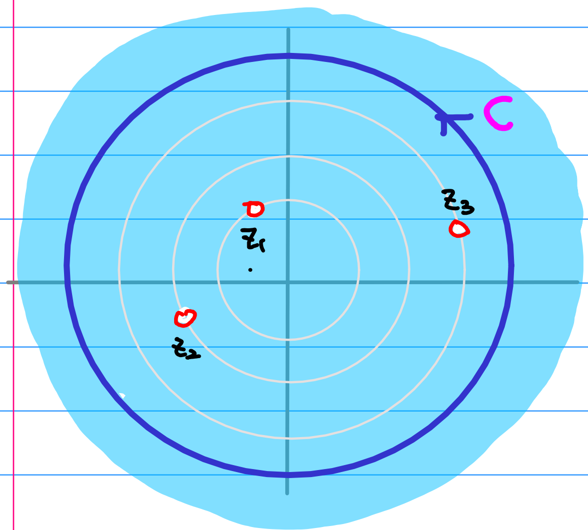
Laurent Series and z-Transform

20170729

.

General Series Expansion at $z=0$



$$f(z) = \sum_{n=n_1}^{\infty} a_n z^n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} dz$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} dz$$

z_k : Poles of $\dfrac{f(z)}{z^{n+1}}$

$n \geq 0$ $\quad z_1, z_2, z_3, 0$

$n < 0$ $\quad z_1, z_2, z_3$

$$f(z) \rightarrow \frac{f(z)}{z^{n+1}} \rightarrow \text{poles } z_1, z_2, \cdots, z_k$$

$$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_1\right) + \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_2\right) + \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

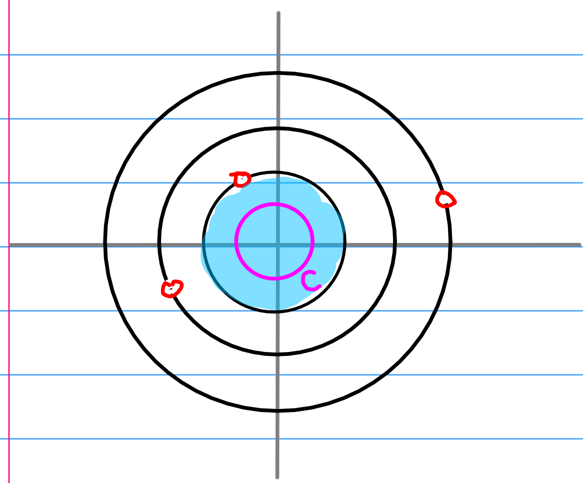
$$\sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

$$a_n$$

$$\sum_{n=n_1}^{\infty} a_n z^n = f(z)$$

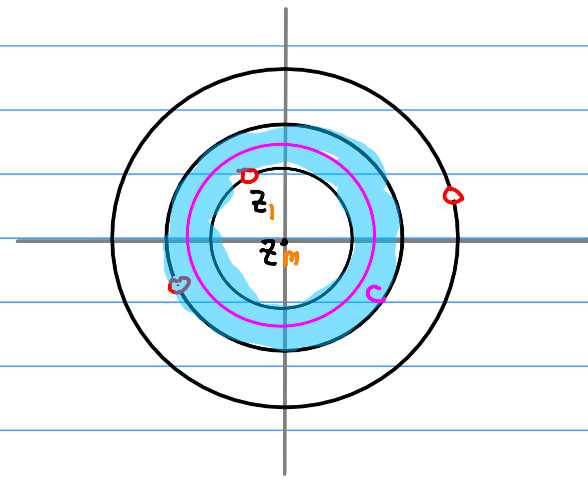
a_n ($n \geq 0$) ROC's of L.S.

Case Ⅰ $f_1(z) = \sum a_n z^n$



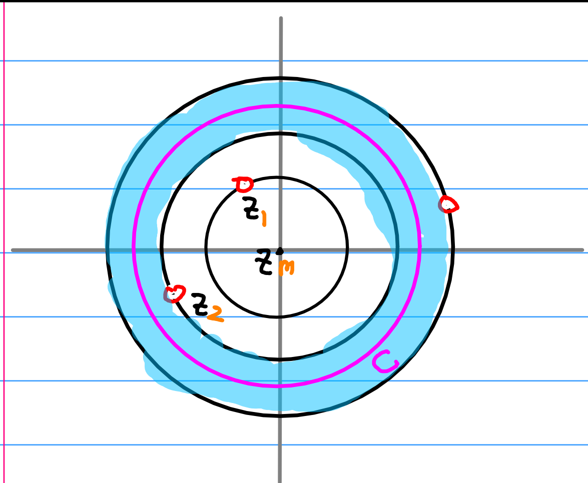
$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\, dz$$

Case Ⅱ $f_2(z) = \sum b_n z^n$



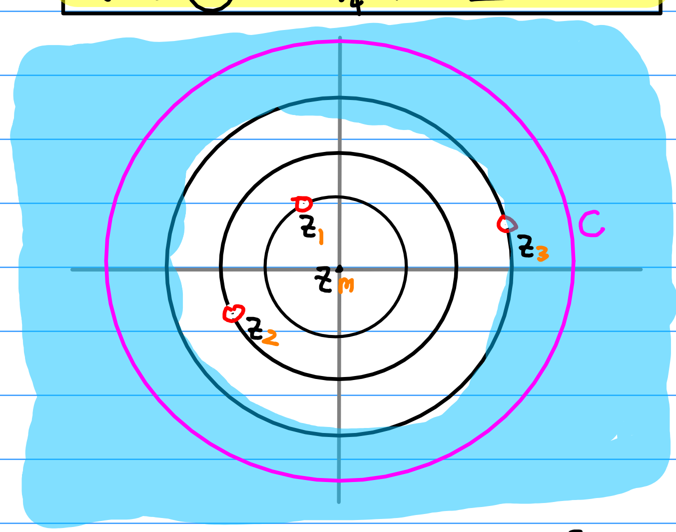
$$b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\, dz$$

Case Ⅲ $f_3(z) = \sum c_n z^n$



$$c_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\, dz$$

Case Ⅳ $f_4(z) = \sum d_n z^n$



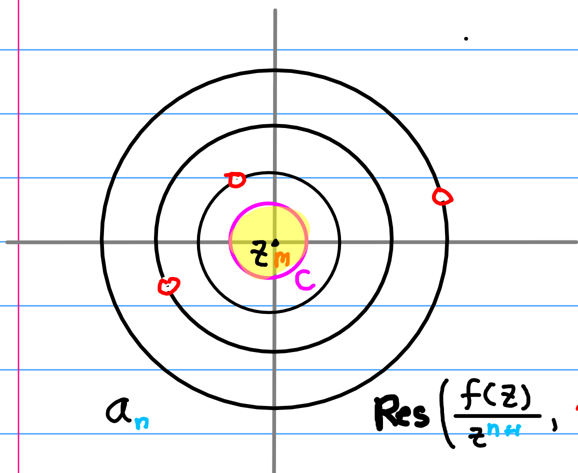
$$d_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\, dz$$

* $f_i(z)$ valid in its ROC

converge

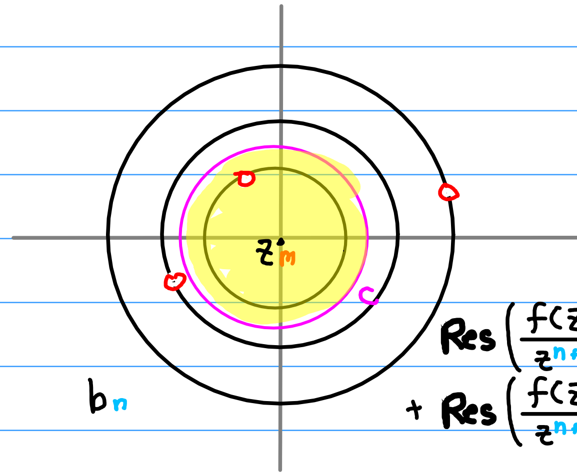
$a_n \ (n \geq 0)$ ROC's of Cauchy's Residue Theorem

Case $\boxed{A}$ $f_1(z) = \sum a_n z^n$



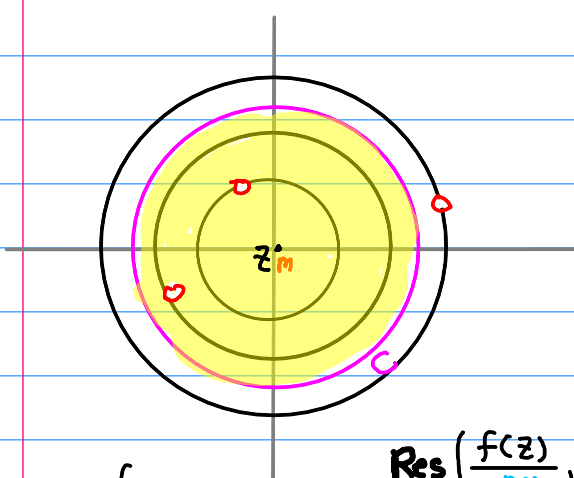
a_n $\quad \text{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$

Case $\boxed{B}$ $f_2(z) = \sum b_n z^n$



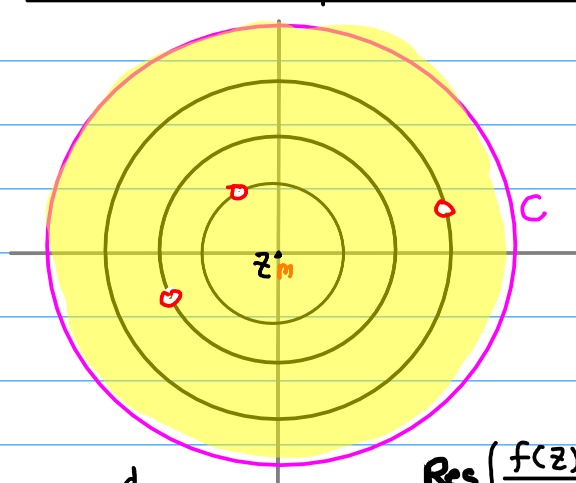
b_n $\quad \text{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right) + \text{Res}\left(\frac{f(z)}{z^{n+1}}, z_1\right)$

Case $\boxed{C}$ $f_3(z) = \sum c_n z^n$



c_n $\quad \text{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right) + \text{Res}\left(\frac{f(z)}{z^{n+1}}, z_1\right) + \text{Res}\left(\frac{f(z)}{z^{n+1}}, z_2\right)$

Case $\boxed{D}$ $f_4(z) = \sum d_n z^n$



d_n $\quad \text{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right) + \text{Res}\left(\frac{f(z)}{z^{n+1}}, z_1\right) + \text{Res}\left(\frac{f(z)}{z^{n+1}}, z_2\right) + \text{Res}\left(\frac{f(z)}{z^{n+1}}, z_3\right)$

* General Series Expansion at $z=0$

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n z^n}$$

$$\boxed{\begin{aligned} a_n &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\,dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right) \end{aligned}}$$

* Z-transform

$$\boxed{X(z) = \sum_{k=0}^{\infty} x_k z^{-k}}$$

$$\boxed{\begin{aligned} x_n &= \frac{1}{2\pi i}\oint_C X(z) z^{n-1}\,dz \\ &= \sum_k \mathrm{Res}\left(X(z) z^{n-1}, z_k\right) \end{aligned}}$$

Signal Processing Applications

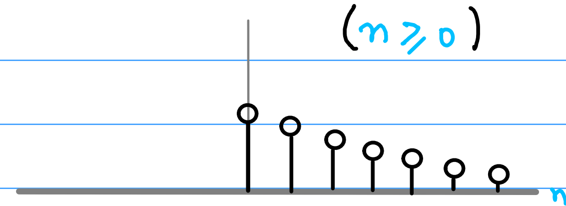
$x_k \rightarrow X(z)$ Z. T.

$a_n \rightarrow f(z)$ L. S.

↑

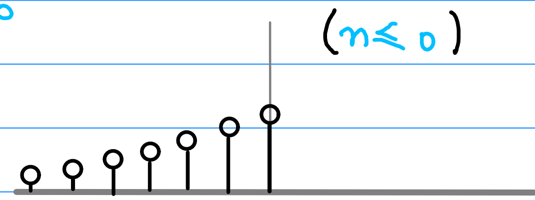
given signal sequence

Causal Signal $(n \geq 0)$



$$\sum_{n=0}^{\infty} \qquad n_1 = 0$$

Anti-causal Signal $(n \leq 0)$



$$\sum_{n=-\infty}^{0} \qquad n_1 = -\infty \rightarrow n_1 = 0$$

Inverse z-Transform $x[n] = \frac{1}{2\pi i}\int_C X(z)\, z^{n-1}\, dz$

$$X(z) = \sum_{k=0}^{\infty} x_k\, z^{-k}$$

$$z^{n-1} X(z) = \left(\sum_{k=0}^{\infty} x_k\, z^{-k}\right) z^{n-1}$$

$$\boxed{\int z^{n-1}\, \text{LHS}\, dz = \int \text{RHS}\, z^{n-1}\, dz}$$

$$= \sum_{k=0}^{\infty} x_k\, z^{-k+n-1}$$

$$\boxed{[0,\infty) = [0,n-1] \cup [n] \cup [n+1,\infty)}$$

$$= \sum_{k=0}^{n-1} x_k\, z^{-k+n-1} + \sum_{k=n}^{n} x_k\, z^{-k+n-1} + \sum_{k=n+1}^{\infty} x_k\, z^{-k+n-1}$$

$$= \sum_{k=0}^{n-1} x_k\, z^{-k+n-1} + \frac{x_n}{z^1} + \sum_{k=n+1}^{\infty} \frac{x_k}{z^{k-n+1}}$$

$$\int_c X(z)\, z^{n-1}\, dz = \int_c \sum_{k=0}^{n-1} x_k\, z^{-k+n-1}\, dz + \int_c \frac{x_n}{z^1}\, dz + \int_c \sum_{k=n+1}^{\infty} \frac{x_k}{z^{k-n+1}}\, dz$$

$$= \sum_{k=0}^{n-1} x_k \int_c z^{-k+n-1}\, dz + x_n \int_c \frac{1}{z^1}\, dz + \sum_{k=n+1}^{\infty} x_k \int_c \frac{1}{z^{k-n+1}}\, dz$$

$$= \sum_{k=0}^{n-1} x_k \cdot 0 + x_n \cdot 2\pi i + \sum_{k=n+1}^{\infty} x_k \cdot 0$$

$$x[n] = \frac{1}{2\pi i}\int_c X(z)\, z^{n-1}\, dz$$

Z - Transform	$X(z)$
Laurent Series	$f(z)$

z-Transform $X(z) \longleftrightarrow x_n$

Laurent Series $f(z) \longleftrightarrow a_n$

$$\boxed{X(z) = f(z^{-1}) \longleftrightarrow x_n = a_n}$$

z-Transform $X(z) \longleftrightarrow x_n$

Laurent Series $f(z) \longleftrightarrow a_n$

$$\boxed{X(z) = f(z) \longleftrightarrow x_n = a_{-n}}$$

$$X(z) = f(z^{-1}), \quad x_n = a_n$$

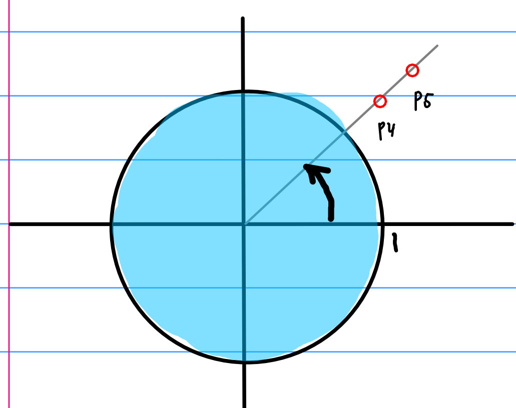
$$f(z) = \cdots + a_{-2} z^{-2} + a_{-1} z^{-1} + a_0 z^0 + a_1 z^1 + a_2 z^2 + \cdots$$

$$f(z^{-1}) = \cdots + a_{-2} z^{2} + a_{-1} z^{1} + a_0 z^0 + a_1 z^{-1} + a_2 z^{-2} + \cdots$$

