

Laurent Series and z-Transform

- Geometric Series

Applications

A

20210115 Fri

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Unshifted Geometric Sequences

Causal

$$\frac{1}{1 - az} \rightarrow u(n)$$

Anti-causal

$$\frac{1}{1 - az^{-1}} \rightarrow u(-n)$$

Positive Exponent

$$az, az^{-1} \rightarrow a^n$$

Negative Exponent

$$a^{-1}z, az^{-1} \rightarrow a^{-n}$$

Positive Exponent

unshifted

$$\frac{1}{1 - az} \quad |z| < a^{-1} \quad a^n u(n)$$

complementary

$$\frac{az}{1 - az} \quad |z| < a^{-1} \quad a^n u(n-1)$$

unshifted

$$-\frac{az^{-1}}{1 - az^{-1}} \quad |z| > a^{-1} \quad a^n u(-n-1)$$

complementary

$$-\frac{1}{1 - az^{-1}} \quad |z| > a^{-1} \quad a^n u(-n)$$

Negative Exponent

unshifted

$$\frac{1}{1 - a^{-1}z} \quad |z| < a \quad a^{-n} u(n)$$

complementary

$$\frac{a^{-1}z}{1 - a^{-1}z} \quad |z| < a \quad a^{-n} u(n-1)$$

unshifted

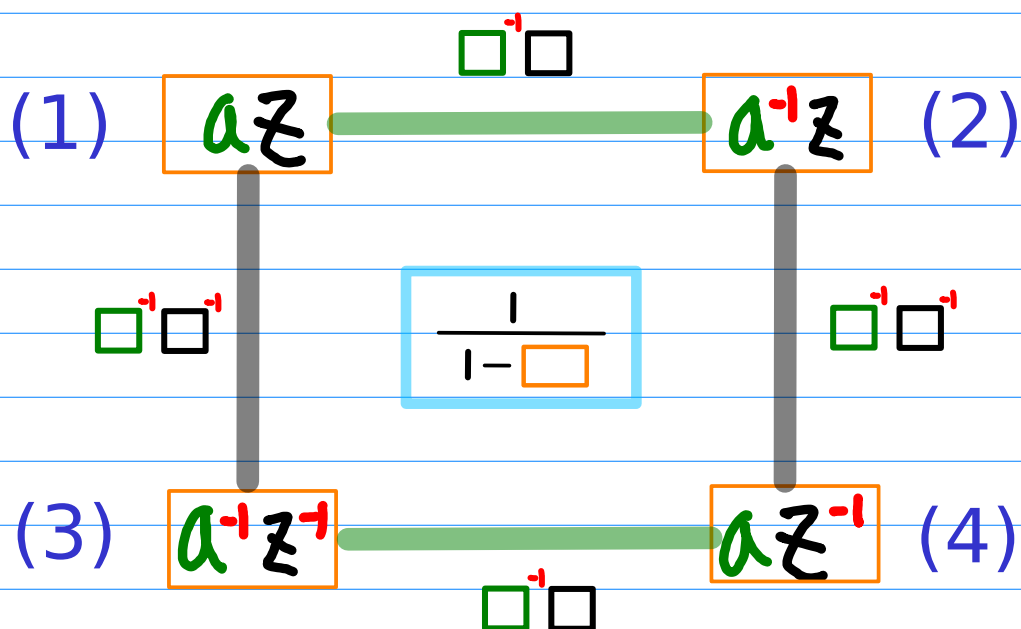
$$-\frac{az^{-1}}{1 - az^{-1}} \quad |z| > a \quad a^{-n} u(-n-1)$$

complementary

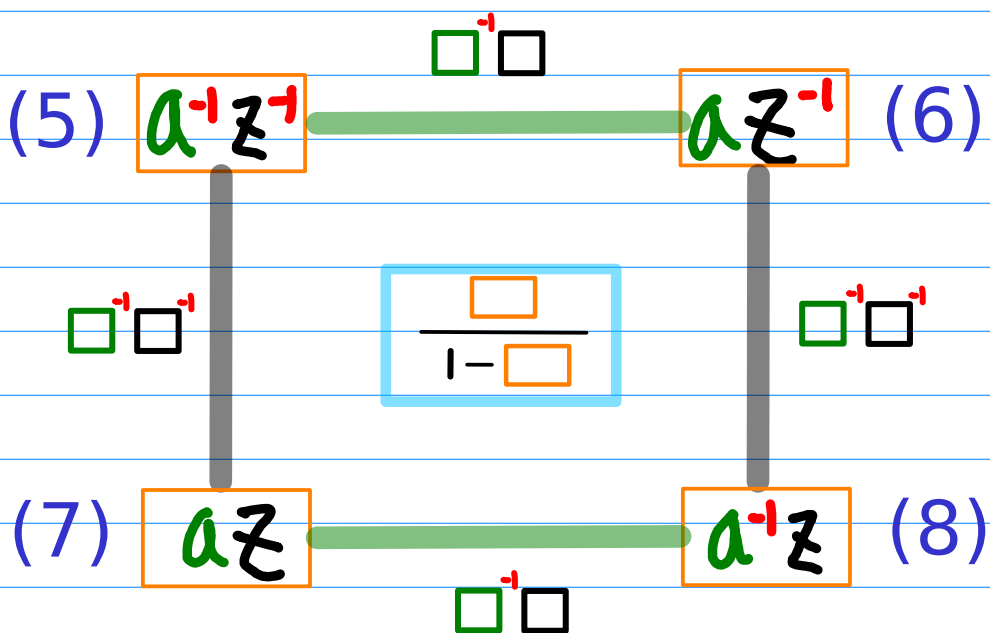
$$-\frac{1}{1 - az^{-1}} \quad |z| > a \quad a^{-n} u(-n)$$

