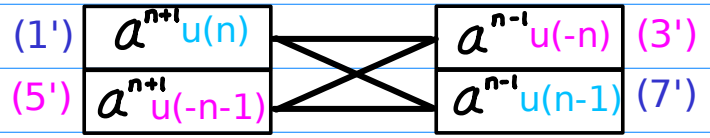
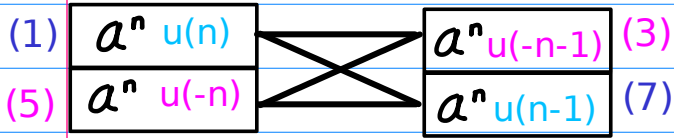
b^n	$b^{-sh2(n)}$
$a^{(n+1)}$	$a^{-(n-1)}$
$a^{-(n+1)}$	$a^{(n-1)}$
$a^{(n-1)}$	$a^{-(n+1)}$
$a^{-(n-1)}$	$a^{(n+1)}$

$R(n)$	$R(-n)$
$u(n)$	$u(-n)$
$u(n-1)$	$u(-n-1)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

$R(n)$	$R(-n)$
$u(n)$	$u(-n)$
$u(n-1)$	$u(-n-1)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

B.I Range Shifting
(1) Range Complementing
(2) Range Flipping

E.I Shifting2
(1) Shifted Range Flipping
(2) Range Complementing



$$R(n) \xleftrightarrow{(1)} R(-n)$$

$$R(n) \xleftrightarrow{(2)} \overline{R(n)}$$

$$R(n) \longleftrightarrow \overline{R(-n)}$$

$$a^n R(n) \xleftrightarrow{(1)} a^{sh2(n)} R(-n)$$

$$R(n) \xleftrightarrow{(2)} \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

b^n	$b^{sh2(n)}$
$a^{(n+1)}$	$a^{(n-1)}$
$a^{-(n+1)}$	$a^{-(n-1)}$
$a^{(n-1)}$	$a^{(n+1)}$
$a^{-(n-1)}$	$a^{-(n+1)}$

$R(n)$	$\overline{R(-n)}$
$u(n)$	$u(n-1)$
$u(n-1)$	$u(n)$
$u(-n)$	$u(-n-1)$
$u(-n-1)$	$u(-n)$

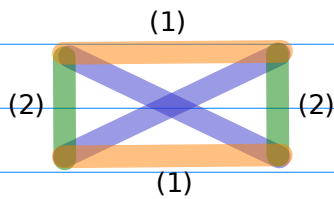
$R(n)$	$\overline{R(-n)}$
$u(n)$	$u(n-1)$
$u(n-1)$	$u(n)$
$u(-n)$	$u(-n-1)$
$u(-n-1)$	$u(-n)$

C.I Complementary Inverting

(1) Base Inverting
(2) Range Complementing

$$\begin{array}{|c|} \hline (1) \ a^n u(n) \\ \hline (5) \ a^n u(-n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{|c|} \hline a^{-n} u(n) \ (2) \\ \hline a^{-n} u(-n-1) \ (6) \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (3) \ a^n u(-n) \\ \hline (7) \ a^n u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{|c|} \hline a^{-n} u(-n) \ (4) \\ \hline a^{-n} u(n-1) \ (8) \\ \hline \end{array}$$



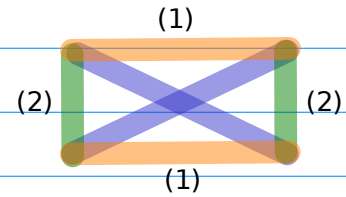
$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\ a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)} \end{array}$$

F.I Complementary Inverting

(1) Base Inverting
(2) Range Complementing

$$\begin{array}{|c|} \hline (1') \ a^{n+1} u(n) \\ \hline (5') \ a^{n+1} u(-n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{|c|} \hline a^{-n-1} u(n) \ (2') \\ \hline a^{-n-1} u(-n-1) \ (6') \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (3') \ a^{n-1} u(-n) \\ \hline (7') \ a^{n-1} u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{|c|} \hline a^{-n+1} u(-n) \ (4') \\ \hline a^{-n+1} u(n-1) \ (8') \\ \hline \end{array}$$



$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\ a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)} \end{array}$$

b^n	b^{-n}
a^n	a^{-n}
a^{-n}	a^n

b^n	b^{-n}
$a^{(n+1)}$	$a^{-(n+1)}$
$a^{-(n+1)}$	$a^{(n+1)}$
$a^{(n-1)}$	$a^{-(n-1)}$
$a^{-(n-1)}$	$a^{(n-1)}$