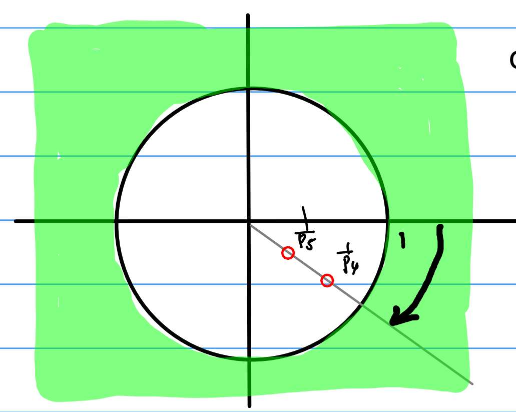
$$X(z) = \cdots + x_{-2} z^{2} + x_{-1} z^{1} + x_0 z^0 + x_1 z^{-1} + x_2 z^{-2} + \cdots$$

$$f(z^{-1}) = X(z) \iff a_n = x_n$$

$f(z)$



$X(z) = f(z^{-1})$



causal

p $\qquad$ p^{-1}

$$X(z) = f(z^{-1}), \quad x_n = a_n$$

$$f(z) = \cdots + a_{-2} z^{-2} + a_{-1} z^{-1} + a_0 z^0 + a_1 z^1 + a_2 z^2 + \cdots$$

$$f(z^{-1}) = \cdots + a_{-2} z^{2} + a_{-1} z^{1} + a_0 z^0 + a_1 z^{-1} + a_2 z^{-2} + \cdots$$

$$f(z^{-1}) = \cdots a_{2} z^{-2} + a_{1} z^{-1} + a_0 z^0 + a_{-1} z^{1} + a_{-2} z^{2} + \cdots$$

	...	z^{-2}	z^{-1}	z^0	z^1	z^2	...
$f(z)$	...	a_{-2}	a_{-1}	a_0	a_1	a_2	...
$f(z^{-1})$	...	a_2	a_1	a_0	a_{-1}	a_{-2}	...

$$X(z) = \cdots + x_{-2} z^{2} + x_{-1} z^{1} + x_0 z^0 + x_1 z^{-1} + x_2 z^{-2} + \cdots$$

$$X(z) = \cdots x_{2} z^{-2} + x_{1} z^{-1} + x_0 z^0 + x_{-1} z^{1} + x_{-2} z^{2} + \cdots$$

	...	z^{-2}	z^{-1}	z^0	z^1	z^2	...
$X(z)$	...	x_2	x_1	x_0	x_{-1}	x_{-2}	...

z-Transform $\quad X(z) \longleftrightarrow x_n$

Laurent Series $\quad f(z) \longleftrightarrow a_n$

$$X(z) = f(z^{-1}) \longleftrightarrow x_n = a_n$$

$$X(z) = f(z^{-1}), \qquad x_n = a_n$$

$$a_n \longleftrightarrow f(z)$$

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\,dz$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

$$x_n \longleftrightarrow X(z)$$

$$a'_n \longleftrightarrow f(z^{-1}) = f(w)$$

$$a'_n = \frac{1}{2\pi i}\oint_C \frac{f(z^{-1})}{z^{n+1}}\,dz$$

$$= \frac{1}{2\pi i}\oint_C f(z^{-1})\, z^{-n-1}\,dz$$

$$\boxed{w = z^{-1}} \qquad \boxed{dw = -z^{-2}\,dz}$$

$$= \frac{-1}{2\pi i}\oint_{C'} f(z^{-1})\, z^{-n+1} z^{-2}\,dz$$

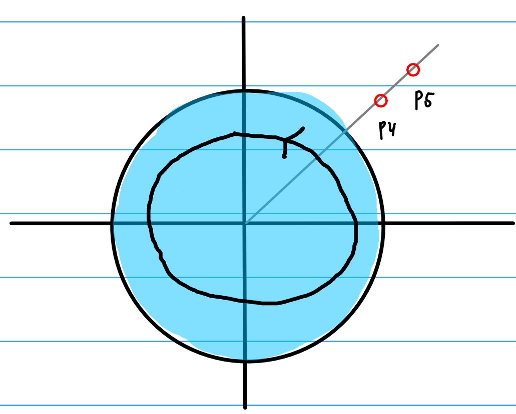
$$= \frac{1}{2\pi i}\oint_C f(w)\, w^{n-1}\,dw$$

$$= \sum_k \mathrm{Res}\left(f(w)\, w^{n-1}, w_k\right)$$

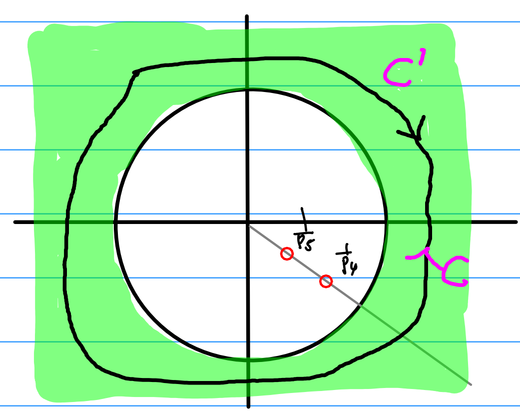
$$\boxed{a_n = \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)}$$

$$\boxed{x_n = \sum_k \mathrm{Res}\left(X(z)\, z^{n-1}, z_k\right)}$$

$f(z)$



$X(z) = f(z^{-1})$



causal

$$X(z) = f(z), \quad x_n = a_{-n}$$

$$X(z) = \cdots + x_{-2} z^{2} + x_{-1} z^{1} + x_0 z^{0} + x_1 z^{-1} + x_2 z^{-2} + \cdots$$

$$= \cdots + x_2 z^{-2} + x_1 z^{-1} + x_0 z^{0} + x_{-1} z^{1} + x_{-2} z^{+2} + \cdots$$

$$f(z) = \cdots + a_{-2} z^{-2} + a_{-1} z^{-1} + a_0 z^{0} + a_1 z^{1} + a_2 z^{2} + \cdots$$

$$f(z) = X(z) \Longleftrightarrow a_{-n} = x_n$$

$$X(z) = f(z), \qquad x_n = a_{-n}$$

$$f(z) = \cdots + a_{-2} z^{-2} + a_{-1} z^{-1} + a_0 z^{0} + a_1 z^{1} + a_2 z^{2} + \cdots$$

	$\cdots$	z^{-2}	z^{-1}	z^{0}	z^{1}	z^{2}	$\cdots$
$f(z)$	$\cdots$	a_{-2}	a_{-1}	a_0	a_1	a_2	$\cdots$

$$X(z) = \cdots + x_{-2} z^{2} + x_{-1} z^{1} + x_0 z^{0} + x_1 z^{-1} + x_2 z^{-2} + \cdots$$

$$X(z) = \cdots\ x_2 z^{-2} + x_1 z^{-1} + x_0 z^{0} + x_{-1} z^{1} + x_{-2} z^{2} + \cdots$$

	$\cdots$	z^{2}	z^{1}	z^{0}	z^{-1}	z^{-2}	$\cdots$
$X(z)$	$\cdots$	x_{-2}	x_{-1}	x_0	x_1	x_2	$\cdots$
	$\cdots$	z^{-2}	z^{-1}	z^{0}	z^{1}	z^{2}	$\cdots$
$X(z)$	$\cdots$	x_2	x_1	x_0	x_{-1}	x_{-2}	$\cdots$

z-Transform $\quad X(z) \longleftrightarrow x_n$

Laurent Series $\quad f(z) \longleftrightarrow a_n$

$$X(z) = f(z) \longleftrightarrow x_n = a_{-n}$$

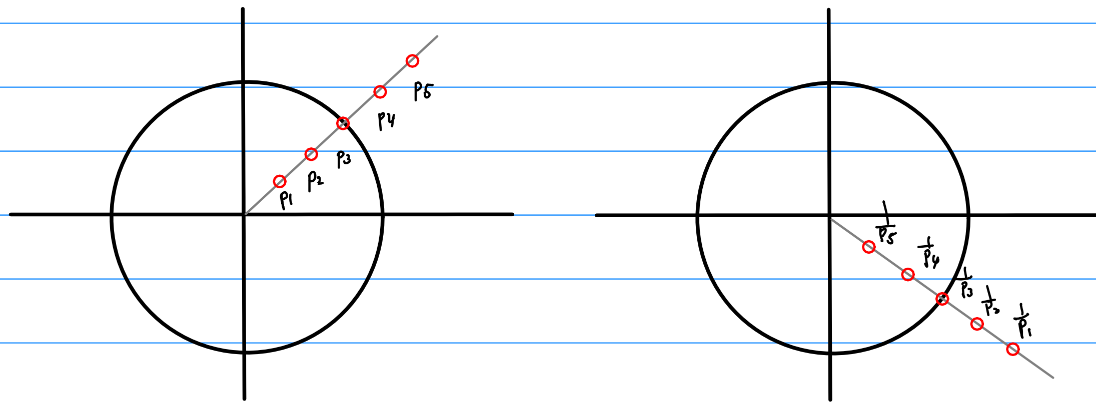
$$X(z) = f(z) \,, \qquad x_n = a_{-n}$$

$$f(z) = X(z) \quad \longleftrightarrow \quad a_{-n} = x_n$$

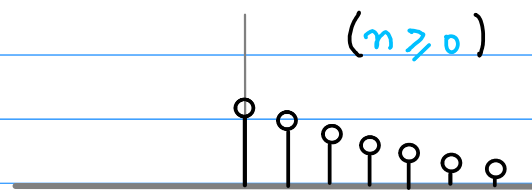
$$\boxed{a_n = \frac{1}{2\pi i} \oint_C \frac{X(z)}{z^{n+1}} \, dz = \sum_k \operatorname{Res}\left(\frac{X(z)}{z^{n+1}} , z_k \right)}$$

$$x_n = a_{-n} = \frac{1}{2\pi i} \oint_C \frac{X(z)}{z^{-n+1}} \, dz = \sum_k \operatorname{Res}\left(\frac{X(z)}{z^{-n+1}} , z_k \right)$$

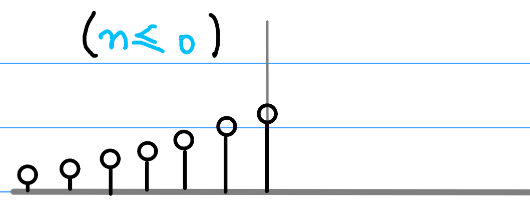
$$\boxed{x_n = \frac{1}{2\pi i} \oint_C X(z) \, z^{n-1} \, dz = \sum_k \operatorname{Res}\left(X(z) \, z^{n-1} , z_k \right)}$$



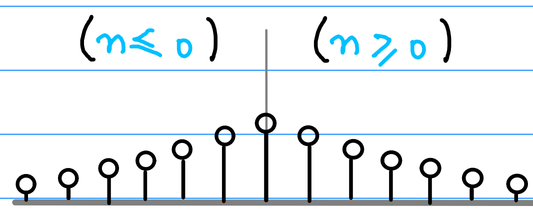
causal

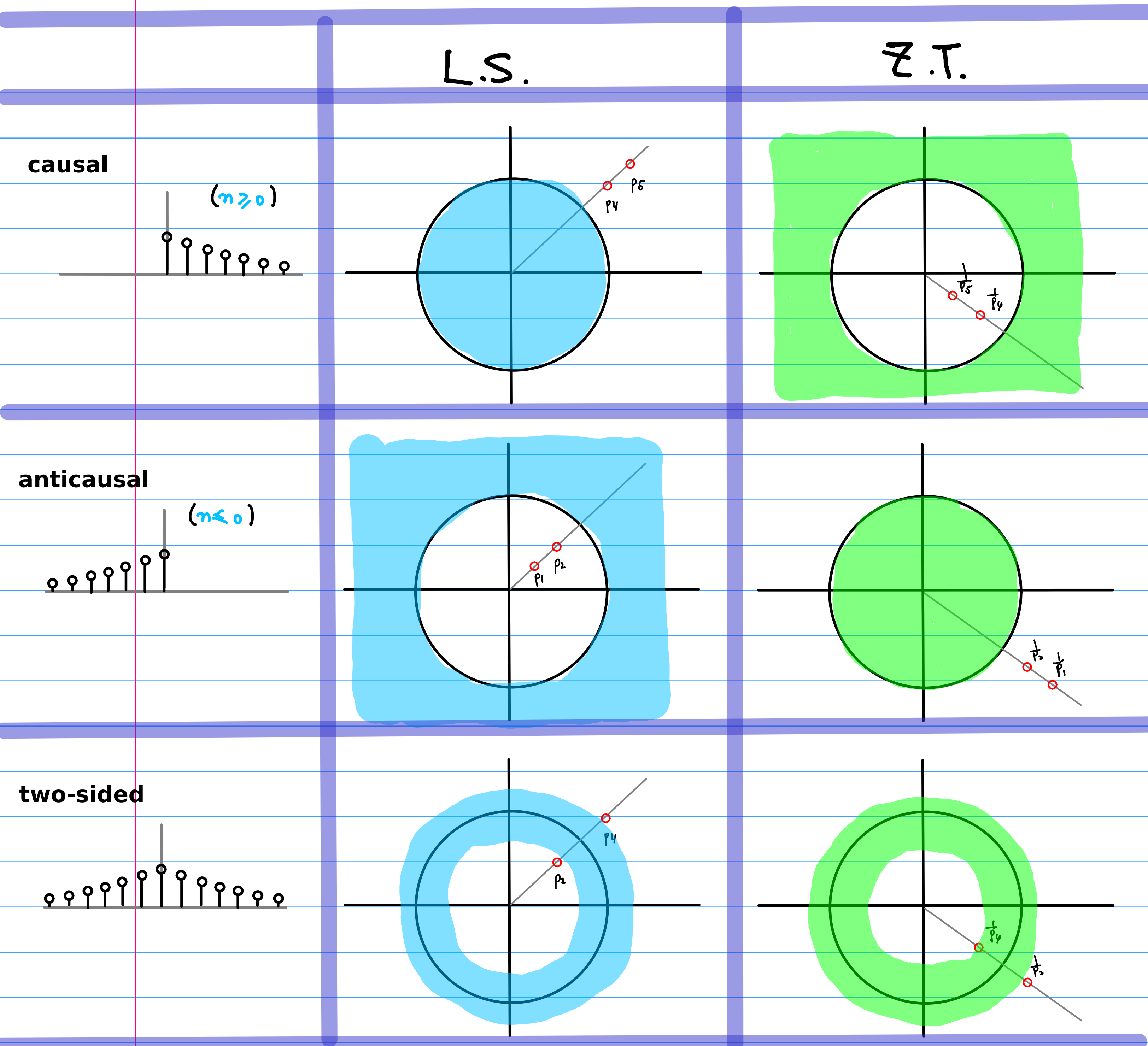


anti-causal



two-sided





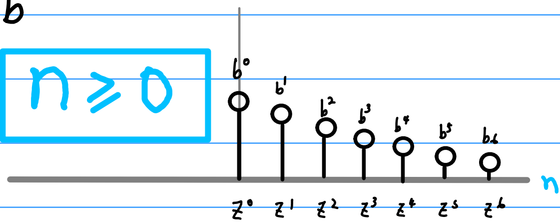
* no poles in ROC's

L.S.	Z.T.