Numbering the basic elements - (1) CR

unshifted geometric sequences

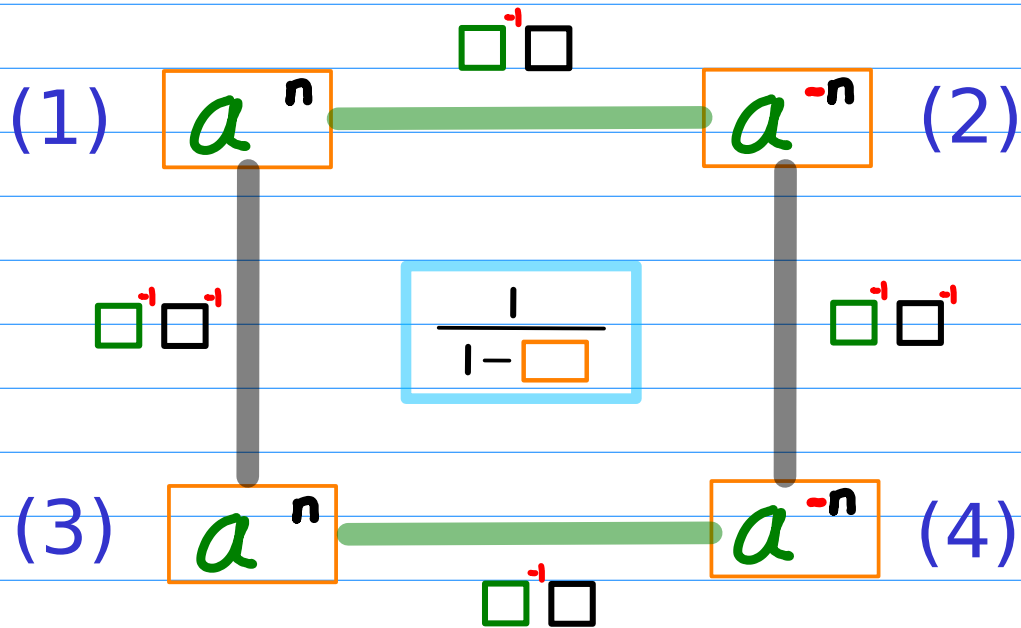


complementary geometric sequences

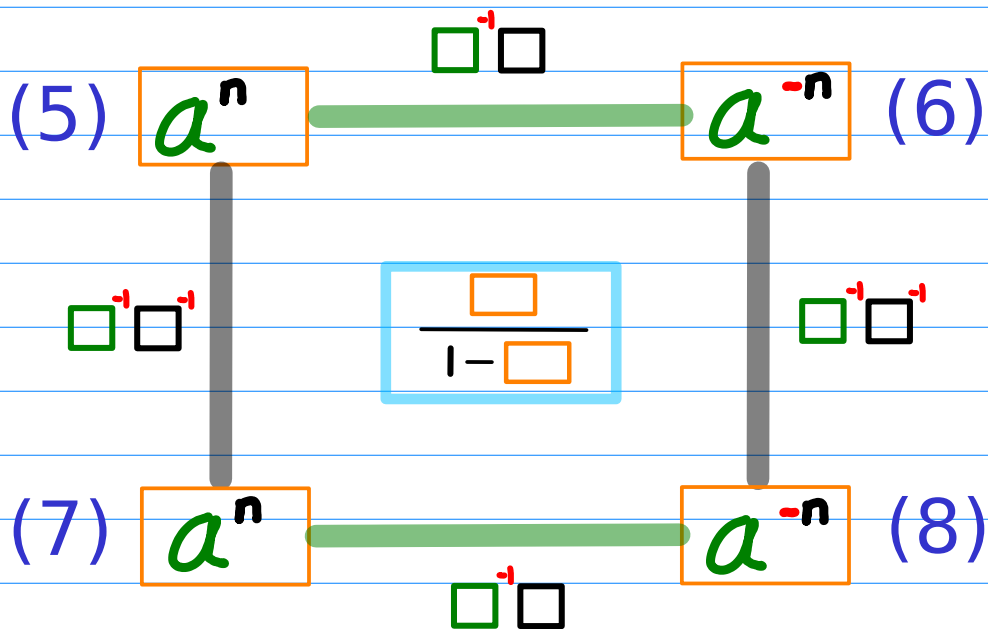


Numbering the basic elements - (2) Power

unshifted geometric sequences

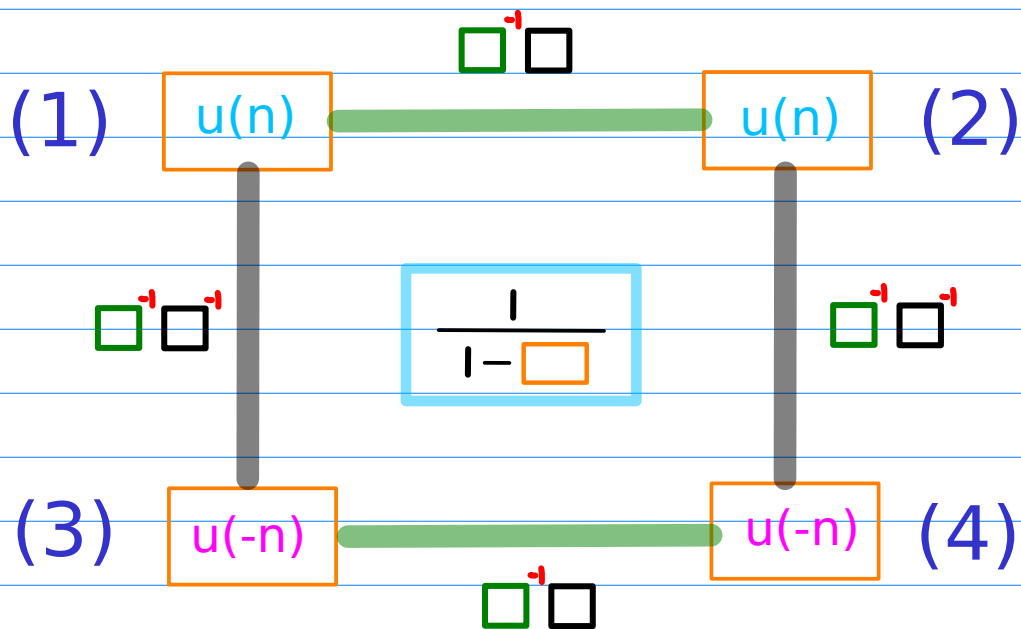


complementary geometric sequences

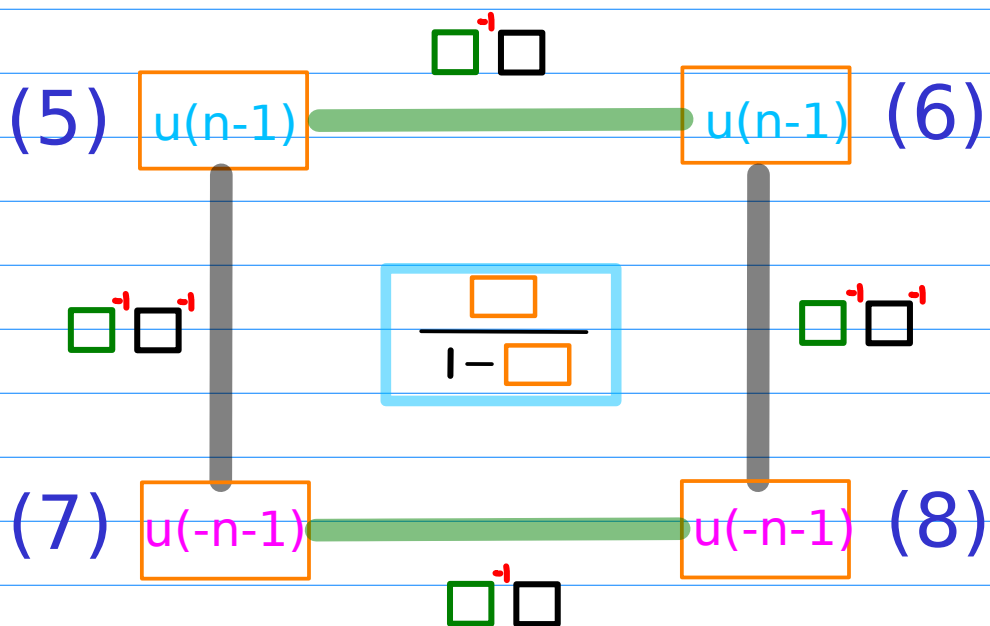


Numbering the basic elements - (3) Range

unshifted geometric sequences

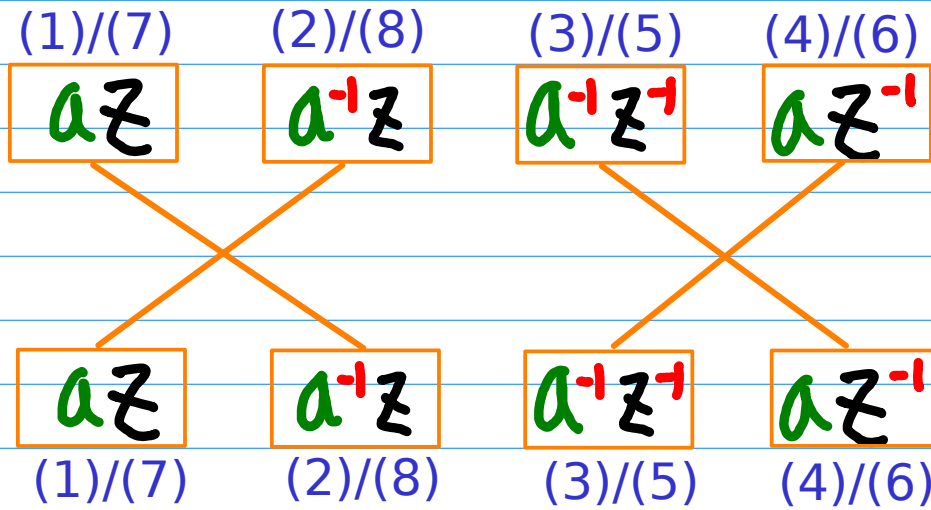


complementary geometric sequences

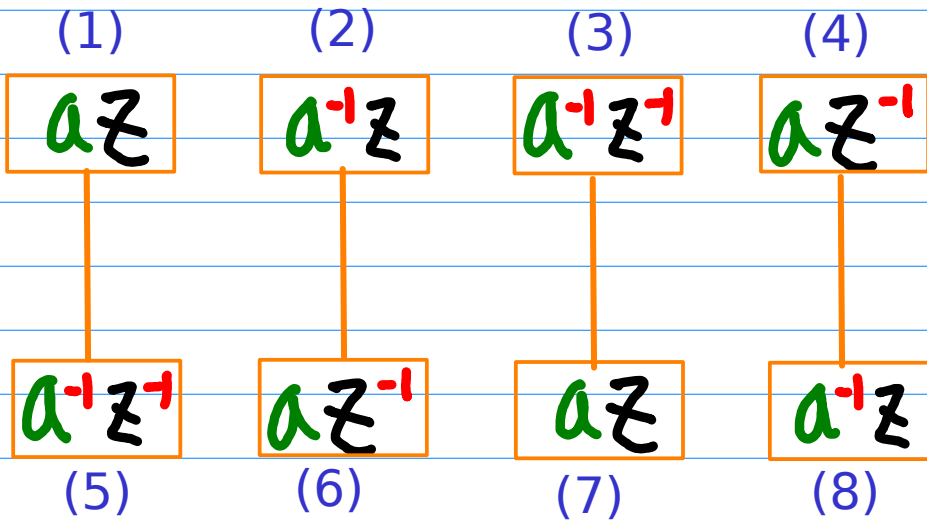


Inverse Relations

inverse power $\square^{\prime} \square$



complementary $\square^{\prime} \square^{\prime}$



Geometric Series Form Combinations with a unit start term unshifted

(1) $a^n u(n)$

$$+ \frac{1}{1 - az}$$

(2) $a^{-n} u(n)$