$R(n)$	$\overline{R(n)}$
$u(n)$	$u(-n-1)$
$u(n-1)$	$u(-n)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

$R(n)$	$\overline{R(n)}$
$u(n)$	$u(-n-1)$
$u(n-1)$	$u(-n)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

A.I Flipping

- (1) Base Inverting
- (2) Range Flipping

$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 R(n) & \xleftrightarrow{(2)} & R(-n) \\
 a^n R(n) & \longleftrightarrow & a^{-n} R(-n)
 \end{array}$$

D.I Flipping2

- (1) Base Inverting
- (2) Shifted Range Flipping

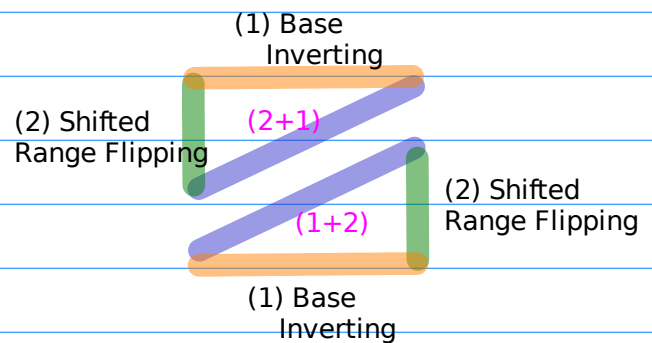
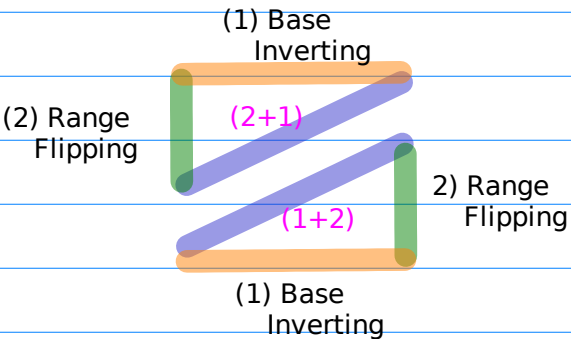
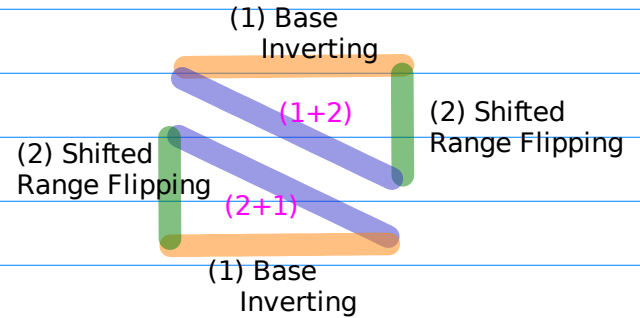
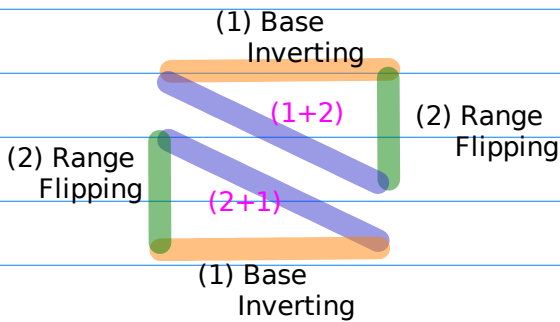
$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 a^n R(n) & \xleftrightarrow{(2)} & a^{sh2(n)} R(-n) \\
 a^n R(n) & \longleftrightarrow & a^{-sh2(n)} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-n} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-sh2(n)} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^n R(-n) \\
 & \xrightarrow{(1)} & a^{-n} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^{sh2(n)} R(-n) \\
 & \xrightarrow{(1)} & a^{-sh2(n)} R(-n)
 \end{array}$$