L.S.

b^n

$\boxed{n \geq 0}$



$$f(z) = 1 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots$$

$a_0 = 1$

$a_1 = \left(\tfrac{1}{2}\right)$

$a_2 = \left(\tfrac{1}{2}\right)^2$

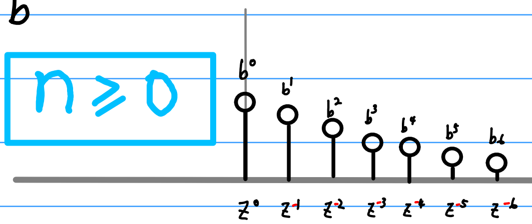
$a_3 = \left(\tfrac{1}{2}\right)^3 \qquad a_n = \left(\tfrac{1}{2}\right)^n$

$$f(z) = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n z^n$$

Z.T.

b^n

$\boxed{n \geq 0}$



$$X(z) = 1 + \left(\tfrac{1}{2}\right) z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \cdots$$

$x_0 = 1$

$x_1 = \left(\tfrac{1}{2}\right)$

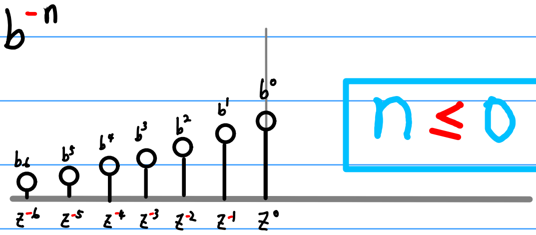
$x_2 = \left(\tfrac{1}{2}\right)^2$

$x_3 = \left(\tfrac{1}{2}\right)^3 \qquad x_n = \left(\tfrac{1}{2}\right)^n$

$$X(z) = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n z^{-n}$$

L.S.

b^{-n}



$\boxed{n \leq 0}$

$$f(z) = 1 + \left(\tfrac{1}{2}\right) z^{-1} + \left(\tfrac{1}{2}\right) z^{-2} + \cdots$$

$a_0 = 1$

$a_{-1} = \left(\tfrac{1}{2}\right)^1 = \left(\tfrac{1}{2}\right)^{-(-1)}$

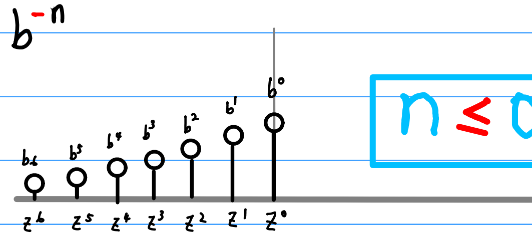
$a_{-2} = \left(\tfrac{1}{2}\right)^2 = \left(\tfrac{1}{2}\right)^{-(-2)}$

$a_{-3} = \left(\tfrac{1}{2}\right)^3 = \left(\tfrac{1}{2}\right)^{-(-3)} \qquad a_n = \left(\tfrac{1}{2}\right)^{-n}$

$$f(z) = \sum_{n=-\infty}^{0} \left(\tfrac{1}{2}\right)^{-n} z^n$$

Z.T.

b^{-n}



$\boxed{n \leq 0}$

$$X(z) = 1 + \left(\tfrac{1}{2}\right) z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots$$

$x_0 = 1$

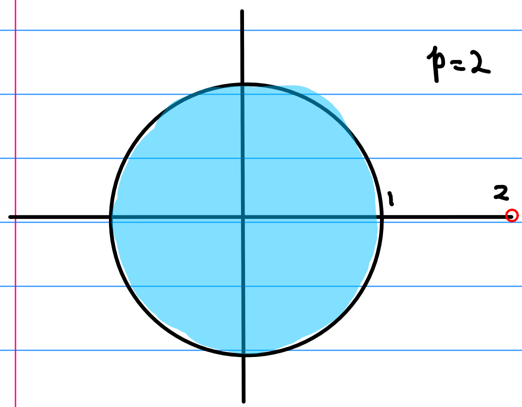
$x_{-1} = \left(\tfrac{1}{2}\right)^1 = \left(\tfrac{1}{2}\right)^{-(-1)}$

$x_{-2} = \left(\tfrac{1}{2}\right)^2 = \left(\tfrac{1}{2}\right)^{-(-2)}$

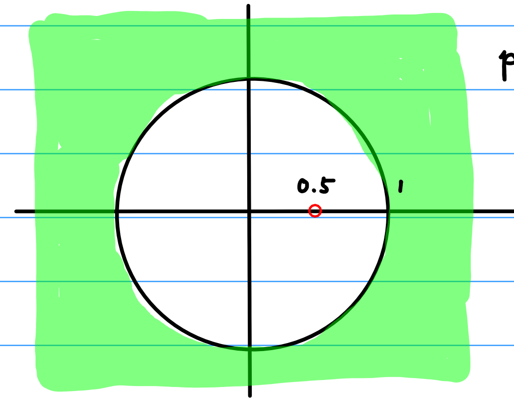
$x_{-3} = \left(\tfrac{1}{2}\right)^3 = \left(\tfrac{1}{2}\right)^{-(-3)} \qquad x_n = \left(\tfrac{1}{2}\right)^{-n}$

$$X(z) = \sum_{n=-\infty}^{0} \left(\tfrac{1}{2}\right)^{-n} z^{-n}$$

Causal



causal



$$\cdots 0, 0, 0, 1, b, b^2, b^3, \cdots$$

$$f(z) = X(z^{-1}) = \frac{z^{-1}}{z^{-1} - 0.5} = \frac{1}{1 - 0.5z} = \frac{2}{2-z}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{b}{z}\right)^n = \frac{1}{1-\frac{b}{z}} = \frac{z}{z-b} = \frac{z}{z-0.5}$$

$$a_n = \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right) = \mathrm{Res}\left(\frac{2}{z^{n+1}(2-z)}, 0\right) = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$x_n = \mathrm{Res}\left(X(z) z^{n-1}, 0.5\right) = \mathrm{Res}\left(\frac{z^n}{z-0.5}, 0.5\right) = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = 1 + \left(\frac{1}{2}\right)^1 z^1 + \left(\frac{1}{2}\right)^2 z^2 + \cdots = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^n = \frac{2}{2-z}$$

$$X(z) = 1 + \left(\frac{1}{2}\right) z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \cdots = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \frac{z}{z-0.5}$$

$$a_n = \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right) = \mathrm{Res}\left(\frac{2}{z^{n+1}(2-z)}, 0\right)$$

$n \geq 0$

causal

$$a_n = \mathrm{Res}\left(\frac{2}{z^{n+1}(2-z)}, 0\right) = \left(\frac{1}{2}\right)^n$$

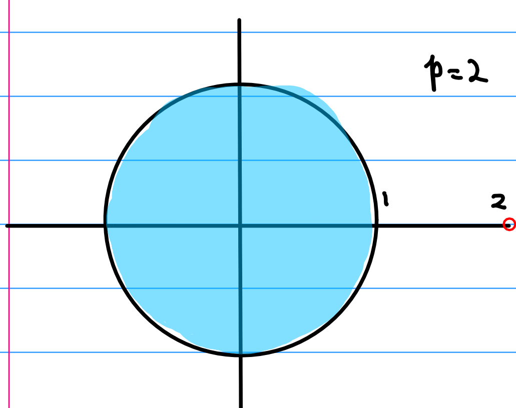
$$a_0 = \mathrm{Res}\left(\frac{2}{z^{1}(2-z)}, 0\right) = 1 \qquad n=0$$

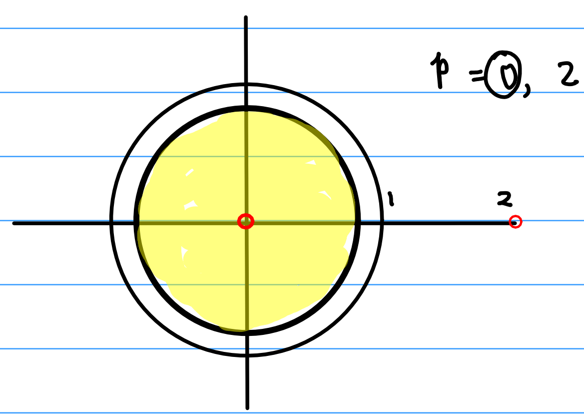
$$a_1 = \mathrm{Res}\left(\frac{2}{z^{2}(2-z)}, 0\right) = \frac{2}{1!}\frac{d}{dz}\frac{1}{2-z}\bigg|_{z=0} = \frac{2}{(2-z)^2} = \left(\frac{1}{2}\right)^1 \qquad n=1$$

$$a_2 = \mathrm{Res}\left(\frac{2}{z^{3}(2-z)}, 0\right) = \frac{2}{2!}\frac{d^2}{dz^2}\frac{1}{2-z}\bigg|_{z=0} = \frac{2}{(2-z)^3} = \left(\frac{1}{2}\right)^2 \qquad n=2$$

$$a_3 = \mathrm{Res}\left(\frac{2}{z^{4}(2-z)}, 0\right) = \frac{2}{3!}\frac{d^3}{dz^3}\frac{1}{2-z}\bigg|_{z=0} = \frac{2}{(2-z)^4} = \left(\frac{1}{2}\right)^3 \qquad n=3$$

$$a_4 = \mathrm{Res}\left(\frac{2}{z^{5}(2-z)}, 0\right) = \frac{2}{4!}\frac{d^4}{dz^4}\frac{1}{2-z}\bigg|_{z=0} = \frac{2}{(2-z)^5} = \left(\frac{1}{2}\right)^4 \qquad n=4$$





the finite number of poles

$$x_n = \sum_k \mathrm{Res}\left(X(z)z^{n-1}, z_k\right)$$

$$X(z) = \frac{z}{z-0.5}$$

$$X(z)z^{n-1} = \frac{z^n}{z-0.5} \qquad \text{pole: } 0.5$$

$$x_n = \mathrm{Res}\left(X(z)z^{n-1}, 0.5\right) = \mathrm{Res}\left(\frac{z^n}{z-0.5}, 0.5\right)$$

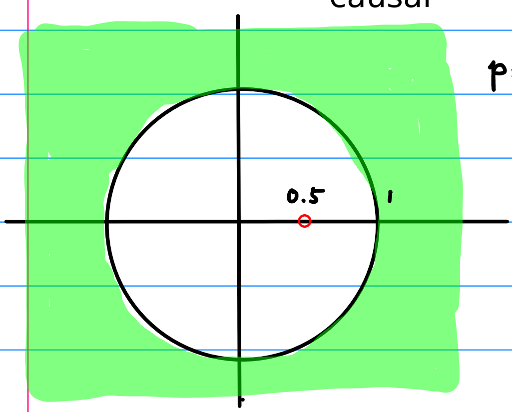
$$x_0 = \mathrm{Res}\left(\frac{z^0}{z-0.5}, 0.5\right) = 1$$

$$x_1 = \mathrm{Res}\left(\frac{z^1}{z-0.5}, 0.5\right) = \left(\frac{1}{2}\right)^1$$

$$x_2 = \mathrm{Res}\left(\frac{z^2}{z-0.5}, 0.5\right) = \left(\frac{1}{2}\right)^2$$