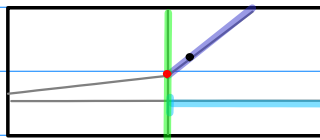
$$+ \frac{1}{1 - a^{-1}z}$$

(3) $-a^n u(-n)$

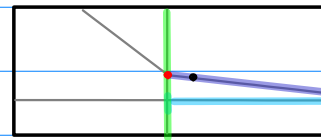
$$- \frac{1}{1 - a^{-1}z^{-1}}$$

(4) $-a^{-n} u(-n)$

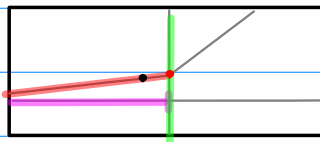
$$- \frac{1}{1 - az^{-1}}$$



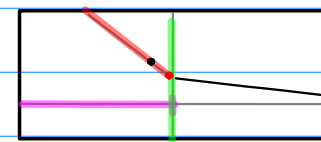
$a^n u(n)$



$a^{-n} u(n)$



$a^n u(-n)$



$a^{-n} u(-n)$

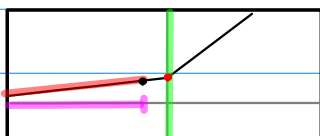
Geometric Series Form Combinations with a common-ratio start term **unshifted, complementary**

$$(5) \quad \frac{a^{-1}z^{-1}}{1 - a^{-1}z^{-1}} \quad -a^n u(-n-1)$$

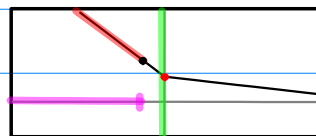
$$(6) \quad \frac{az^{-1}}{1 - az^{-1}} \quad -a^{-n} u(-n-1)$$

$$(7) \quad \frac{az}{1 - az} \quad a^n u(n-1)$$

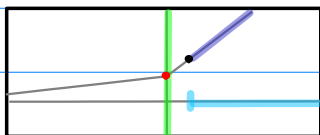
$$(8) \quad \frac{a^{-1}z}{1 - a^{-1}z} \quad a^{-n} u(n-1)$$