B.I Range Shifting
(1) Range Complementing
(2) Range Flipping

E.I Shifting2
(1) Shifted Range Flipping
(2) Range Complementing

$$R(n) \xleftrightarrow{(1)} R(-n)$$

$$R(n) \xleftrightarrow{(2)} \overline{R(n)}$$

$$R(n) \longleftrightarrow \overline{R(-n)}$$

$$a^n R(n) \xleftrightarrow{(1)} a^{sh2(n)} R(-n)$$

$$R(n) \xleftrightarrow{(2)} \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

$$a^n R(n) \xrightarrow{(1)} a^{-n} R(n)$$

$$\xrightarrow{(2)} a^{-n} \overline{R(n)}$$

$$a^n R(n) \xrightarrow{(1)} a^{-n} R(n)$$

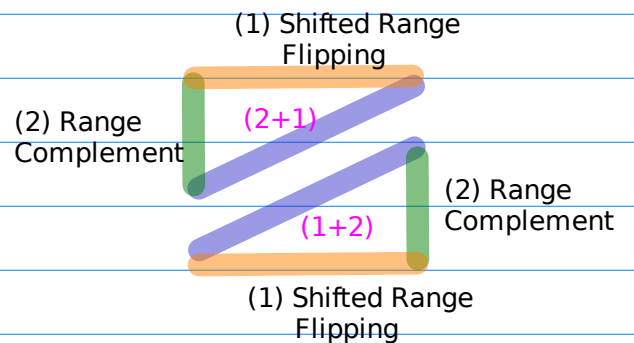
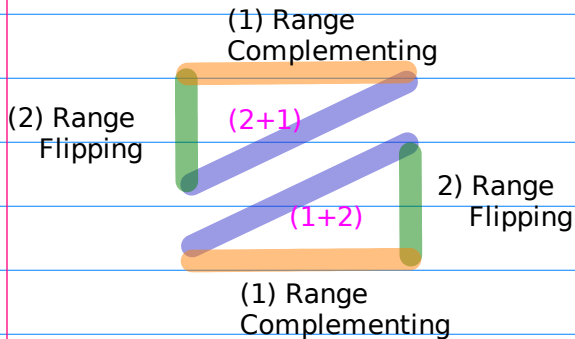
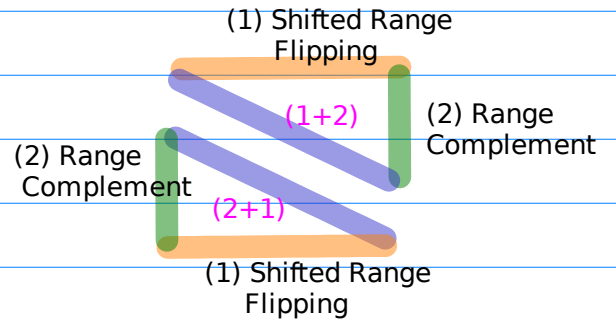
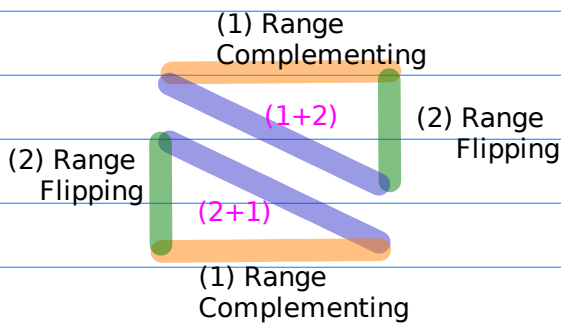
$$\xrightarrow{(2)} a^{-sh2(n)} R(-n)$$

$$a^n rng(n) \xrightarrow{(2)} a^n \overline{R(n)}$$

$$\xrightarrow{(1)} a^{-n} \overline{R(n)}$$

$$a^n R(n) \xrightarrow{(2)} a^{sh2(n)} R(-n)$$

$$\xrightarrow{(1)} a^{-sh2(n)} R(-n)$$



C.I Complementary Inverting

(1) Base Inverting
(2) Range Complementing

F.I Complementary Inverting

(1) Base Inverting
(2) Range Complementing

$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\
 a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)}
 \end{array}$$