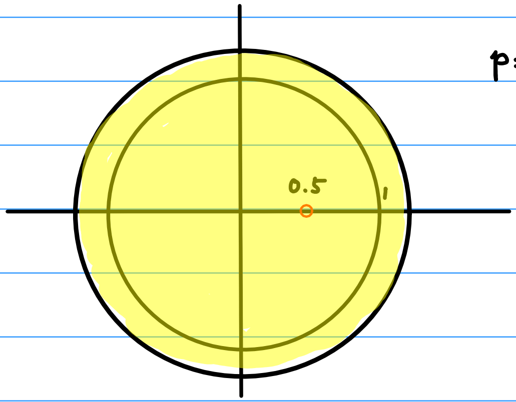
$$x_3 = \mathrm{Res}\left(\frac{z^3}{z-0.5}, 0.5\right) = \left(\frac{1}{2}\right)^3$$

causal



$p = 0.5$

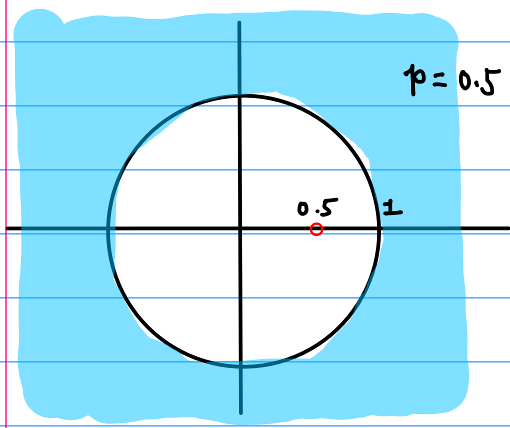
causal



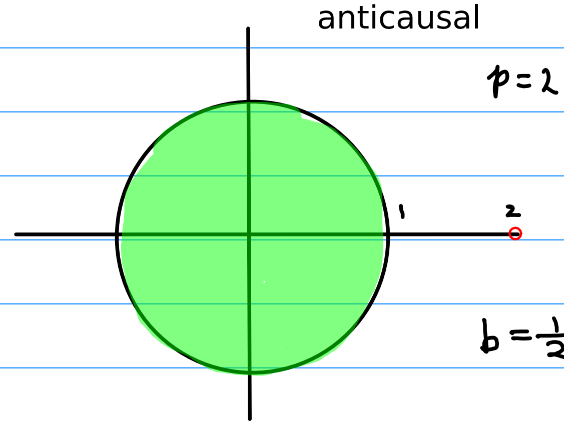
$p = 0.5$

the finite number of poles

Anti-causal



anticausal



$b = \frac{1}{2}$

$\cdots, b^3, b^2, b^1, 1, 0, 0, 0, \cdots$

$$f(z) = X(z^{-1}) = \frac{2}{2 - z^{-1}} = \frac{2z}{2z - 1} = \frac{z}{z - 0.5}$$

$$X(z) = \sum_{n=-\infty}^{0} x_n z^{-n} = \sum_{n=0}^{\infty} (bz)^n = \frac{1}{1 - bz} = \frac{b^{-1}}{b^{-1} - z} = \frac{2}{2 - z}$$

$$a_n = \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, \frac{1}{2}\right) \quad n \leq 0$$

$$= \mathrm{Res}\left(\frac{z}{z^{n+1}(z - 0.5)}, \frac{1}{2}\right)$$

$$= \left(\frac{1}{2}\right)^{-n} \quad (n \leq 0)$$

$$f(z) = 1 + \left(\frac{1}{2}\right)^{-1} z^{-1} + \left(\frac{1}{2}\right)^{-2} z^{-2} + \left(\frac{1}{2}\right)^{-3} z^{-3} + \cdots$$

$$= \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^{-n} z^{n} = \frac{z}{z - 0.5}$$

$$X(z) = 1 + \left(\frac{1}{2}\right) z^{1} + \left(\frac{1}{2}\right)^{2} z^{2} + \left(\frac{1}{2}\right)^{3} z^{3} + \cdots$$

$$= \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^{-n} z^{-n} = \frac{2}{2 - z}$$

$$a_n = \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, \frac{1}{2}\right) = \mathrm{Res}\left(\frac{z}{z^{n+1}(z-0.5)}, \frac{1}{2}\right) \qquad n \le 0$$

anti-causal

* $\boxed{z=0}$ ~~pole~~ for $n \le 0$

$$\frac{z}{z^{n+1}(z-0.5)} = \frac{1}{z^n (z-0.5)} \qquad \frac{1}{z^0}, \frac{1}{z^{-1}}, \frac{1}{z^{-2}}, \cdots = 1, z^1, z^2, \cdots$$

* $\boxed{z = \frac{1}{2}}$ the only pole

$$a_n = \mathrm{Res}\left(\frac{1}{z^n (z-0.5)}, \frac{1}{2}\right) = \left(\frac{1}{2}\right)^{-n} \qquad n \le 0$$

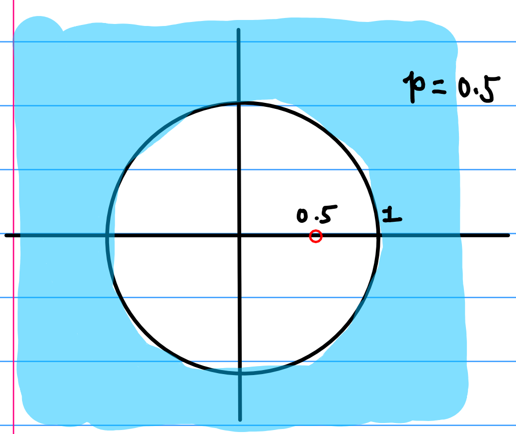
$$a_0 = \mathrm{Res}\left(\frac{1}{z^0 (z-0.5)}, \frac{1}{2}\right) = 1 \qquad n = 0$$

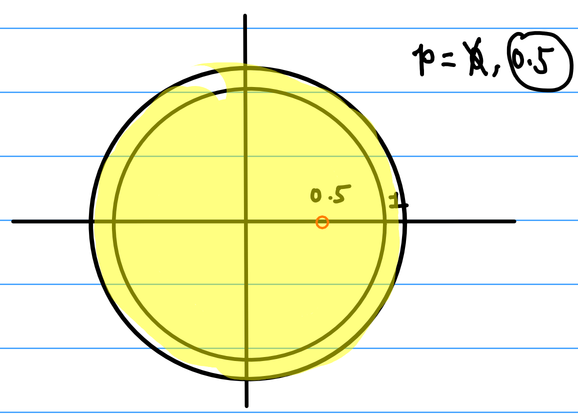
$$a_{-1} = \mathrm{Res}\left(\frac{1}{z^{-1} (z-0.5)}, \frac{1}{2}\right) = \left(\frac{1}{2}\right)^1 \qquad n = -1$$

$$a_{-2} = \mathrm{Res}\left(\frac{1}{z^{-2} (z-0.5)}, \frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 \qquad n = -2$$

$$a_{-3} = \mathrm{Res}\left(\frac{1}{z^{-3} (z-0.5)}, \frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 \qquad n = -3$$

$$a_{-4} = \mathrm{Res}\left(\frac{1}{z^{-4} (z-0.5)}, \frac{1}{2}\right) = \left(\frac{1}{2}\right)^4 \qquad n = -4$$





$$x_n = \sum_k \mathrm{Res}\left(X(z) z^{n-1}, z_k\right)$$

$$X(z) = \frac{2}{2-z}$$

$$X(z) z^{n-1} = \frac{2z^{n-1}}{2-z} \qquad \text{pole: } 2$$

$$x_n = \mathrm{Res}\left(\frac{2z^{n-1}}{2-z}, 0\right) = \mathrm{Res}\left(\frac{2}{z^{1-n}(2-z)}, 0\right) \qquad (n \le 0)$$

$$x_0 = \mathrm{Res}\left(\frac{2}{z^1(2-z)}, 0\right) = 1 \qquad n=0$$

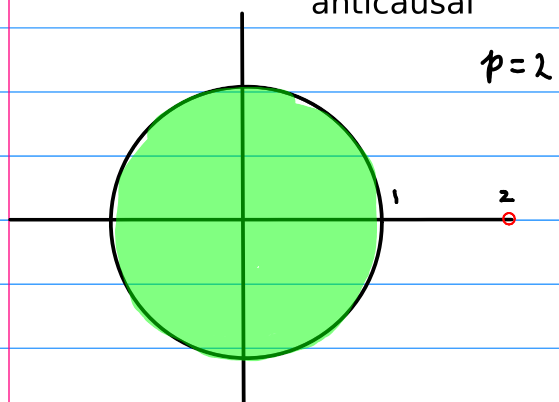
$$x_{-1} = \mathrm{Res}\left(\frac{2}{z^2(2-z)}, 0\right) = \frac{2}{1!}\frac{d}{dz}\frac{1}{2-z}\Big|_{z=0} = \frac{2}{(2-z)^2} = \left(\frac{1}{2}\right)^1 \qquad n=1$$

$$x_{-2} = \mathrm{Res}\left(\frac{2}{z^3(2-z)}, 0\right) = \frac{2}{2!}\frac{d^2}{dz^2}\frac{1}{2-z}\Big|_{z=0} = \frac{2}{(2-z)^3} = \left(\frac{1}{2}\right)^2 \qquad n=2$$

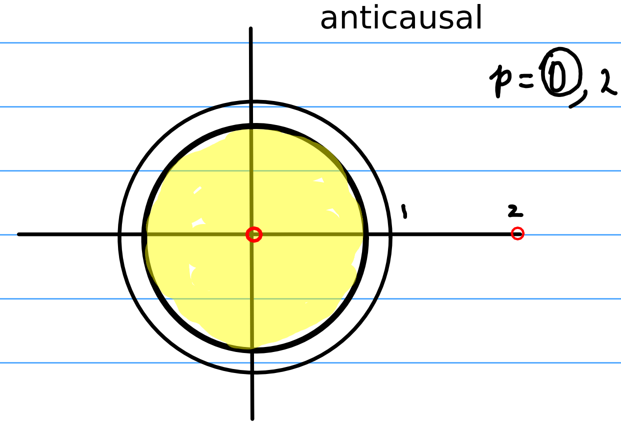
$$x_{-3} = \mathrm{Res}\left(\frac{2}{z^4(2-z)}, 0\right) = \frac{2}{3!}\frac{d^3}{dz^3}\frac{1}{2-z}\Big|_{z=0} = \frac{2}{(2-z)^4} = \left(\frac{1}{2}\right)^3 \qquad n=3$$

$$x_{-4} = \mathrm{Res}\left(\frac{2}{z^4(2-z)}, 0\right) = \frac{2}{4!}\frac{d^4}{dz^4}\frac{1}{2-z}\Big|_{z=0} = \frac{2}{(2-z)^5} = \left(\frac{1}{2}\right)^4 \qquad n=4$$

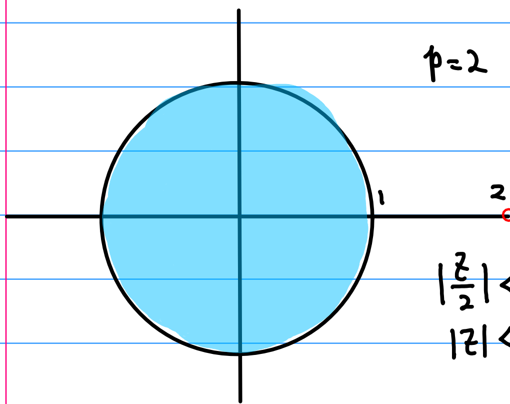
anticausal



anticausal



Summary



$p=2$

$\left|\frac{z}{2}\right|<1$

$|z|<2$

$$X(z^{-1}) = \frac{z^{-1}}{z^{-1}-0.5} = \frac{1}{1-(z/2)}$$

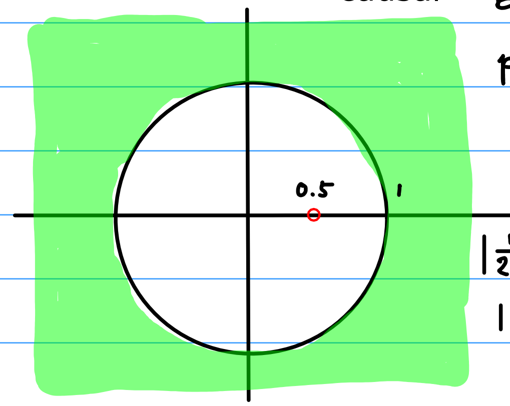
$$f(z) = \frac{2}{2-z} = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^n$$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$= p^{-n} \quad (n \geq 0) \quad p=2$$

causal $b=\frac{1}{2}$

$p=0.5$



$\left|\frac{1}{2z}\right|<1$

$|z|>\frac{1}{2}$

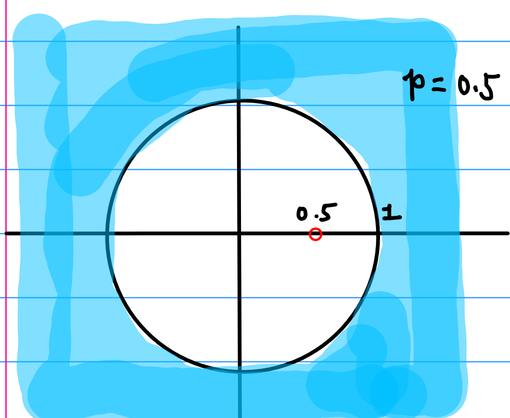
$\cdots 0, 0, 0, 1, b, b^2, b^3, \cdots$

$$X(z) = \frac{z}{z-0.5} = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n}$$

$$x_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$= p^n \quad (n \geq 0) \quad p=\frac{1}{2}$$

$p=0.5$



$\left|\frac{1}{2z}\right|<1$

$|z|>\frac{1}{2}$

$$X(z^{-1}) = \frac{2}{2-z^{-1}}$$

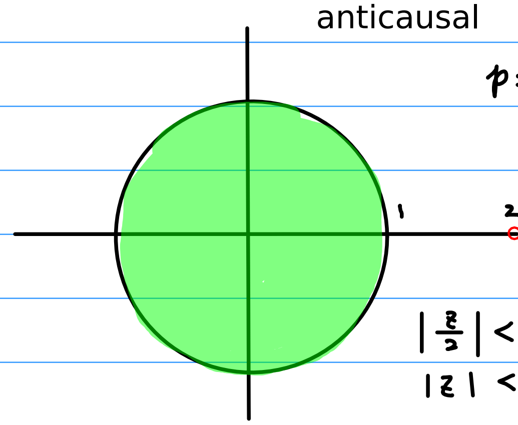
$$f(z) = \frac{z}{z-0.5} = \sum_{n=-\infty}^{0}\left(\frac{1}{2}\right)^{-n} z^n$$

$$a_n = \left(\frac{1}{2}\right)^{-n} \quad (n \leq 0)$$

$$= p^{n} \quad (n \leq 0) \quad p=\frac{1}{2}$$

anticausal

$p=2$



$\left|\frac{z}{2}\right|<1$

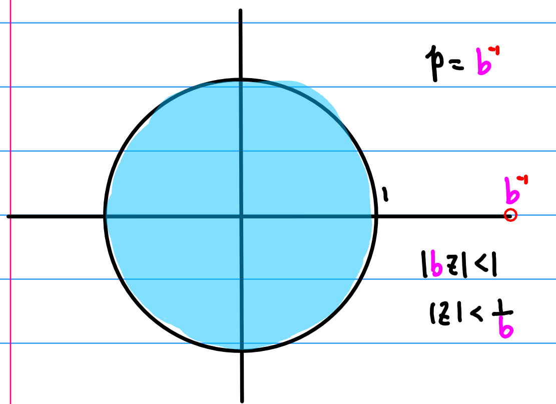
$|z|<2$

$\cdots, b^3, b^2, b^1, 1, 0, 0, 0, \cdots$

$$X(z) = \frac{2}{2-z} = \sum_{n=-\infty}^{0}\left(\frac{1}{2}\right)^{-n} z^{-n}$$