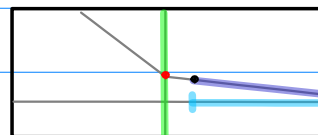
$$a^n u(-n-1)$$



$$a^{-n} u(-n-1)$$

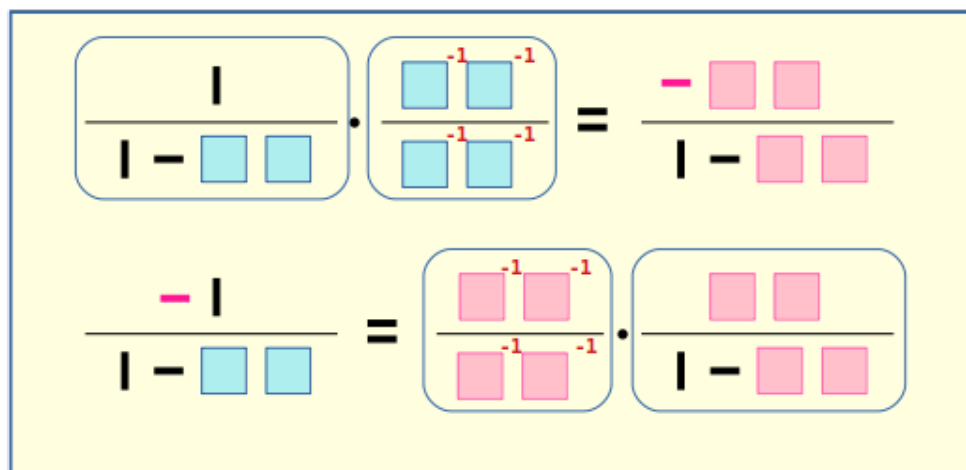
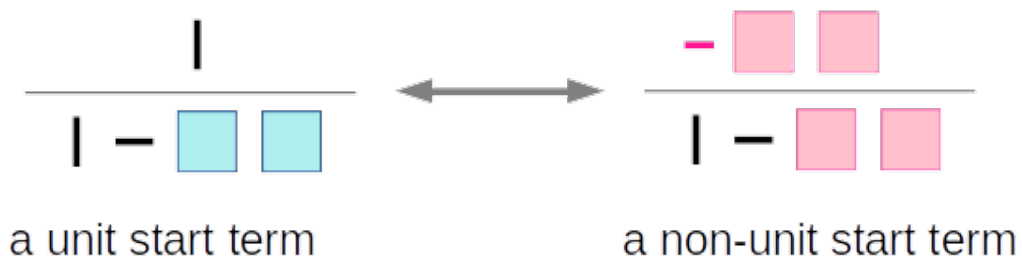
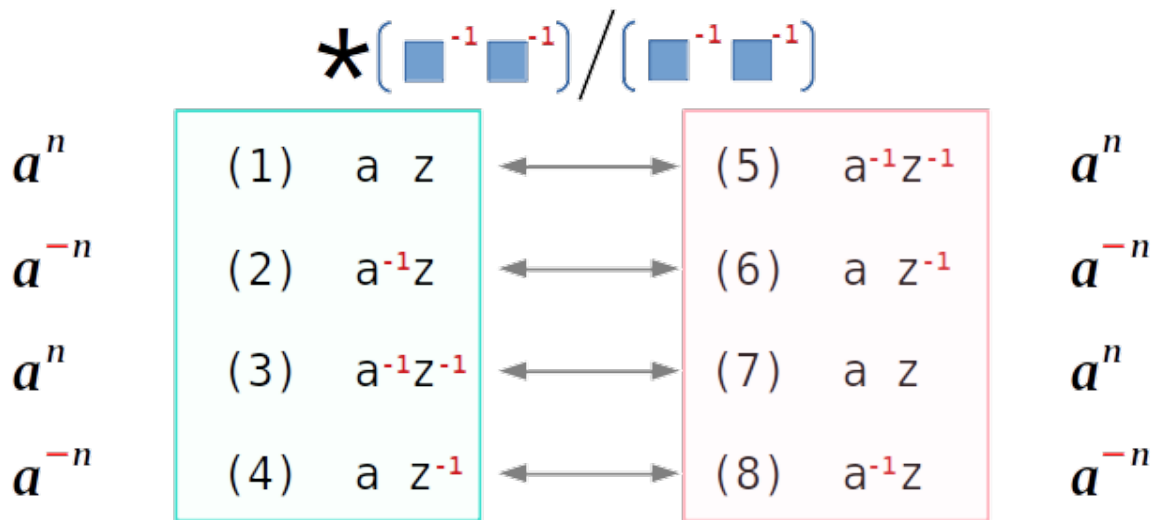


$$a^n u(n-1)$$



$$a^{-n} u(n-1)$$

Complementary Relations



$$* \left(\begin{array}{cc} \square^{-1} & \square^{-1} \end{array} \right) / \left(\begin{array}{cc} \square^{-1} & \square^{-1} \end{array} \right)$$

$$(1) \quad a^{+1} z^{+1} \quad a^{+n} \cdot u(n)$$

$$(2) \quad a^{-1} z^{+1} \quad a^{-n} \cdot u(n)$$

$$(3) \quad a^{-1} z^{-1} \quad a^{+n} \cdot u(-n)$$

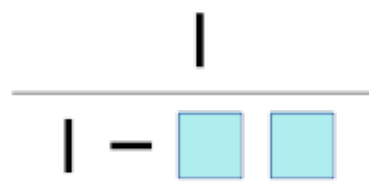
$$(4) \quad a^{+1} z^{-1} \quad a^{-n} \cdot u(-n)$$

$$(5) \quad a^{-1} z^{-1} \quad a^{+n} \cdot u(-n-1)$$

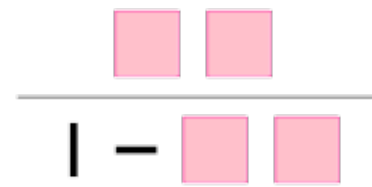
$$(6) \quad a^{+1} z^{-1} \quad a^{-n} \cdot u(-n-1)$$

$$(7) \quad a^{+1} z^{+1} \quad a^{+n} \cdot u(n-1)$$

$$(8) \quad a^{-1} z^{+1} \quad a^{-n} \cdot u(n-1)$$



a unit start term



a non-unit start term

| Positive Exponent | Negative Exponent |
|-------------------|-------------------|
| (1) | (2) |
| (3) | (4) |
| (5) | (6) |
| (7) | (8) |

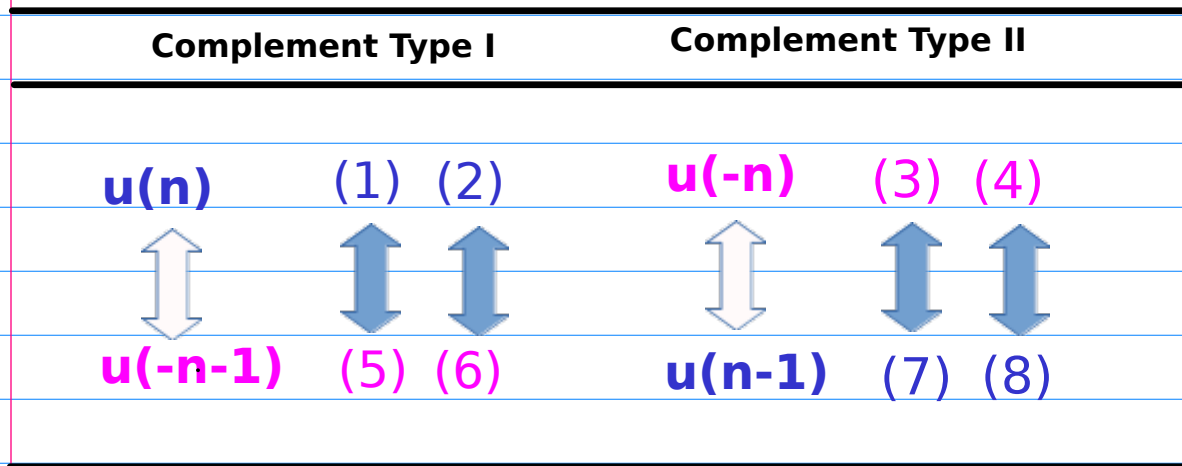
| | | | |
|-----------|-----|-----|-------------|
| $u(n)$ | (1) | (2) | Causal |
| $u(-n)$ | (3) | (4) | Anti-Causal |
| $u(-n-1)$ | (5) | (6) | Anti-Causal |
| $u(n-1)$ | (7) | (8) | Causal |

| | | | |
|---------------------------|-----|-----|-----------|
| ranges include the origin | (1) | (2) | $u(n)$ |
| | (3) | (4) | $u(-n)$ |
| ranges exclude the origin | (5) | (6) | $u(-n-1)$ |
| | (7) | (8) | $u(n-1)$ |