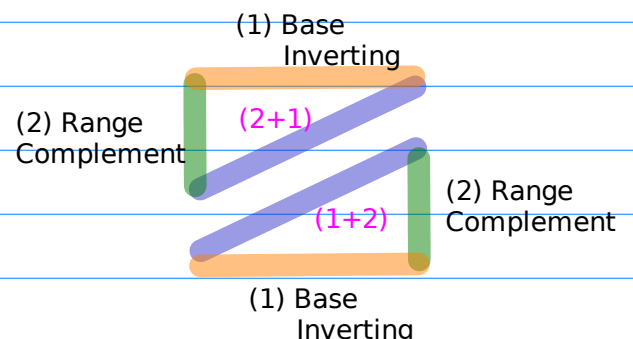
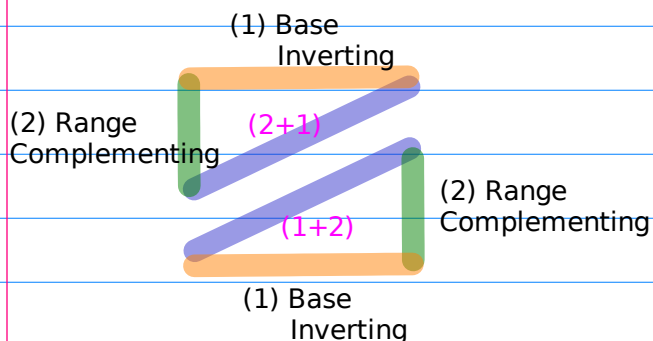
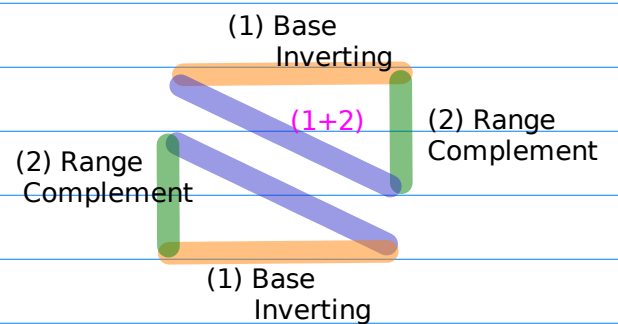
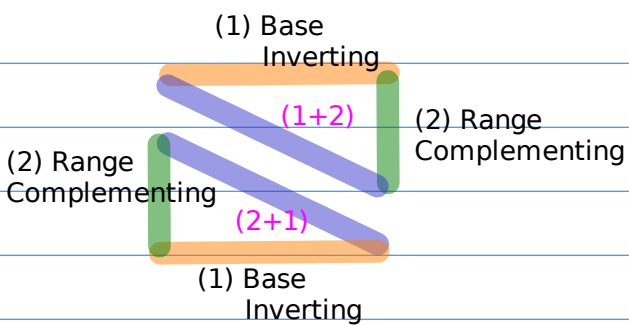
$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\
 a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)}
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-n} \overline{R(n)}
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-n} \overline{R(n)}
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^n \overline{R(n)} \\
 & \xrightarrow{(1)} & a^{-n} \overline{R(n)}
 \end{array}$$

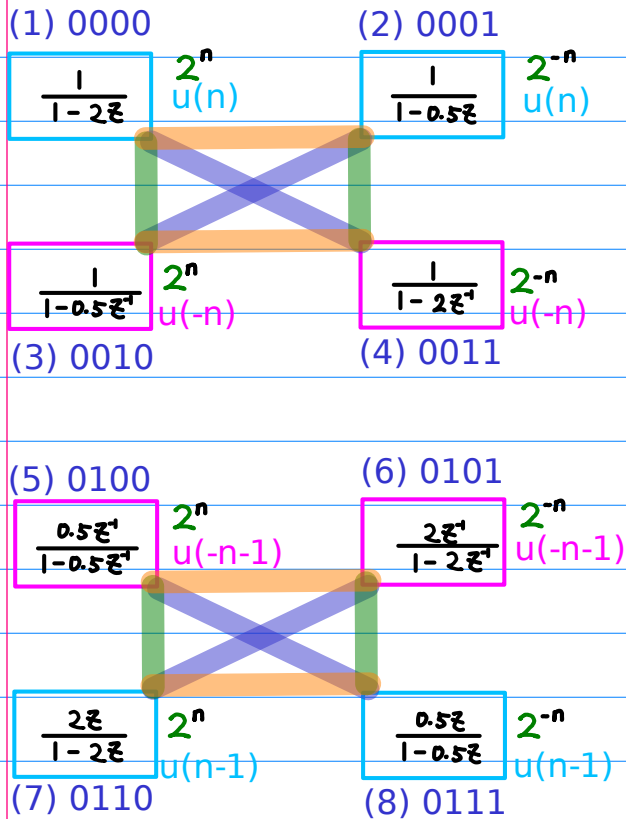
$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^n \overline{R(n)} \\
 & \xrightarrow{(1)} & a^{-n} \overline{R(n)}
 \end{array}$$