$$x_n = \left(\frac{1}{2}\right)^{-n} \quad (n \leq 0)$$

$$= p^n \quad (n \leq 0) \quad p=2$$



$$X(z^{-1}) = \frac{z^{-1}}{z^{-1} - b} = \frac{1}{1 - bz}$$

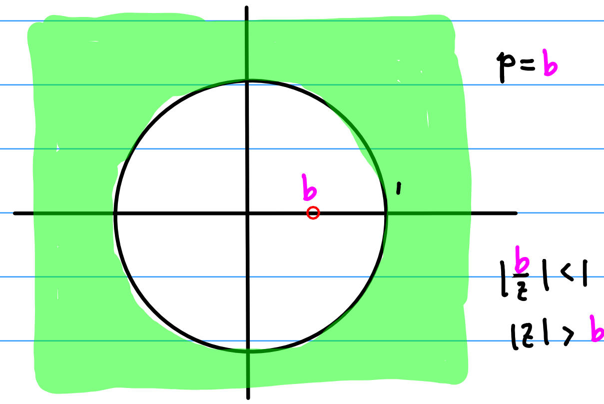
$$= (bz)^0 + (bz)^1 + (bz)^2 + \cdots$$

$$f(z) = \frac{b^{-1}}{b^{-1} - z} = \sum_{n=0}^{\infty} b^n z^n$$

$$a_n = b^n \quad (n \geq 0)$$

$$p = b^{-1} \quad |z| < p \quad |pz| < 1$$

causal



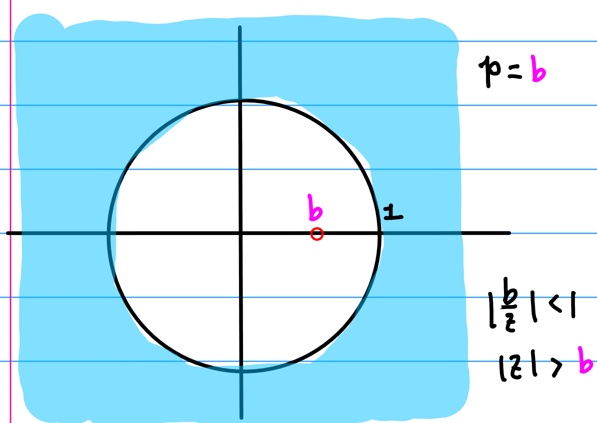
$$\cdots 0, 0, 0, 1, b, b^2, b^3, \cdots$$

$$\left(\frac{b}{z}\right)^0 + \left(\frac{b}{z}\right)^1 + \left(\frac{b}{z}\right)^2 + \cdots$$

$$X(z) = \frac{1}{1 - \frac{b}{z}} = \frac{z}{z - b}$$

$$x_n = b^n \quad (n \geq 0)$$

$$p = b \quad |z| > p \quad \left|\frac{p}{z}\right| < 1$$



$$X(z^{-1}) = \frac{b^{-1}}{b^{-1} - z^{-1}} = \frac{1}{1 - \frac{b}{z}}$$

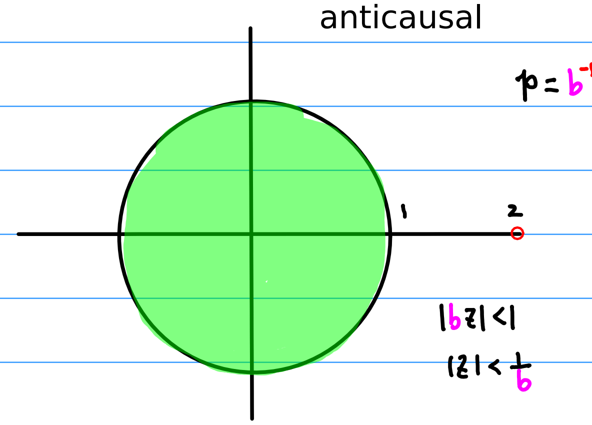
$$= \left(\frac{b}{z}\right)^0 + \left(\frac{b}{z}\right)^1 + \left(\frac{b}{z}\right)^2 + \cdots$$

$$f(z) = \frac{z}{z - b} = \sum_{n=-\infty}^{0} b^{-n} z^n$$

$$a_n = b^{-n} \quad (n \leq 0)$$

$$p = b$$

anticausal



$$\cdots, b^3, b^2, b^1, 1, 0, 0, 0, \cdots$$

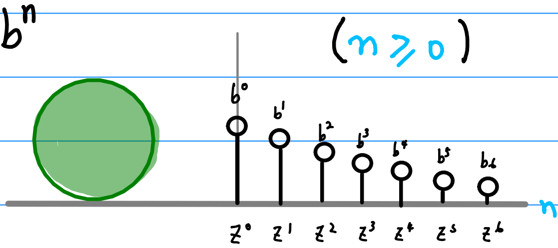
$$(bz)^0 + (bz)^1 + (bz)^2 + \cdots$$

$$X(z) = \frac{1}{1 - bz} = \frac{b^{-1}}{b^{-1} - z}$$

$$x_n = b^{-n} \quad (n \leq 0)$$

$$p = b^{-1} \quad |z| < p \quad \left|\frac{z}{p}\right| < 1$$

b^n $(n \geqslant 0)$

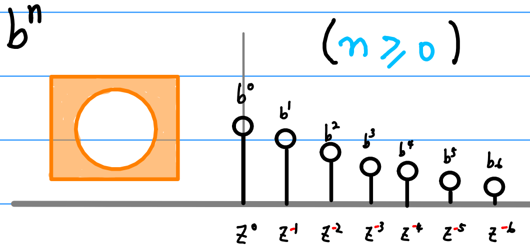


$$X(z^{-1}) = \frac{z^{-1}}{z^{-1} - 0.5} \qquad |z| < 2$$

$$f(z) = \frac{2}{2-z} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^n$$

$$a_n = \left(\frac{1}{2}\right)^n = p^{-n} \qquad \boxed{p = 2}$$

b^n $(n \geqslant 0)$

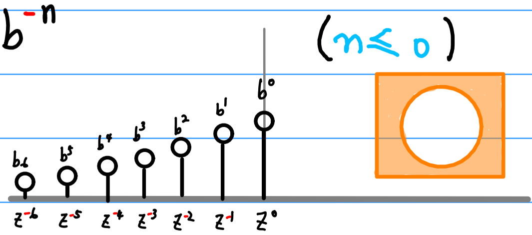


$$|z| > \frac{1}{2}$$

$$X(z) = \frac{z}{z - 0.5} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

$$x_n = \left(\frac{1}{2}\right)^n = p^n \qquad \boxed{p = \frac{1}{2}}$$

b^{-n} $(n \leqslant 0)$

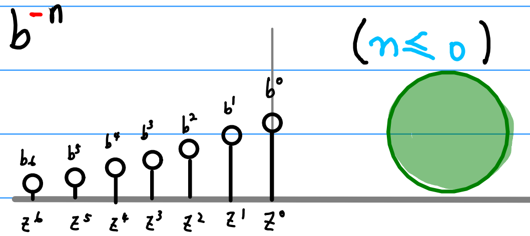


$$X(z^{-1}) = \frac{2}{2 - z^{-1}} \qquad |z| > \frac{1}{2}$$

$$f(z) = \frac{z}{z - 0.5} = \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^{-n} z^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

$$a_n = \left(\frac{1}{2}\right)^{-n} = p^{-n} \qquad \boxed{p = \frac{1}{2}}$$

b^{-n} $(n \leqslant 0)$



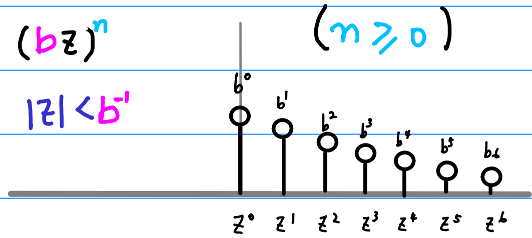
$$|z| < 2$$

$$X(z) = \frac{2}{2 - z} = \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^{-n} z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^n$$

$$x_n = \left(\frac{1}{2}\right)^{-n} = p^n \qquad \boxed{p = 2}$$

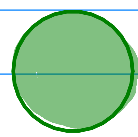
$(bz)^n$ $\quad (n \geqslant 0)$

$|z| < b^{-1}$



$$f(z) = \frac{1}{1 - bz} = \frac{b^{-1}}{b^{-1} - z}$$

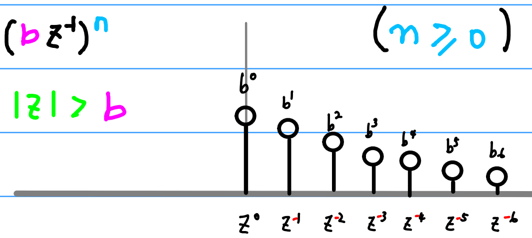
$$a_n = b^n$$
$$= p^{-n}$$



$\boxed{p = b^{-1}}$

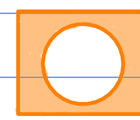
$(bz^{-1})^n$ $\quad (n \geqslant 0)$

$|z| > b$



$$X(z) = \frac{1}{1 - b/z} = \frac{z}{z - b}$$

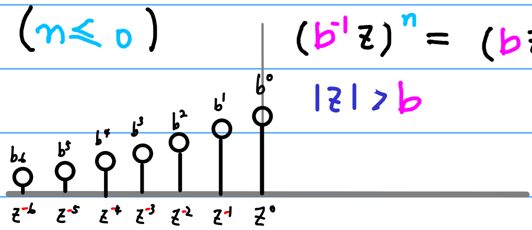
$$x_n = b^n$$
$$= p^n$$



$\boxed{p = b}$

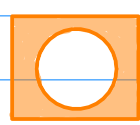
$(n \leqslant 0)$ $\quad (b^{-1}z)^n = (bz^{-1})^{-n}$

$|z| > b$



$$f(z) = \frac{1}{1 - (bz^{-1})} = \frac{z}{z - b}$$

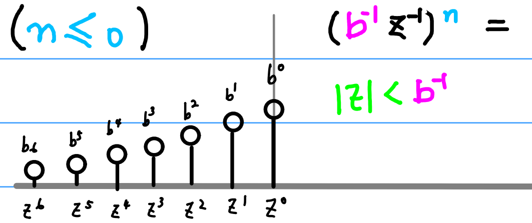
$$a_n = b^{-n}$$
$$= p^{-n}$$



$\boxed{p = b}$

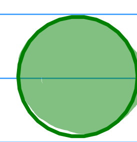
$(n \leqslant 0)$ $\quad (b^{-1}z^{-1})^n = (bz)^{-n}$

$|z| < b^{-1}$



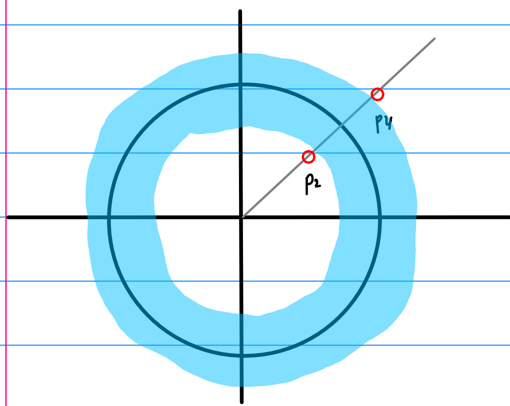
$$X(z) = \frac{1}{1 - (bz)} = \frac{b^{-1}}{b^{-1} - z}$$

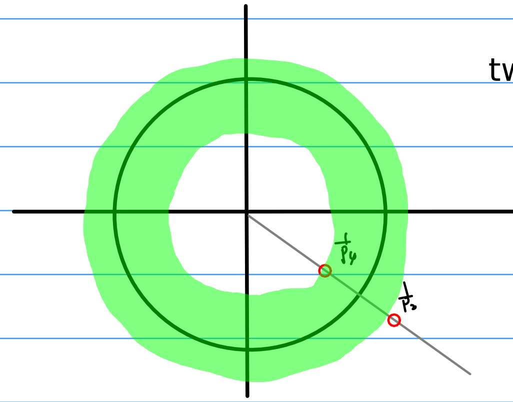
$$x_n = b^{-n}$$
$$= p^n$$



$\boxed{p = b^{-1}}$

Two-sided





two-sided

$$\frac{1}{2} < |z| < 2 \Rightarrow \left|\frac{1}{2z}\right| < 1, \ \left|\frac{z}{2}\right| < 1$$

$$\frac{1}{1-\frac{1}{2z}} + \frac{1}{1-\frac{z}{2}} = \frac{2z}{2z-1} + \frac{2}{2-z}$$

$$= \frac{z}{z-0.5} - \frac{2}{z-2}$$

$$\boxed{\frac{1}{1-\frac{1}{2z}} + \frac{1}{1-\frac{z}{2}} - 1 = \frac{z}{z-0.5} - \frac{2}{z-2} \quad -1}$$

$$\frac{1}{1-\frac{1}{2z}} = \left(\frac{1}{2z}\right)^0 + \left(\frac{1}{2z}\right)^1 + \left(\frac{1}{2z}\right)^2 + \left(\frac{1}{2z}\right)^3 + \cdots = \frac{2z}{2z-1} = \frac{z}{z-0.5}$$

$$\left(\frac{1}{2z}\right)^1 + \left(\frac{1}{2z}\right)^2 + \left(\frac{1}{2z}\right)^3 + \cdots = \frac{z}{z-0.5} - 1 = \frac{0.5}{z-0.5}$$

$$\frac{1}{1-\frac{z}{2}} = \left(\frac{z}{2}\right)^0 + \left(\frac{z}{2}\right)^1 + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots = \frac{2}{2-z}$$

$$f(z) = \frac{0.5}{z-0.5} - \frac{2}{z-2} = \frac{\frac{1}{2}z \cancel{-1} - 2z \cancel{+1}}{(z-0.5)(z-2)} = \frac{-\frac{3}{2}z}{(z-0.5)(z-2)}$$

$$f(z) = \frac{0.5}{z-0.5} - \frac{2}{z-2} = \frac{\frac{1}{2}z \cancel{-1} - 2z \cancel{+1}}{(z-0.5)(z-2)} = \frac{-\frac{3}{2}z}{(z-0.5)(z-2)}$$

$$f(z^{-1}) = \frac{0.5}{z^{-1}-0.5} - \frac{2}{z^{-1}-2} = \frac{\frac{1}{2}z \cancel{-1} - 2z \cancel{+1}}{(z-0.5)(z-2)} = \frac{-\frac{3}{2}z}{(z-0.5)(z-2)}$$

$$= \frac{0.5z}{1-0.5z} - \frac{2z}{1-2z}$$

$$= \frac{z}{2-z} - \frac{z}{0.5-z}$$

$$= \frac{-z}{z-2} + \frac{z}{z-0.5} = \frac{-z^2+0.5z+z^2-2z}{(z-0.5)(z-2)} = \frac{-\frac{3}{2}z}{(z-0.5)(z-2)}$$

$$f(z) = f(z^{-1}) = X(z)$$

$$\left(\frac{1}{2z}\right)^1+\left(\frac{1}{2z}\right)^2+\left(\frac{1}{2z}\right)^3+\cdots = \frac{0.5}{z-0.5} = \sum_{n=1}^{\infty}\left(\frac{1}{2z}\right)^n$$

$$\left(\frac{z}{2}\right)^0+\left(\frac{z}{2}\right)^1+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots = \frac{2}{2-z} = \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$\cdots+\left(\frac{z}{2}\right)^3+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^1+\left(\frac{z}{2}\right)^0+\left(\frac{1}{2z}\right)^1+\left(\frac{1}{2z}\right)^2+\left(\frac{1}{2z}\right)^3+\cdots = \boxed{\frac{2}{2-z}}+\boxed{\frac{0.5}{z-0.5}}$$

$$= \frac{0.5}{z-0.5}+\frac{2}{2-z}$$

$$= \frac{0.5}{z-0.5}-\frac{2}{z-2}$$

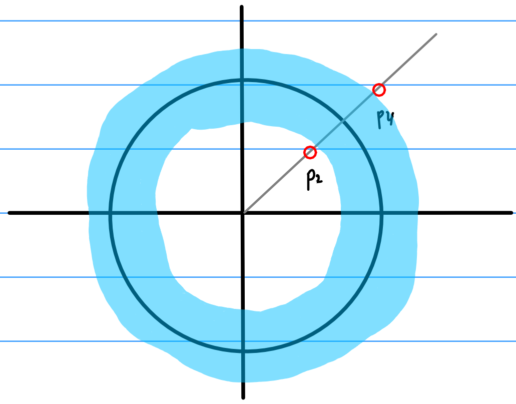
$$= \frac{\frac{1}{2}z \cancel{-1} - 2z \cancel{+1}}{(z-0.5)(z-2)}$$