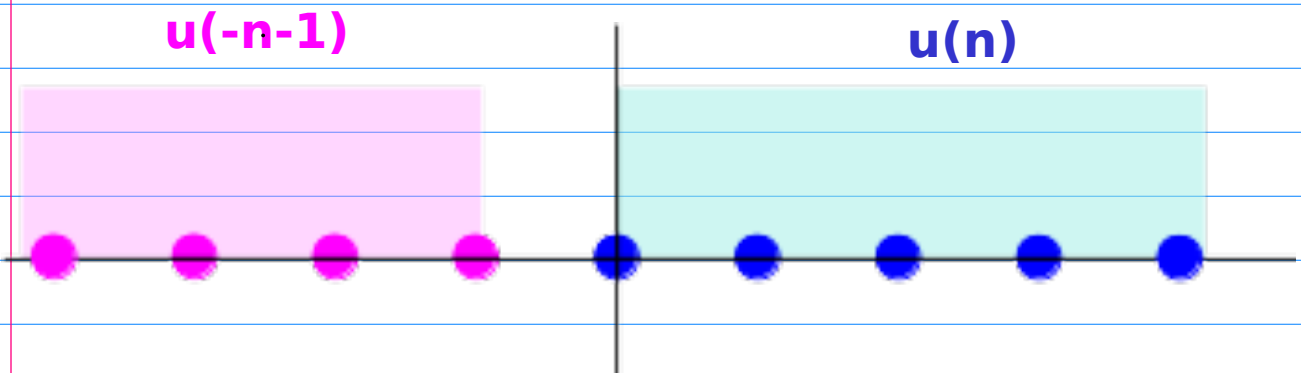
Complement Type I

| | | | |
|-----------|-----|-----|-------------|
| $u(n)$ | (1) | (2) | Causal |
| $u(-n)$ | (3) | (4) | Anti-Causal |
| $u(-n-1)$ | (5) | (6) | Anti-Causal |
| $u(n-1)$ | (7) | (8) | Causal |

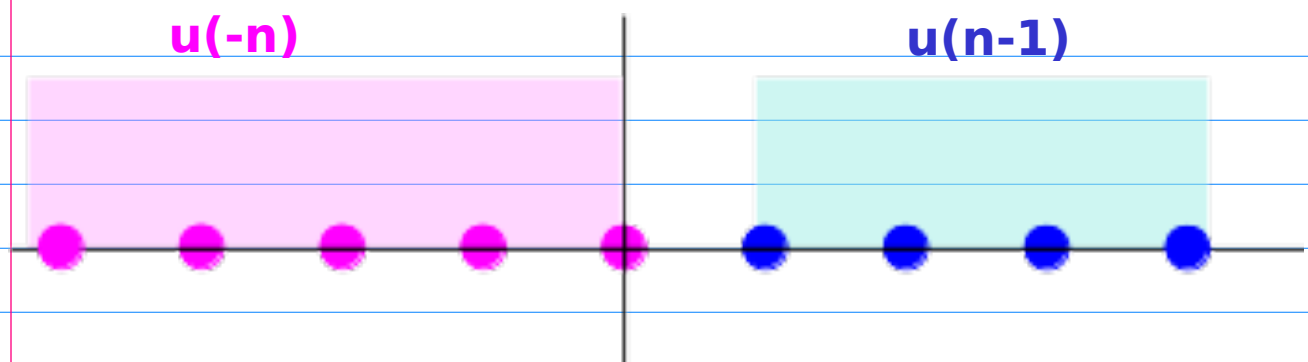
Complement Type II



Complement Type I



Complement Type II



| | | | |
|--------------------|-----------|-----|-----|
| Causal | $u(n)$ | (1) | (2) |
| | $u(n-1)$ | (7) | (8) |
| Anti-Causal | $u(-n-1)$ | (5) | (6) |
| | $u(-n)$ | (3) | (4) |

| | | | |
|---------------------------|-----------|-----|-----|
| Complement Type I | $u(n)$ | (1) | (2) |
| | $u(-n-1)$ | (5) | (6) |
| Complement Type II | $u(-n)$ | (3) | (4) |
| | $u(n-1)$ | (7) | (8) |

| | | Positive Exponent | Negative Exponent |
|---------------------------|-----------|--------------------------|--------------------------|
| Complement Type I | $u(n)$ | (1) | (2) |
| | $u(-n-1)$ | (5) | (6) |
| Complement Type II | $u(-n)$ | (3) | (4) |
| | $u(n-1)$ | (7) | (8) |

Shifted Geometric Sequences

Exponent Shifting

$$* a$$

$$a^{n+1} \leftarrow a^n$$

Left Shift

$$a^{-n+1} \leftarrow a^{-n}$$

Right Shift

$$* a^{-1}$$

$$a^{n-1} \leftarrow a^n$$

Right Shift

$$a^{-n-1} \leftarrow a^{-n}$$

Left Shift

Exponent & Range Shifting

$$* z$$

$$n \leftarrow n-1$$

Right Shift

$$* z^{-1}$$

$$n \leftarrow n+1$$

Left Shift

Positive Exponent

Left Shifted

$$\frac{a}{1-az} \quad |z| < a^{-1} \quad a^{n+1} u(n)$$

Right Shifted

$$\frac{z}{1-az} \quad |z| < a^{-1} \quad a^{n-1} u(n-1)$$

Left Shifted

$$- \frac{z^{-1}}{1-az^{-1}} \quad |z| > a^{-1} \quad a^{n+1} u(-n-1)$$

Right Shifted

$$- \frac{a^{-1}}{1-az^{-1}} \quad |z| > a^{-1} \quad a^{n-1} u(-n)$$

Negative Exponent

$$\frac{a^{-1}}{1-a^{-1}z} \quad |z| < a \quad a^{-n-1} u(n)$$

$$\frac{z}{1-a^{-1}z} \quad |z| < a \quad a^{-n+1} u(n-1)$$

$$- \frac{z^{-1}}{1-az^{-1}} \quad |z| > a \quad a^{-n-1} u(-n-1)$$

$$- \frac{a}{1-az^{-1}} \quad |z| > a \quad a^{-n+1} u(-n)$$

multiplying a or a^{-1}

multiplying z or z^{-1}

Exponent Shifting

$$* a$$

$$a^{n+1} \leftarrow a^n$$

Left Shift

$$a^{-n+1} \leftarrow a^{-n}$$

Right Shift

$$* a^{-1}$$

$$a^{n-1} \leftarrow a^n$$

Right Shift

$$a^{-n-1} \leftarrow a^{-n}$$

Left Shift

Exponent & Range Shifting

$$* z$$

$$n \leftarrow n-1$$

Right Shift

$$* z^{-1}$$

$$n \leftarrow n+1$$

Left Shift

Combinations of Shifted Geometric Series (1)