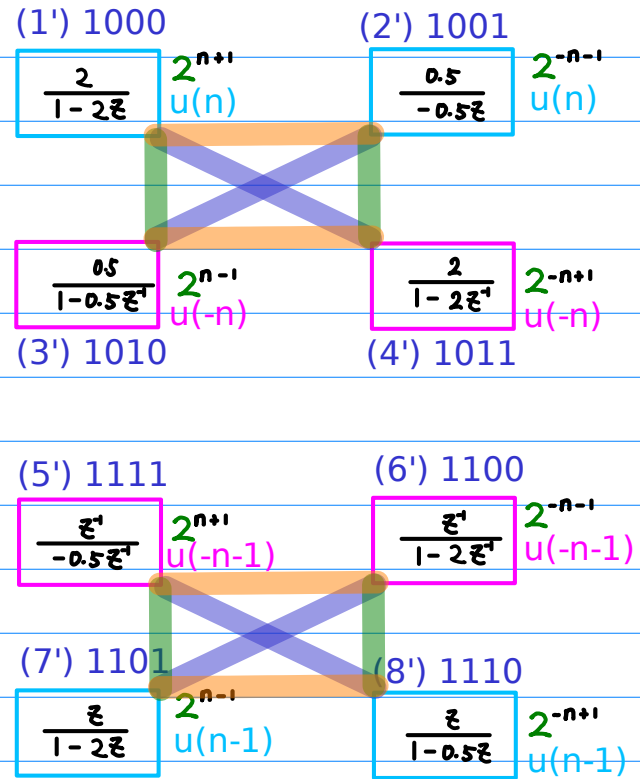




A.I Flipping Base Inverting Range Flipping



D.I Flipping2 Base Inverting Shifted Range Flipping



Shifted Range Flipping
= Exponent Shifting2
+ Range Flipping

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow R(-n)$$

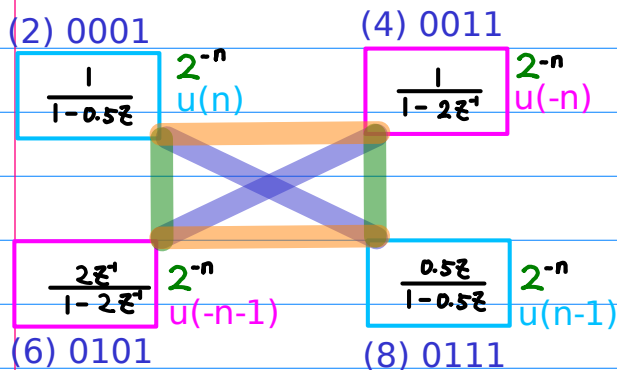
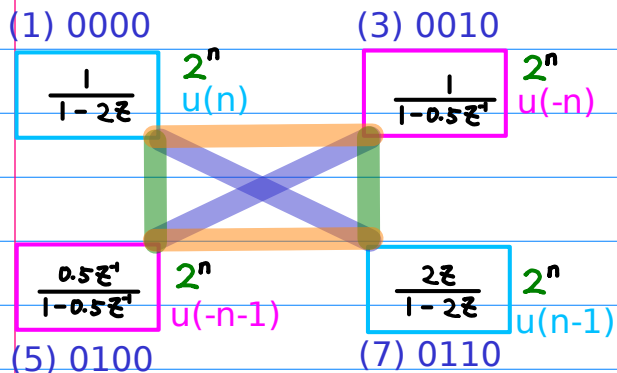
$$a^n R(n) \longleftrightarrow a^{-n} R(-n)$$

$$a^n \longleftrightarrow a^{-n}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