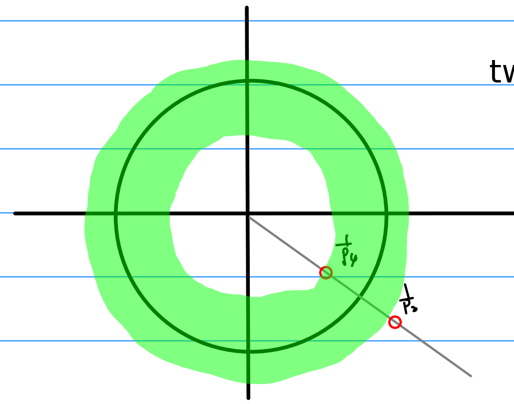
$$= \frac{-\frac{3}{2}z}{(z-0.5)(z-2)}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{2z}\right)^n + \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^n$$

two-sided

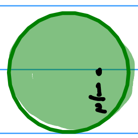




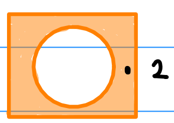
$$\frac{1}{2} < |z| < 2 \qquad\qquad \frac{1}{2} < |z| < 2$$

$$\left|\frac{1}{2z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1 \qquad\qquad \left|\frac{1}{2z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$$

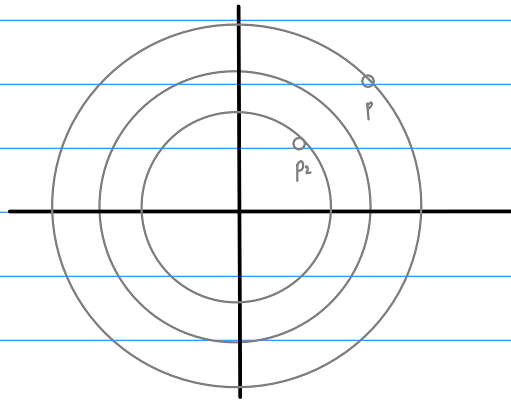
$$\frac{z}{z-2} + \frac{0.5}{0.5-z} - 1$$

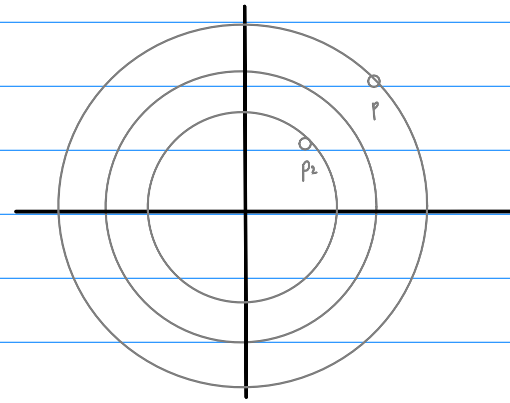


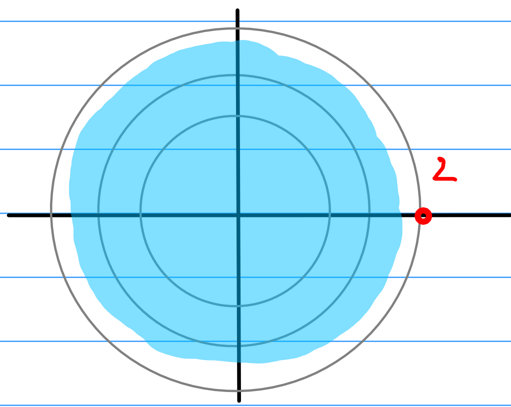
$$\frac{b^{-1}}{b^{-1}-z} \qquad \frac{2}{2-z}$$



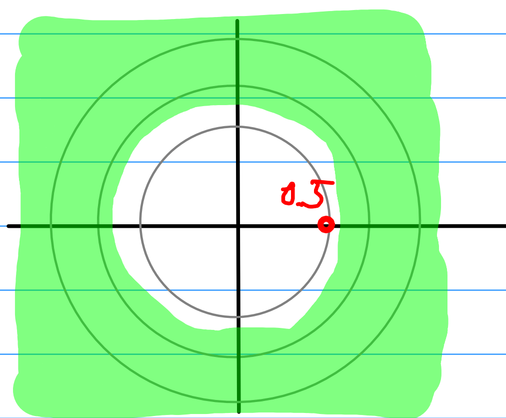
$$\frac{z}{z-b} \qquad \frac{z}{z-0.5}$$



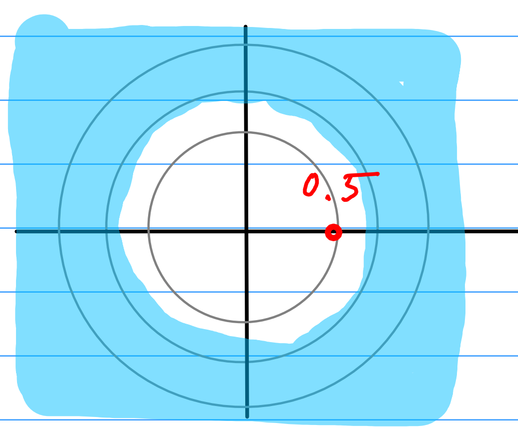




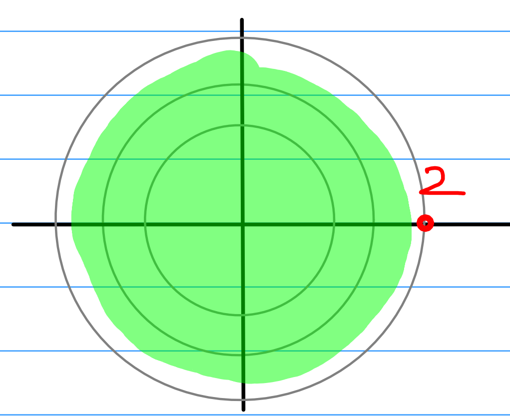
$\left|\frac{z}{2}\right| < 1$ $\qquad \frac{2}{2-z}$



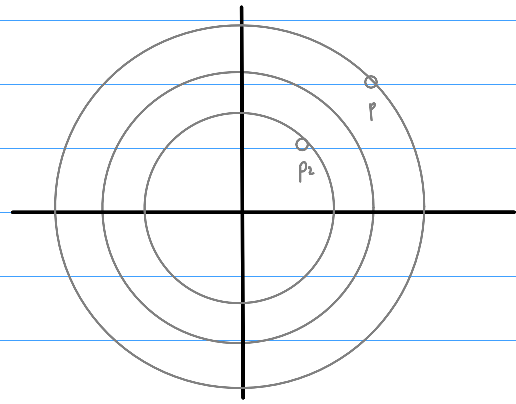
$\left|\frac{0.5}{z}\right| < 1$ $\qquad \frac{z}{z-0.5}$

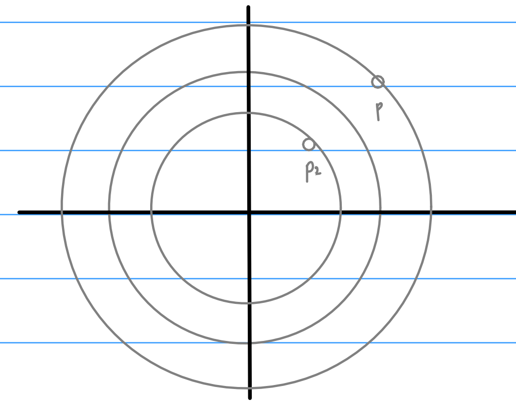


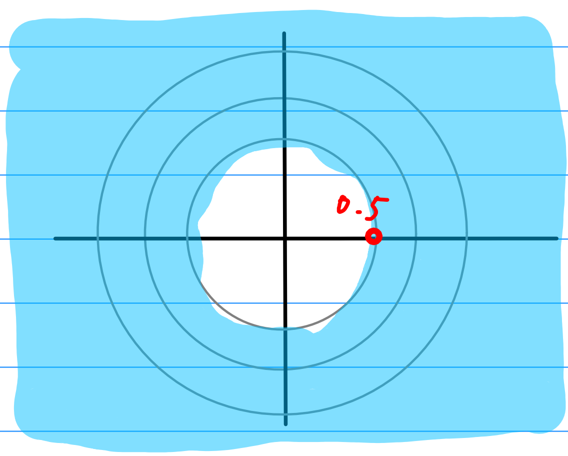
$\left|\frac{0.5}{z}\right| < 1$ $\qquad \frac{z}{z-0.5}$

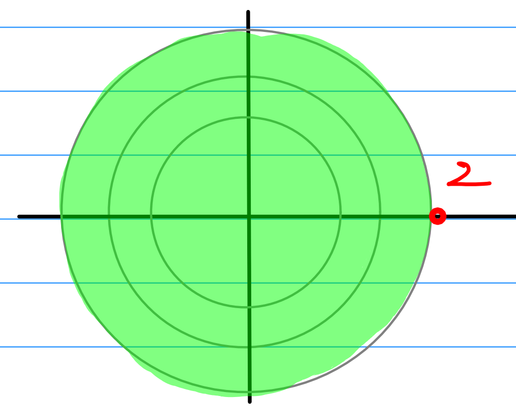


$\left|\frac{z}{2}\right| < 1$ $\qquad \frac{2}{2-z}$


p
p_2


p
p_2


0.5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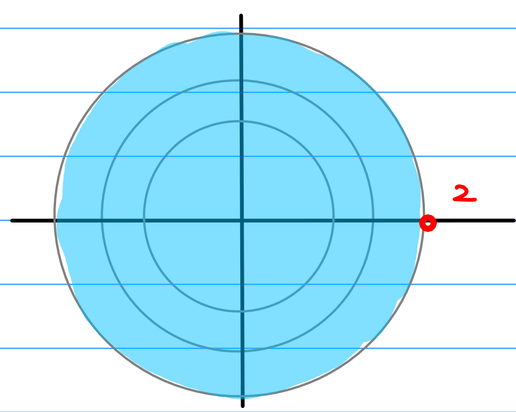
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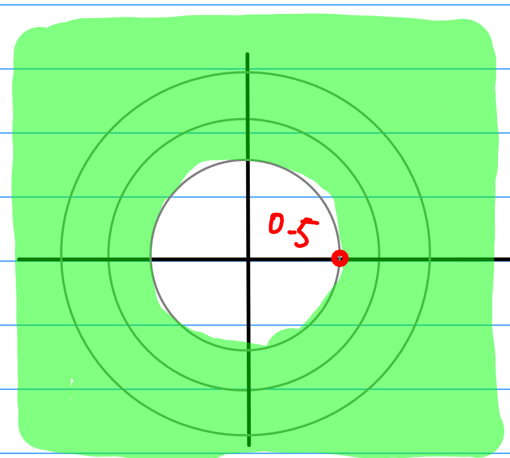

0.5

$$\cdots\ 0,\ 0,\ 0,\ 1,\ \left(\tfrac{1}{2}\right),\ \left(\tfrac{1}{2}\right)^2,\ \left(\tfrac{1}{2}\right)^3,\ \cdots$$

$$X(z) = \frac{1}{1-\frac{0.5}{z}} = \frac{z}{z-0.5}$$

$$\left|\frac{0.5}{z}\right|<1 \qquad |z|>0.5$$

$$\cdots\ 0,\ 0,\ 0,\ \left(\tfrac{1}{2}\right)^1,\ \left(\tfrac{1}{2}\right)^2,\ \left(\tfrac{}{2}\right)^3,\ \left(\tfrac{1}{2}\right)^4,\ \cdots$$

$$X(z) = \frac{0.5}{1-\frac{0.5}{z}} = \frac{0.5z}{z-0.5}$$

$$\left|\frac{0.5}{z}\right|<1 \qquad |z|>0.5$$

$$\cdots\ 0,\ 0,\ 0,\ 0,\ \left(\tfrac{1}{2}\right),\ \left(\tfrac{1}{2}\right)^2,\ \left(\tfrac{1}{2}\right)^3,\ \cdots$$

$$X(z) = \frac{0.5}{1-\frac{0.5}{z}}\cdot z^{-1} = \frac{0.5}{z-0.5}$$

$$\left|\frac{0.5}{z}\right|<1 \qquad |z|>0.5$$

$$\cdots,\ \left(\tfrac{1}{2}\right)^3,\ \left(\tfrac{1}{2}\right)^2,\ \left(\tfrac{1}{2}\right),\ 1,\ 0,\ 0,\ 0,\ \cdots$$

$$X(z) = \frac{1}{1-\frac{z}{2}} = \frac{2}{2-z}$$

$$\left|\frac{z}{2}\right|<1 \qquad |z|<2$$

$$b^3,\ b^2,\ b^1,\ 1,\ a,\ a^2,\ a^3$$

$$\cdots\ 0,\ 0,\ 0,\ 0,\ a,\ a^2,\ a^3,\ \cdots$$

$$\cdots,\ b^3,\ b^2,\ b^1,\ 1,\ 0,\ 0,\ 0,\ \cdots$$

$$X(z) = \frac{0.5}{z-0.5} + \frac{2}{2-z}$$

$$= \frac{0.5}{z-0.5} - \frac{2}{z-2}$$

$$= \frac{\frac{1}{2}z - \cancel{1} - 2z + \cancel{1}}{(z-0.5)(z-2)}$$

$$= \frac{-\frac{3}{2}z}{(z-0.5)(z-2)}$$

$$\cdots,\ b^3,\ b^2,\ b^1,\ 1,\ 0,\ 0,\ 0,\ \cdots$$

$$\cdots\ 0,\ 0,\ 0,\ 1,\ a,\ a^2,\ a^3,\ \cdots$$

$$\frac{z}{z-0.5} + \frac{2}{2-z} - 1$$

$$= \frac{z^2-2z-2z+1}{(z-0.5)(z-2)} - 1$$

$$= \frac{\cancel{z^2}-4z+\cancel{1}-\cancel{z^2}+2.5z-\cancel{1}}{(z-0.5)(z-2)}$$

$$= \frac{-1.5z}{(z-0.5)(z-2)}$$

$$f(z) = \frac{-\frac{3}{2} z}{(z-0.5)(z-2)} \qquad X(z) = \frac{-\frac{3}{2} z}{(z-0.5)(z-2)}$$

$$a_n = \begin{cases} \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, \frac{1}{2}\right) + \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right) & (n \geq 0) \\ \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, \frac{1}{2}\right) & (n < 0) \end{cases}$$

$$a_n = \begin{cases} \mathrm{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^{n}}, \frac{1}{2}\right) + \mathrm{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^{n}}, 0\right) & (n \geq 0) \\ \mathrm{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^{n}}, \frac{1}{2}\right) & (n < 0) \end{cases}$$