Positive Exponent

/z $n \leftarrow n+1$

*z $n \leftarrow n-1$

(1) $\frac{1}{1-az} \quad |z| < a^{-1}$ $\xrightarrow{*a}$ $\frac{a}{1-az} \quad |z| < a^{-1}$ **Left Shifted**
 $a^n u(n)$ $\xrightarrow{/z}$ $a^{n+1} u(n)$

(7) $\frac{az}{1-az} \quad |z| < a^{-1}$ $\xrightarrow{/a}$ $\frac{z}{1-az} \quad |z| < a^{-1}$ **Right Shifted**
 $a^n u(n-1)$ $\xrightarrow{*z}$ $a^{n-1} u(n-1)$

(5) $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$ $\xrightarrow{*a}$ $-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$ **Left Shifted**
 $a^n u(-n-1)$ $\xrightarrow{/z}$ $a^{n+1} u(-n-1)$

(3) $-\frac{1}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$ $\xrightarrow{/a}$ $-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$ **Right Shifted**
 $a^n u(-n)$ $\xrightarrow{*z}$ $a^{n-1} u(-n)$

| | | | |
|-------------|-----------|-----|-----|
| Causal | $u(n)$ | (1) | (2) |
| | $u(n-1)$ | (7) | (8) |
| Anti-Causal | $u(-n-1)$ | (5) | (6) |
| | $u(-n)$ | (3) | (4) |

Combinations of Shifted Geometric Series (2)

Negative Exponent

/z $n \leftarrow n+1$

*z $n \leftarrow n-1$

(2) $\frac{1}{1-a^1z} \quad |z| < a$ $\xrightarrow{/a}$ $\frac{a^{-1}}{1-a^1z} \quad |z| < a$ **Left Shifted**

$a^{-n} u(n)$ $\xrightarrow{/z}$ $a^{-n-1} u(n)$

(8) $\frac{a^1z}{1-a^1z} \quad |z| < a$ $\xrightarrow{*a}$ $\frac{z}{1-a^1z} \quad |z| < a$ **Right Shifted**

$a^{-n} u(n-1)$ $\xrightarrow{*z}$ $a^{-n+1} u(n-1)$

(6) $-\frac{a^1z^{-1}}{1-a^1z^{-1}} \quad |z| > a$ $\xrightarrow{/a}$ $-\frac{z^{-1}}{1-a^1z^{-1}} \quad |z| > a$ **Left Shifted**

$a^{-n} u(-n-1)$ $\xrightarrow{/z}$ $a^{-n-1} u(-n-1)$

(4) $-\frac{1}{1-a^1z^{-1}} \quad |z| > a$ $\xrightarrow{*a}$ $-\frac{a}{1-a^1z^{-1}} \quad |z| > a$ **Right Shifted**

$a^{-n} u(-n)$ $\xrightarrow{*z}$ $a^{-n+1} u(-n)$

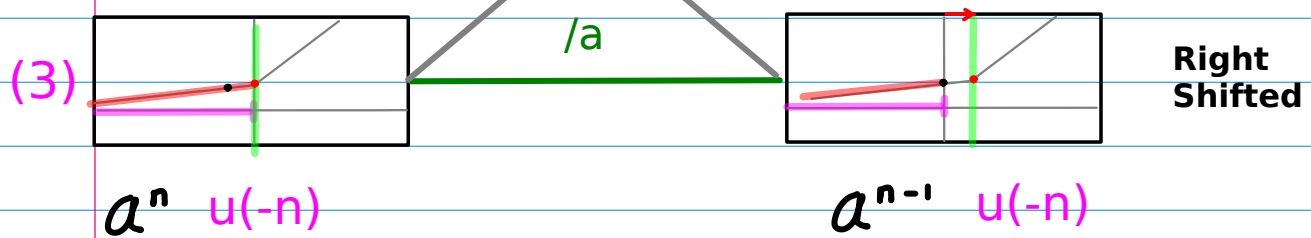
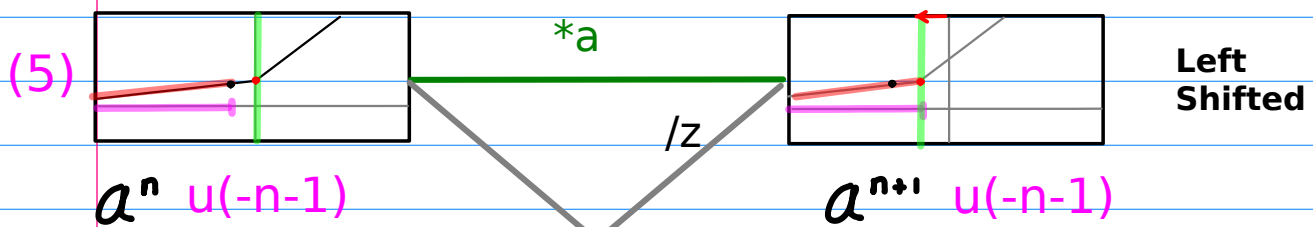
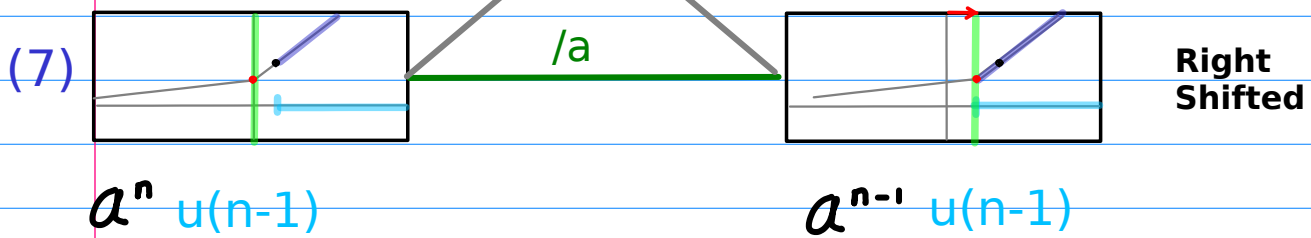
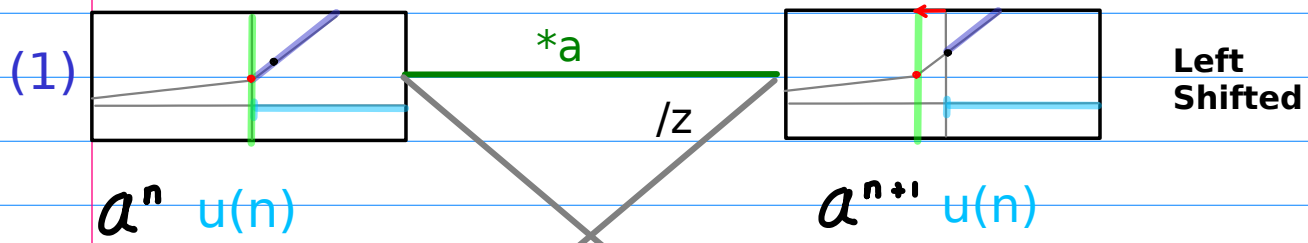
| | | | |
|-------------|-----------|-----|-----|
| Causal | $u(n)$ | (1) | (2) |
| | $u(n-1)$ | (7) | (8) |
| Anti-Causal | $u(-n-1)$ | (5) | (6) |
| | $u(-n)$ | (3) | (4) |