$$a^n R(n) \longleftrightarrow a^{-sh2(n)} R(-n)$$

B.I Range Shifting Range Flipping Range Complementing



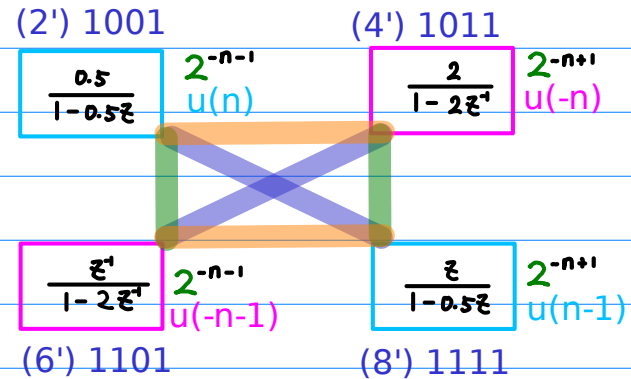
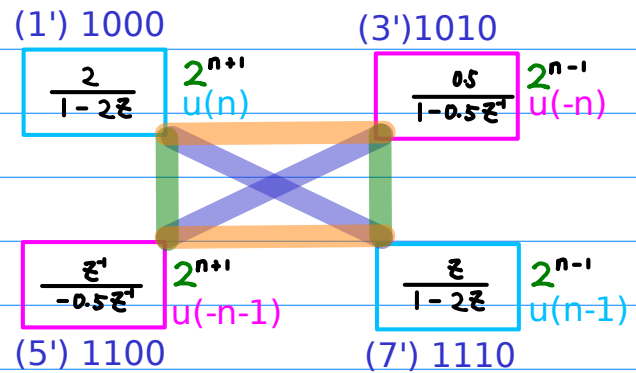
Range Shifting
= Range Flipping
+ Range Complementing

$$R(n) \longleftrightarrow R(-n)$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$R(n) \longleftrightarrow \overline{R(-n)}$$

E.I Shifting2 Shifted Range Flipping Range Complementing



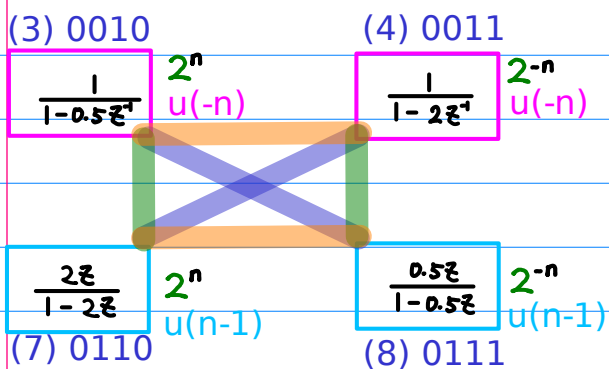
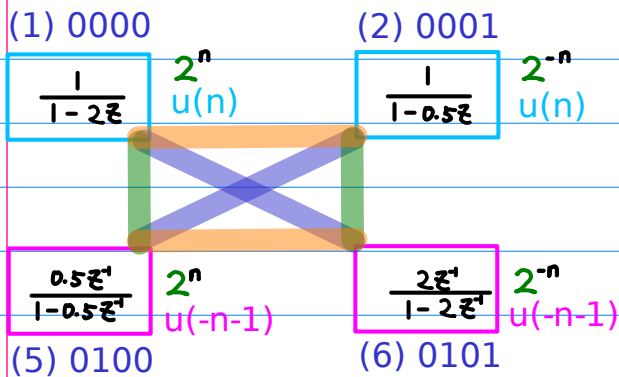
Shifted Range Flipping
= Exponent Shifting2
+ Range Flipping

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

C.I Complementary Inverting Base Inverting Range Complementing

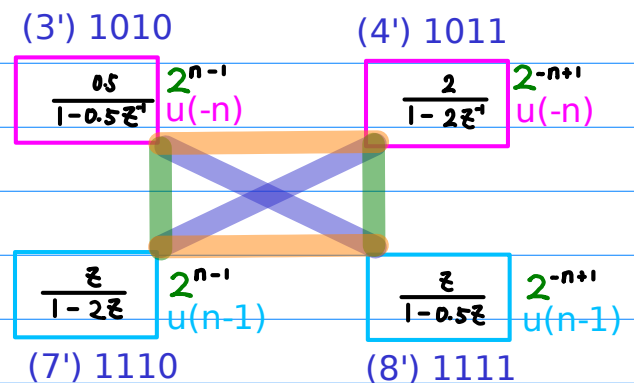
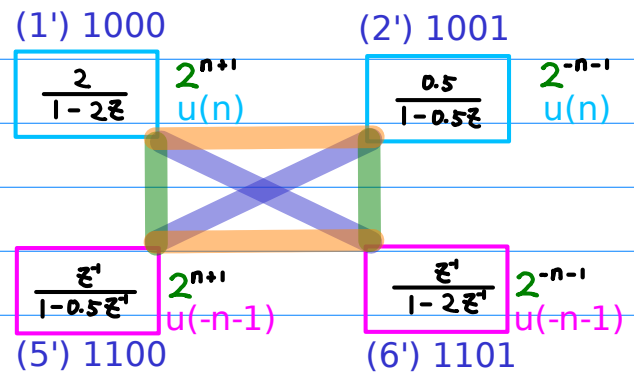