$n=-3$ $\quad \mathrm{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^{-3}}, \frac{1}{2}\right) = \left(\frac{1}{2}\right)^{3} = \left(\frac{1}{2}\right)^{-(-3)}$

$n=-2$ $\quad \mathrm{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^{-2}}, \frac{1}{2}\right) = \left(\frac{1}{2}\right)^{2} = \left(\frac{1}{2}\right)^{-(-2)}$

$n=-1$ $\quad \mathrm{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^{-1}}, \frac{1}{2}\right) = \left(\frac{1}{2}\right)^{1} = \left(\frac{1}{2}\right)^{-(-1)}$

$n=0$ $\quad \mathrm{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^{0}}, \frac{1}{2}\right) = \left(\frac{1}{2}\right)^{0} = \left(\frac{1}{2}\right)^{0}$

$n=1$ $\quad \mathrm{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^{1}}, \frac{1}{2}\right) + \mathrm{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^{1}}, 0\right) = \left(\frac{1}{2}\right)^{-1} + \left(\frac{-3}{2}\right)$

$= \left(\frac{1}{2}\right)$

$n=2$ $\quad \mathrm{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^{2}}, \frac{1}{2}\right) + \mathrm{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^{2}}, 0\right) = \left(\frac{1}{2}\right)^{-2} - \frac{15}{4}$

$= \left(\frac{1}{2}\right)^{2}$

$n=3$ $\quad \mathrm{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^{3}}, \frac{1}{2}\right) + \mathrm{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^{3}}, 0\right) = \left(\frac{1}{2}\right)^{-3} - \frac{63}{8}$

$= \left(\frac{1}{2}\right)^{3}$

$$\mathrm{Res}(G(z), z_0) \begin{cases} \boxed{\lim_{z \to z_0} (z-z_0)\, G(z) = a_{-1}} & \text{simple pole } z_0 \\ \boxed{\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n\, G(z) = a_{-1}} & \text{n-th order pole } z_0 \end{cases}$$

$$\mathrm{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^n}, 0 \right) ?$$

$$\frac{-\frac{3}{2}}{(z-0.5)(z-2)} = \frac{A}{(z-0.5)} + \frac{B}{(z-2)}$$

$$A = \left. \frac{-\frac{3}{2}}{z-2} \right|_{z=0.5} = 1 \qquad -\frac{3}{2}$$

$$B = \left. \frac{-\frac{3}{2}}{z-0.5} \right|_{z=2} = \frac{-\frac{3}{2}}{\frac{3}{2}} = -1$$

$$\frac{-\frac{3}{2}}{(z-0.5)(z-2)} = \frac{1}{(z-0.5)} - \frac{1}{(z-2)}$$

$$\frac{d}{dz} \frac{-\frac{3}{2}}{(z-0.5)(z-2)} = \frac{-1}{(z-0.5)^2} + \frac{1}{(z-2)^2}$$

$$\frac{d^2}{dz^2} \frac{-\frac{3}{2}}{(z-0.5)(z-2)} = \frac{2}{(z-0.5)^3} - \frac{2}{(z-2)^3}$$

$$\frac{d^3}{dz^3} \frac{-\frac{3}{2}}{(z-0.5)(z-2)} = \frac{-6}{(z-0.5)^4} + \frac{6}{(z-2)^4}$$

$$\begin{cases} \frac{d}{dz} \frac{-\frac{3}{2}}{(z-0.5)(z-2)} = \frac{-1}{(z-0.5)^2} + \frac{1}{(z-2)^2} \\ \frac{1}{2!} \frac{d^2}{dz^2} \frac{-\frac{3}{2}}{(z-0.5)(z-2)} = \frac{1}{(z-0.5)^3} - \frac{1}{(z-2)^3} \\ \frac{1}{3!} \frac{d^3}{dz^3} \frac{-\frac{3}{2}}{(z-0.5)(z-2)} = \frac{-1}{(z-0.5)^4} + \frac{1}{(z-2)^4} \end{cases}$$

$$\operatorname{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^{1}},\ 0\right) = \left.\frac{-\frac{3}{2}}{(z-0.5)(z-2)}\right|_{z=0} = \left[\frac{1}{(z-0.5)} - \frac{1}{(z-2)}\right]_{z=0}$$

$$= -2 + \frac{1}{2} = -\frac{3}{2}$$

$$\operatorname{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^{2}},\ 0\right) = \left.\frac{d}{dz}\frac{-\frac{3}{2}}{(z-0.5)(z-2)}\right|_{z=0} = \left[\frac{-1}{(z-0.5)^2} + \frac{1}{(z-2)^2}\right]_{z=0}$$

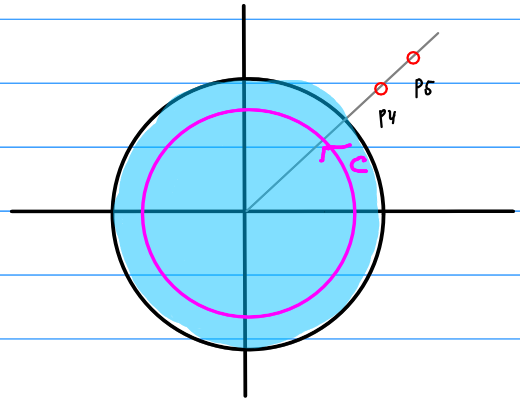
$$= -4 + \frac{1}{4} = -\frac{15}{4}$$

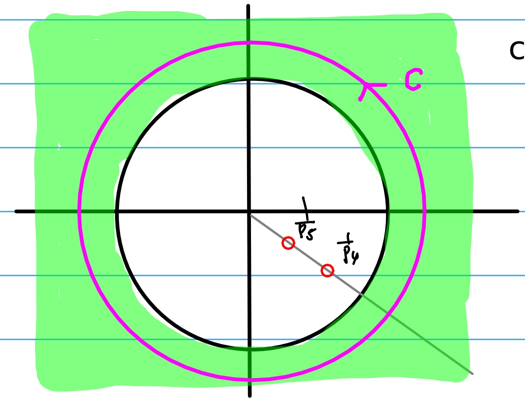
$$\operatorname{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^{3}},\ 0\right) = \left.\frac{1}{2!}\frac{d^2}{dz^2}\frac{-\frac{3}{2}}{(z-0.5)(z-2)}\right|_{z=0} = \left[\frac{1}{(z-0.5)^3} - \frac{1}{(z-2)^3}\right]_{z=0}$$

$$= \left(-8 + \frac{1}{8}\right) = -\frac{63}{8}$$

$$a_n = \begin{cases} \operatorname{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^{n}},\ \frac{1}{2}\right) \underset{-n}{+} \operatorname{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^{n}},\ 0\right) = \left(\frac{1}{2}\right)^{n} & (n \geq 0) \\ \operatorname{Res}\left(\frac{-\frac{3}{2}}{(z-0.5)(z-2)z^{n}},\ \frac{1}{2}\right) = \left(\frac{1}{2}\right)^{-n} & (n < 0) \end{cases}$$

$$a_n = \begin{cases} \left(\frac{1}{2}\right)^{n} & (n \geq 0) \\ \left(\frac{1}{2}\right) & (n < 0) \end{cases}$$





causal

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} z^n$$

$$X(z) = \sum_{k=0}^{\infty} x_k z^{-k}$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} dz = \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

$$x_n = \frac{1}{2\pi i} \oint_C X(z) z^{n-1} dz = \sum_k \mathrm{Res}\left(X(z) z^{n-1}, z_k\right)$$

Poles z_k

$n \geq 0$ $z_1, z_2, z_3,$ o

$n < 0$ z_1, z_2, z_3

Poles z_k

$n > 0$ z_1, z_2, z_3

$n \leq 0$ $z_1, z_2, z_3,$ o

Z - transform

$\boxed{z_m = 0}$

$$\boxed{\begin{aligned} x[n] &= \frac{1}{2\pi i}\oint_C f(z)\, z^{n-1}\, dz \\ &= \sum_k \mathrm{Res}\left(f(z)\, z^{n-1},\, z_k\right) \end{aligned}}$$

$n > 0$ $\quad z_k$: poles of $f(z)$

$n = 0$ $\quad z_k$: poles of $f(z)$ + $z = 0$

$z^{n-1} = z^{-1} = \frac{1}{z}$

$x[n]$ includes $u[n]$ $\rightarrow$ $X[z]$ contains z on its numerator

Also, think about modified partial fraction $\frac{X[z]}{z}$

Laurent Expansion

expansion at z_m

$$\boxed{\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_m)^{n+1}}\, dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}},\, z_k\right) \end{aligned}}$$

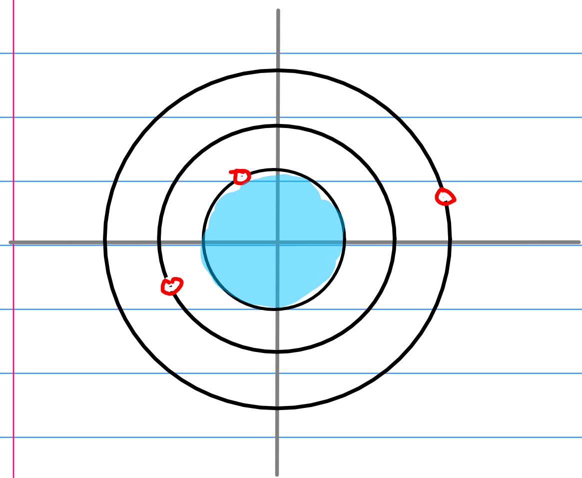
$\boxed{z_m = 0}$

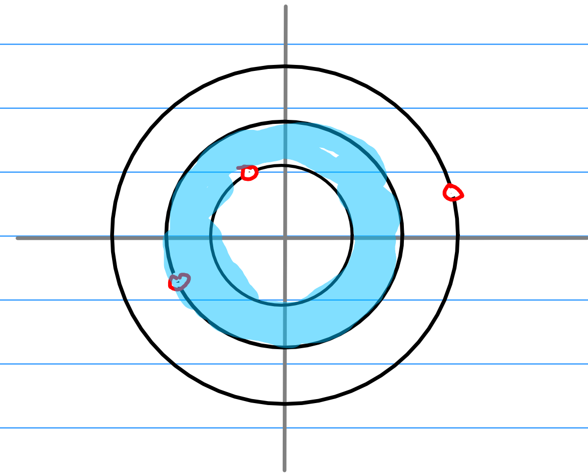
$$\boxed{\begin{aligned} a_n^{\{0\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\, dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}},\, z_k\right) \end{aligned}}$$

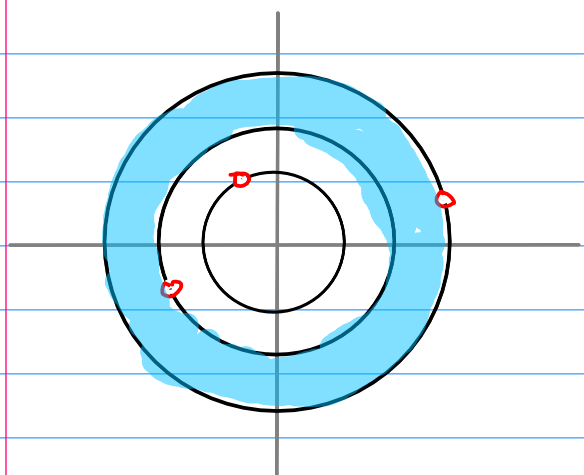
$$\begin{aligned} a_{-n}^{\{0\}} &= \frac{1}{2\pi i}\oint_C f(z)\, z^{n-1}\, dz \\ &= \sum_k \mathrm{Res}\left(f(z)\, z^{n-1},\, z_k\right) \end{aligned}$$

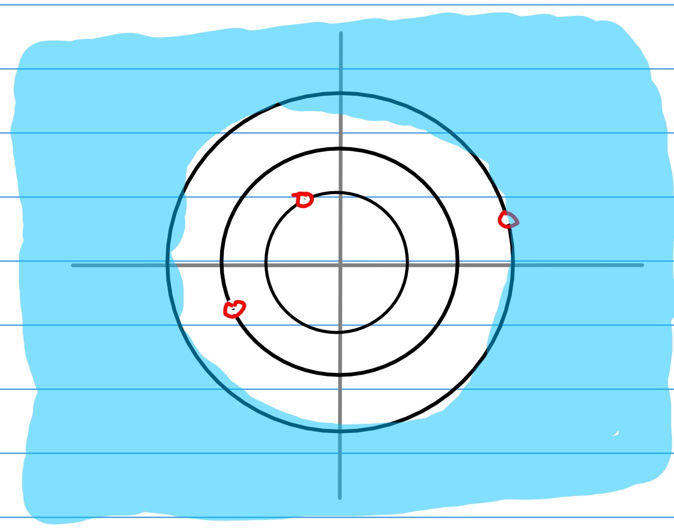
$$\begin{aligned} a_{-n}^{\{0\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{-n+1}}\, dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{-n+1}},\, z_k\right) \end{aligned}$$