Graphs of Shifted Geometric Series (1)

Positive Exponent

$/z \quad n \leftarrow n+1$

$*z \quad n \leftarrow n-1$



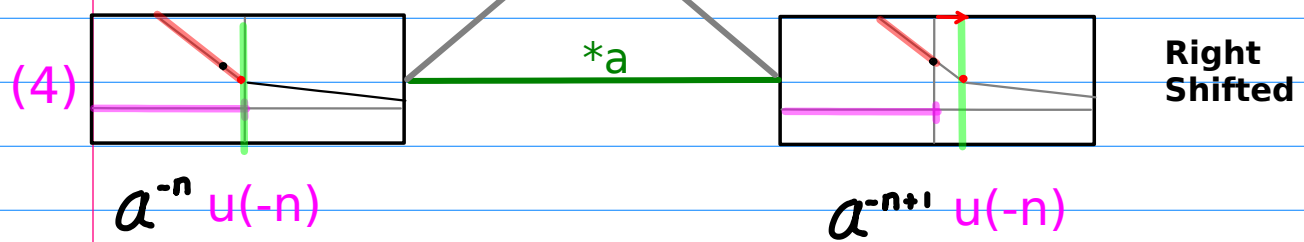
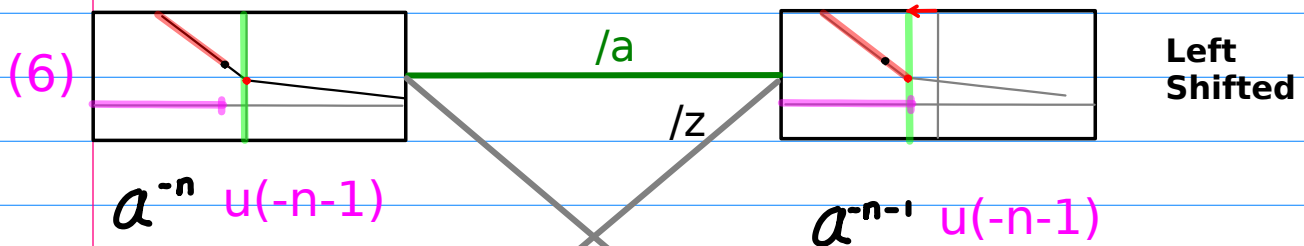
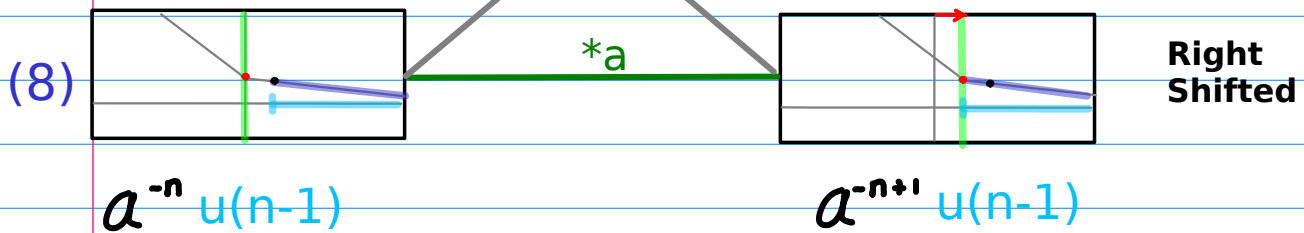
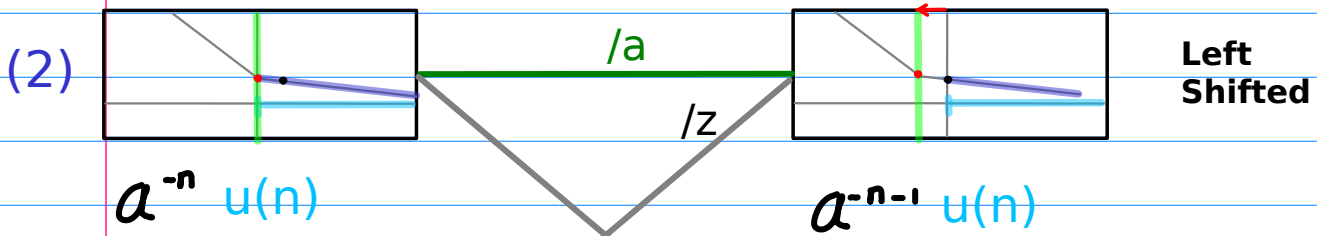
| | | | |
|-------------|-----------|-----|-----|
| Causal | $u(n)$ | (1) | (2) |
| | $u(n-1)$ | (7) | (8) |
| Anti-Causal | $u(-n-1)$ | (5) | (6) |
| | $u(-n)$ | (3) | (4) |

Graphs of Shifted Geometric Series (2)

Negative Exponent

$/z \quad n \leftarrow n+1$

$*z \quad n \leftarrow n-1$

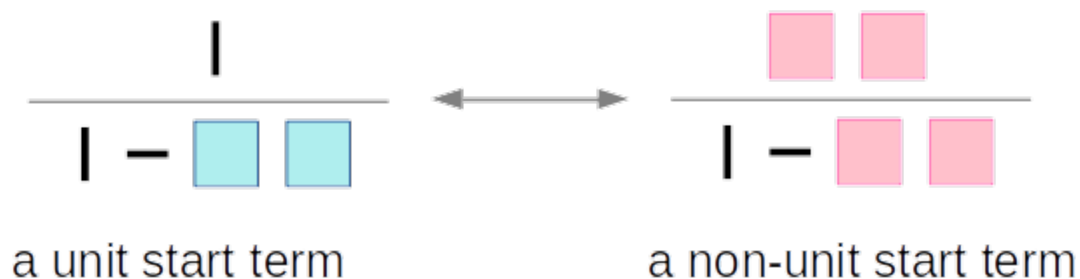


| | | | |
|-------------|-----------|-----|-----|
| Causal | $u(n)$ | (1) | (2) |
| | $u(n-1)$ | (7) | (8) |
| Anti-Causal | $u(-n-1)$ | (5) | (6) |
| | $u(-n)$ | (3) | (4) |

Shifting Geometric Series by $*a$ or $/a$

$$\star \left(\boxed{}^{-1} \boxed{}^{-1} \right) / \left(\boxed{}^{-1} \boxed{}^{-1} \right)$$

| | |
|---|---|
| <p>(1) $a^{+1} z^{+1}$ $a^{+n} \cdot u(n)$ $\boxed{* a}$ $\boxed{a^{+n+1} \cdot u(n)}$</p> <p>(2) $a^{-1} z^{+1}$ $a^{-n} \cdot u(n)$ $\boxed{/ a}$ $\boxed{a^{-n-1} \cdot u(n)}$</p> | <p>(5) $a^{-1} z^{-1}$ $a^{+n} \cdot u(-n-1)$ $\boxed{* a}$ $\boxed{a^{+n+1} \cdot u(-n-1)}$</p> <p>(6) $a^{+1} z^{-1}$ $a^{-n} \cdot u(-n-1)$ $\boxed{/ a}$ $\boxed{a^{-n-1} \cdot u(-n-1)}$</p> |
| <p>(3) $a^{-1} z^{-1}$ $a^{+n} \cdot u(-n)$ $\boxed{/ a}$ $\boxed{a^{+n-1} \cdot u(-n)}$</p> <p>(4) $a^{+1} z^{-1}$ $a^{-n} \cdot u(-n)$ $\boxed{* a}$ $\boxed{a^{-n+1} \cdot u(-n)}$</p> | <p>(7) $a^{+1} z^{+1}$ $a^{+n} \cdot u(n-1)$ $\boxed{/ a}$ $\boxed{a^{+n-1} \cdot u(n-1)}$</p> <p>(8) $a^{-1} z^{+1}$ $a^{-n} \cdot u(n-1)$ $\boxed{* a}$ $\boxed{a^{-n+1} \cdot u(n-1)}$</p> |