$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

F.I Complementary Inverting Base Inverting Range Complementing



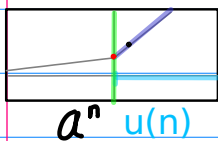
$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

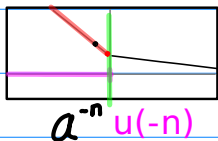
$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

A.II Flipping Base Inverting Range Flipping

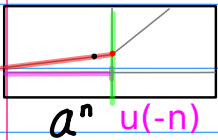
(1) 0000



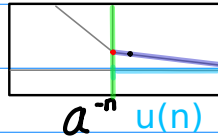
(4) 0011



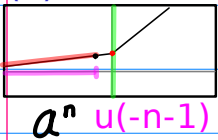
(3) 0010



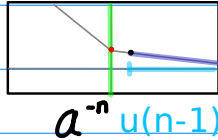
(2) 0001



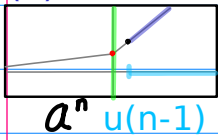
(5) 0100



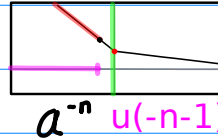
(8) 0111



(7) 0110

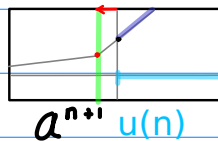


(6) 0101

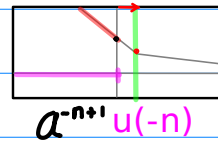


D.II Flipping2 Base Inverting Shifted Range Flipping

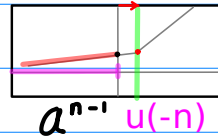
(1') 1000



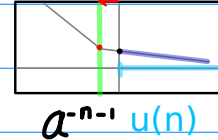
(4') 1011



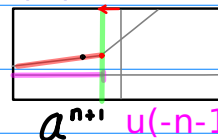
(3') 1010



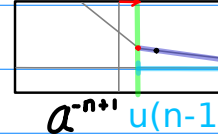
(2') 1001



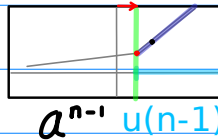
(5') 1100



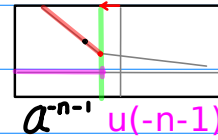
(8') 1111



(7') 1110



(6') 1101



Shifted Range Flipping
= Exponent Shifting2
+ Range Flipping

$$a^n R(n) \longleftrightarrow a^{-n} R(-n)$$

$$a^n R(n) \longleftrightarrow a^{-sh2(n)} R(-n)$$

$$a^n \longleftrightarrow a^{-n}$$

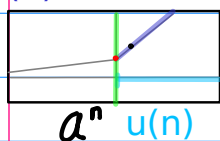
$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow R(-n)$$

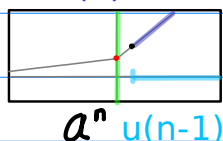
$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

B.II Range Shifting Range Flipping Range Complementing

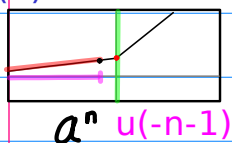
(1) 0000



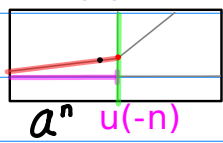
(7) 0110



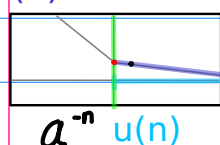
(5) 0100



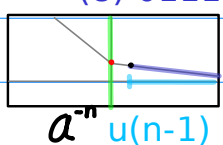
(3) 0010



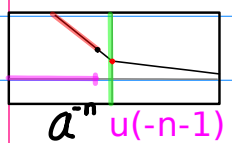
(2) 0001



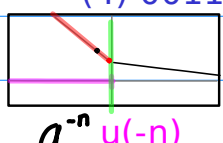
(8) 0111



(6) 0101



(4) 0011



Range Shifting
= Range Flipping
+ Range Complementing

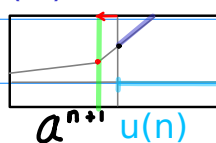
$$R(n) \longleftrightarrow \overline{R(-n)}$$

$$R(n) \longleftrightarrow R(-n)$$

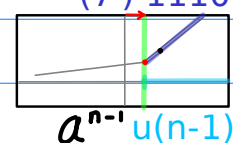
$$R(n) \longleftrightarrow \overline{\overline{R(n)}}$$