Different D, Different Laurent Series



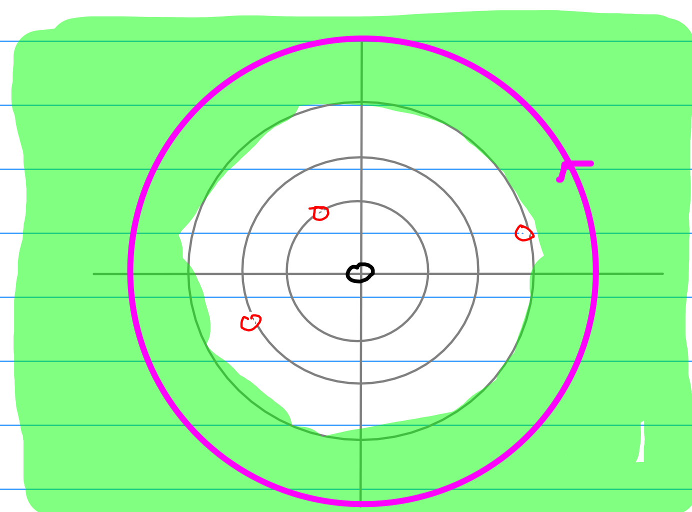






$$x[n] = \frac{1}{2\pi i}\int_C X(z)\, z^{n-1}\, dz$$

$$= \sum_k \mathrm{Res}\left(X(z) z^{n-1}, z_k\right)$$



z-transform

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

Complex Variables and Ap
Brown & Churchill

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

$D_1 : |z| < 1$

$D_2 : 1 < |z| < 2$

$D_3 : 2 < |z|$

① D_1 $\quad |z| < 1, \quad \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \left(2^{-n-1} - 1\right) z^n \quad |z| < 1$$

② D_2 $\quad 1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\left(\frac{1}{z}\right)} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

③ D_3 $\quad 2 < |z| \quad \left|\frac{2}{z}\right| < 1 \quad \left|\frac{1}{z}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)}$$

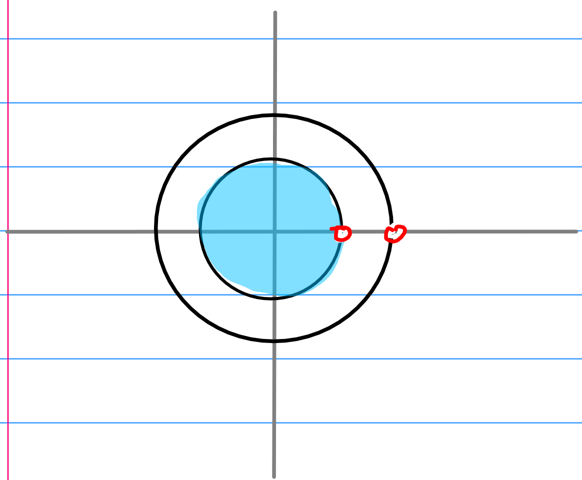
$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

① D_1 $\quad |z|<1, \quad \left|\frac{z}{2}\right|<1$

$$\frac{f(z)}{z^{n+1}} = \frac{-1}{(z-1)(z-2)z^{n+1}}$$

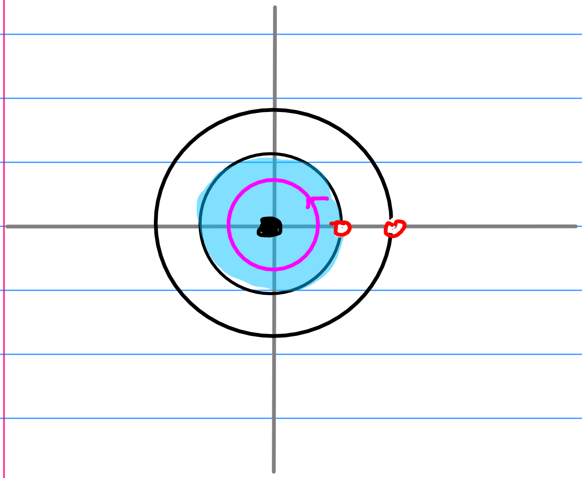


$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n \quad |z|<1$$

$$a_n = \frac{f(z)}{z^{n+1}} = \frac{1}{(z-1)(z-2)z^{n+1}} \qquad \frac{1}{z-1} - \frac{1}{z-2}$$

$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$



$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_n)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$n \geq 0$ then the pole $z=0$

$$\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \quad (\text{order } n)$$

$$\frac{d}{dz}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)\left((z-1)^{-2} - (z-2)^{-2}\right)$$
$$\frac{d^2}{dz^2}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)(-2)\left((z-1)^{-3} - (z-2)^{-3}\right)$$
$$\frac{d^3}{dz^3}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)(-2)(-3)\left((z-1)^{-4} - (z-2)^{-4}\right)$$

$$\frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n n! \left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

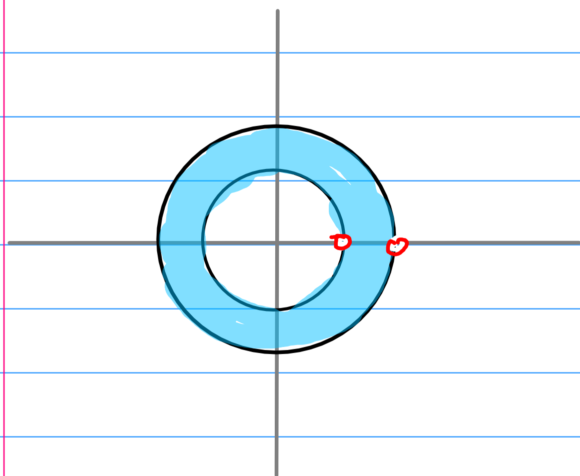
$$\frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n \lim_{z \to 0} \left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$
$$= (-1)^n \left((-1)^{-n-1} - (-2)^{-n-1}\right)$$
$$= -1 + 2^{-n-1}$$

$$a_n = -1 + 2^{-n-1} \qquad (n \geq 0)$$

$$f(z) = \sum_{n=n_1}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \left(2^{-n-1} - 1\right) z^n$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

② D_2 $\quad 1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$

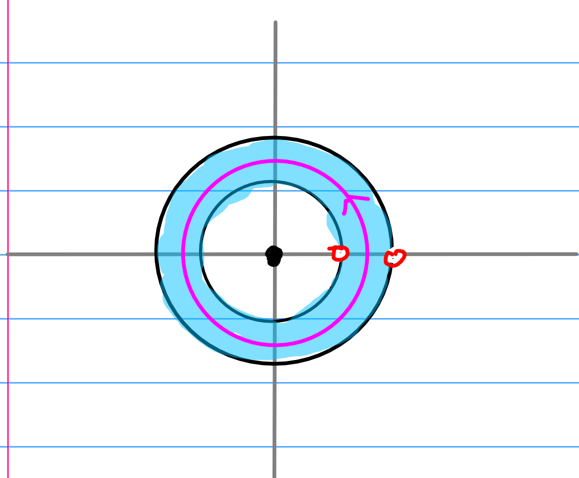


$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\left(\frac{1}{z}\right)} + \frac{1}{2} \frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$a_n = \sum_{k=1}^{M} \operatorname{Res}\left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k\right) = \operatorname{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$
$$+ \operatorname{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$



$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$+ \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$

$$\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \quad (\text{order } n)$$

$$\frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n \lim_{z \to 0} \left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

$$= (-1)^n \left((-1)^{-n-1} - (-2)^{-n-1}\right)$$

$$= -1 + 2^{-n-1}$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) = -1 + 2^{-n-1} \quad (n \geq 0)$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) = \lim_{z \to 1} (z-1) \frac{-1}{(z-1)(z-2)z^{n+1}} = 1$$

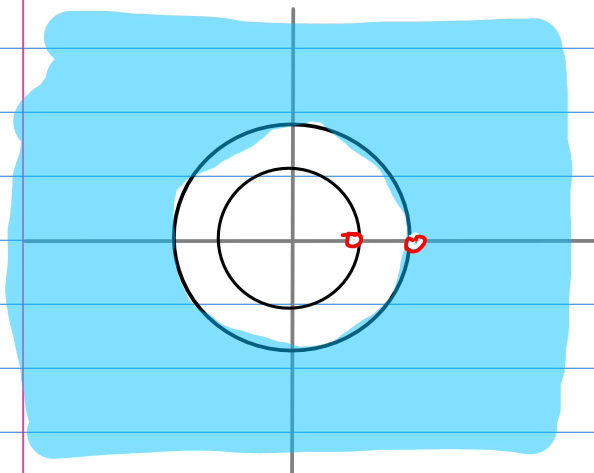
$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
0	0	0	$-1+2^{-1}$	$-1+2^{-2}$	$-1+2^{-3}$	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$
1	1	1	1	1	1	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 1\right)$
1	1	1	2^{-1}	2^{-2}	2^{-3}	

$$\begin{cases} a_n = 2^{-n-1} & n \geq 0 \\ a_n = 1 & n < 0 \end{cases} \qquad \begin{cases} 2^{-n-1} z^n \\ z^{-n} \end{cases}$$

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

③ D_3 $\quad 2 < |z|$ $\quad \left|\frac{2}{z}\right| < 1$ $\quad \left|\frac{1}{z}\right| < 1$



$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)}$$

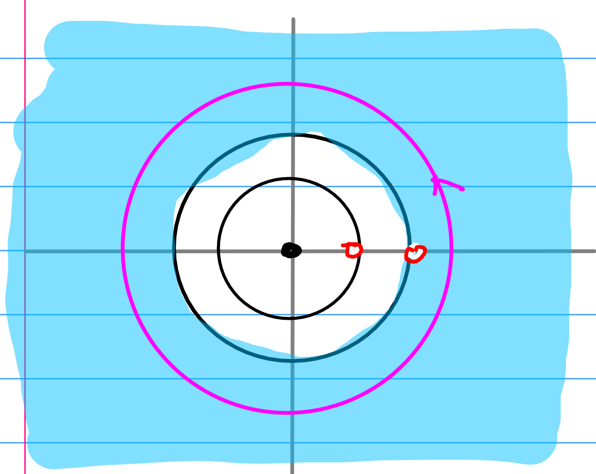
$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$+ \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$

$$+ \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 2\right)$$



$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) = -1 + 2^{-n-1} \qquad (n \geq 0)$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) = \lim_{z\to 1} (z-1)\frac{-1}{(z-1)(z-2)z^{n+1}} = 1$$

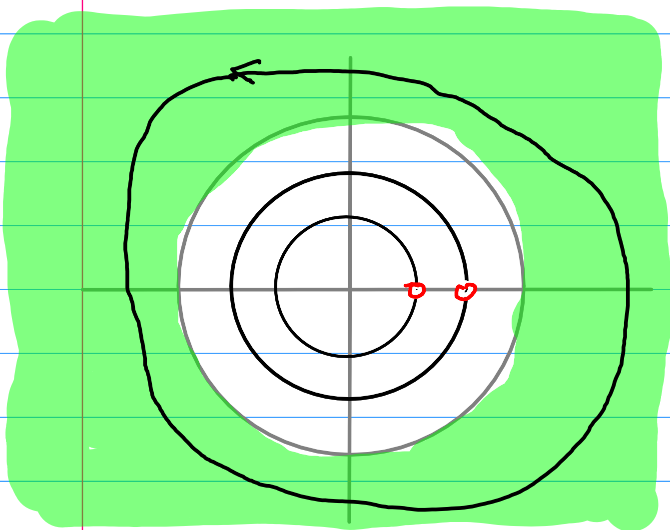
$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 2\right) = \lim_{z\to 2} (z-2)\frac{-1}{(z-1)(z-2)z^{n+1}} = -\frac{1}{2^{n+1}}$$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
0	0	0	$-1+2^{-1}$	$-1+2^{-2}$	$-1+2^{-3}$	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$
1	1	1	1	1	1	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 1\right)$
-2^2	-2	-1	-2^{-1}	-2^{-2}	-2^{-3}	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 2\right)$
$1-2^2$	$1-2$	0	0	0	0	

$$a_n = 1 - 2^{-n+1} \qquad n < 0 \qquad\qquad = \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$f(z) = \sum_{n=-1}^{-\infty} (1-2^{-n+1}) z^n = \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$



$$x[n] = \frac{1}{2\pi i} \int_C \boxed{X(z)\, z^{n-1}}\, dz = \sum_{j=1}^{k} \mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, z_j\right)$$

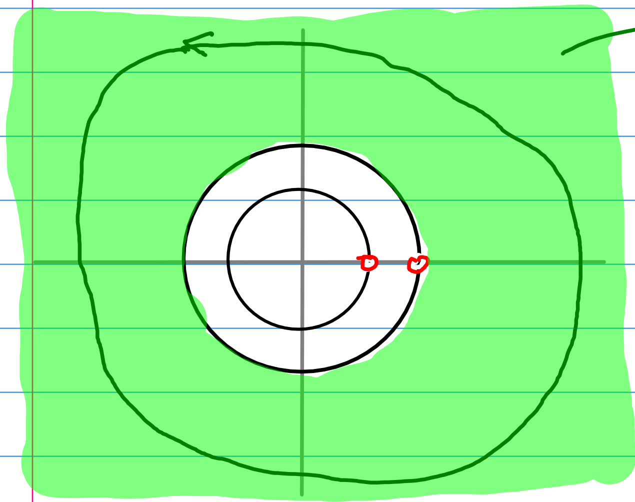
$$X(z) = \frac{-1}{(z-1)(z-2)}$$

$$X(z)\, z^{n-1} = \frac{-1}{(z-1)(z-2)}\, z^{n-1}$$

$$\mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, 1\right) = (z-1)\frac{-1}{(z-1)(z-2)}\, z^{n-1}\Big|_{z=1} = 1$$

$$\mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, 2\right) = (z-2)\frac{-1}{(z-1)(z-2)}\, z^{n-1}\Big|_{z=2} = -2^{n-1}$$

$$x[n] = 1 - 2^{n-1}$$



ROC (Region of Convergence)

$|z| > 2 \Rightarrow \frac{2}{|z|} < 1$

$\left(\frac{2}{z}\right)^0 + \left(\frac{2}{z}\right)^1 + \left(\frac{2}{z}\right)^2 + \cdots \xrightarrow{\text{converge}} \frac{1}{1 - \frac{2}{z}}$

$|z| > 2 \Rightarrow \frac{1}{|z|} < 1$

$\left(\frac{1}{z}\right)^0 + \left(\frac{1}{z}\right)^1 + \left(\frac{1}{z}\right)^2 + \cdots \xrightarrow{\text{converge}} \frac{1}{1 - \frac{1}{z}}$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$\left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots + \frac{1}{2}\left\{\left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots\right\} \xrightarrow{\text{converge}} \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{(z-1)(z-2)}$$

$$(1-2^0)z^{-1} + (1-2^1)z^{-2} + (1-2^2)z^{-3} + \cdots \xrightarrow{\text{converge}} \frac{-1}{(z-1)(z-2)} \quad (|z| > 2)$$

$$x[n] = 1 - 2^{n-1} \longleftrightarrow X(z) = \frac{-1}{(z-1)(z-2)} \quad (|z| > 2)$$