E.II Shifting2 Shifted Range Flipping Range Complementing

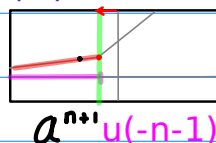
(1') 1000



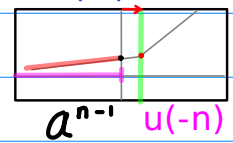
(7') 1110



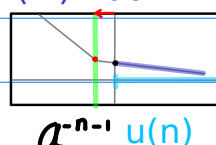
(5') 1100



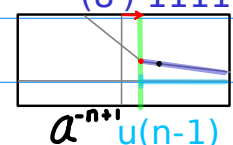
(3') 1010



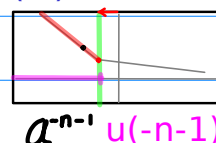
(2') 1001



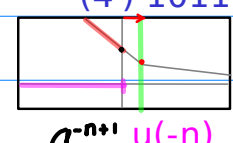
(8') 1111



(6') 1101



(4') 1011



Shifted Range Flipping
= Exponent Shifting2
+ Range Flipping

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

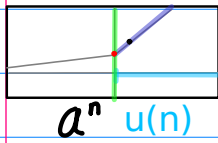
$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

$$R(n) \longleftrightarrow \overline{\overline{R(n)}}$$

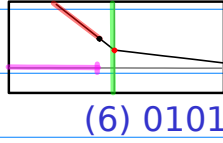
C.II Complementary Inverting Base Inverting Range Complementing

F.II Complementary Inverting Base Inverting Range Complementing

(1) 0000

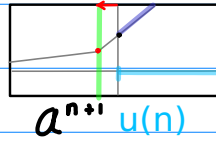


$a^{-n} u(-n-1)$



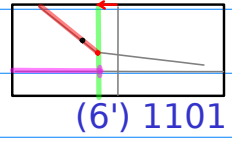
(6) 0101

(1') 1000



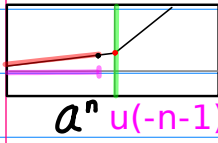
$a^{n+1} u(n)$

$a^{-n-1} u(-n-1)$



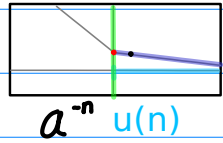
(6') 1101

(5) 0100



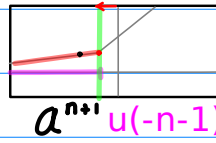
$a^n u(-n-1)$

(2) 0001



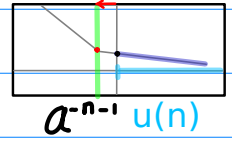
$a^{-n} u(n)$

(5') 1100



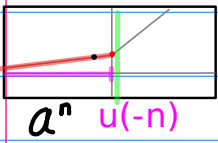
$a^{n+1} u(-n-1)$

(2') 1001



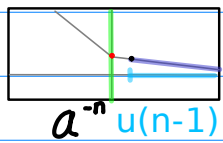
$a^{-n-1} u(n)$

(3) 0010



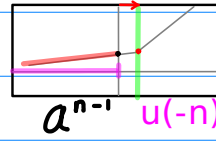
$a^n u(-n)$

(8) 0111



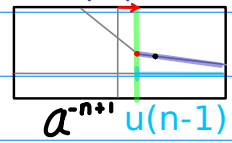
$a^{-n} u(n-1)$

(3') 1010



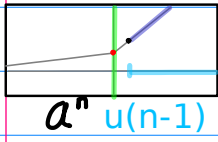
$a^{n-1} u(-n)$

(8') 1111



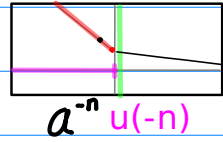
$a^{-n+1} u(n-1)$

(7) 0110



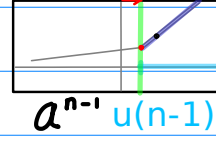
$a^n u(n-1)$

(4) 0011



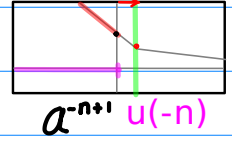
$a^{-n} u(-n)$

(7') 1110



$a^{n-1} u(n-1)$

(4') 1011



$a^{-n+1} u(-n)$

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