Shifting Geometric Series by $*z$ or $/z$

$$* \left(\begin{array}{c} \blacksquare^{-1} \blacksquare^{-1} \end{array} \right) / \left(\begin{array}{c} \blacksquare^{-1} \blacksquare^{-1} \end{array} \right)$$

(1) $a^{+1} z^{+1}$ $a^{+n} \cdot u(n)$
 $\boxed{* z}$ $\boxed{a^{+n-1} \cdot u(n-1)}$

(2) $a^{-1} z^{+1}$ $a^{-n} \cdot u(n)$
 $\boxed{* z}$ $\boxed{a^{-n+1} \cdot u(n-1)}$

(5) $a^{-1} z^{-1}$ $a^{+n} \cdot u(-n-1)$
 $\boxed{* z}$ $\boxed{a^{+n-1} \cdot u(-n)}$

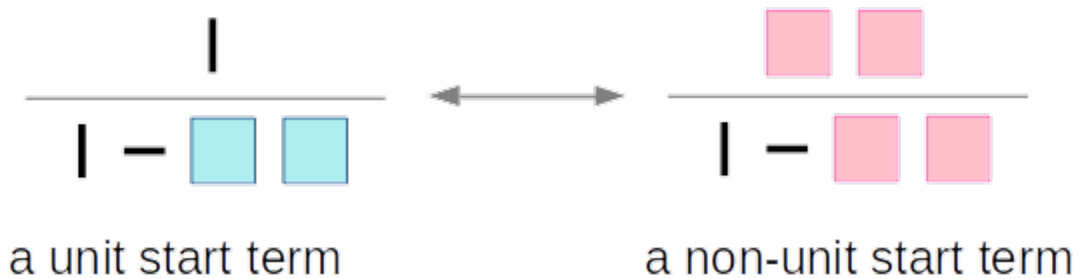
(6) $a^{+1} z^{-1}$ $a^{-n} \cdot u(-n-1)$
 $\boxed{* z}$ $\boxed{a^{-n+1} \cdot u(-n)}$

(3) $a^{-1} z^{-1}$ $a^{+n} \cdot u(-n)$
 $\boxed{/ z}$ $\boxed{a^{+n+1} \cdot u(-n-1)}$

(4) $a^{+1} z^{-1}$ $a^{-n} \cdot u(-n)$
 $\boxed{/ z}$ $\boxed{a^{-n-1} \cdot u(-n-1)}$

(7) $a^{+1} z^{+1}$ $a^{+n} \cdot u(n-1)$
 $\boxed{/ z}$ $\boxed{a^{+n+1} \cdot u(n)}$

(8) $a^{-1} z^{+1}$ $a^{-n} \cdot u(n-1)$
 $\boxed{/ z}$ $\boxed{a^{-n-1} \cdot u(n)}$



| | | |
|-----------|-----|-----|
| $u(n)$ | (1) | (2) |
| $u(-n)$ | (3) | (4) |
| $u(-n-1)$ | (5) | (6) |
| $u(n-1)$ | (7) | (8) |

| | | | |
|---------------------|--------------------------|---------------------|--------------------------|
| (1) $a^{+1} z^{+1}$ | $a^{+n} \cdot u(n)$ | (2) $a^{-1} z^{+1}$ | $a^{-n} \cdot u(n)$ |
| $* a$ | $a^{+n+1} \cdot u(n)$ | $/ a$ | $a^{-n-1} \cdot u(n)$ |
| (3) $a^{-1} z^{-1}$ | $a^{+n} \cdot u(-n)$ | (4) $a^{+1} z^{-1}$ | $a^{-n} \cdot u(-n)$ |
| $/ a$ | $a^{+n-1} \cdot u(-n)$ | $* a$ | $a^{-n+1} \cdot u(-n)$ |
| (5) $a^{-1} z^{-1}$ | $a^{+n} \cdot u(-n-1)$ | (6) $a^{+1} z^{-1}$ | $a^{-n} \cdot u(-n-1)$ |
| $* a$ | $a^{+n+1} \cdot u(-n-1)$ | $/ a$ | $a^{-n-1} \cdot u(-n-1)$ |
| (7) $a^{+1} z^{+1}$ | $a^{+n} \cdot u(n-1)$ | (8) $a^{-1} z^{+1}$ | $a^{-n} \cdot u(n-1)$ |
| $/ a$ | $a^{+n-1} \cdot u(n-1)$ | $* a$ | $a^{-n+1} \cdot u(n-1)$ |

| | | | |
|---------------------|--------------------------|---------------------|--------------------------|
| (1) $a^{+1} z^{+1}$ | $a^{+n} \cdot u(n)$ | (2) $a^{-1} z^{+1}$ | $a^{-n} \cdot u(n)$ |
| $* z$ | $a^{+n-1} \cdot u(n-1)$ | $* z$ | $a^{-n+1} \cdot u(n-1)$ |
| (3) $a^{-1} z^{-1}$ | $a^{+n} \cdot u(-n)$ | (4) $a^{+1} z^{-1}$ | $a^{-n} \cdot u(-n)$ |
| $/ z$ | $a^{+n+1} \cdot u(-n-1)$ | $/ z$ | $a^{-n-1} \cdot u(-n-1)$ |
| (5) $a^{-1} z^{-1}$ | $a^{+n} \cdot u(-n-1)$ | (6) $a^{+1} z^{-1}$ | $a^{-n} \cdot u(-n-1)$ |
| $* z$ | $a^{+n-1} \cdot u(-n)$ | $* z$ | $a^{-n+1} \cdot u(-n)$ |
| (7) $a^{+1} z^{+1}$ | $a^{+n} \cdot u(n-1)$ | (8) $a^{-1} z^{+1}$ | $a^{-n} \cdot u(n-1)$ |
| $/ z$ | $a^{+n+1} \cdot u(n)$ | $/ z$ | $a^{-n-1} \cdot u(n)$ |

| | | | |
|--------------------|-----------|-----|-----|
| Causal | $u(n)$ | (1) | (2) |
| | $u(n-1)$ | (7) | (8) |
| Anti-Causal | $u(-n-1)$ | (5) | (6) |
| | $u(-n)$ | (3) | (4) |

| | | | | | |
|-----|-----------------|--------------------------|-----|-----------------|--------------------------|
| (1) | $a^{+1} z^{+1}$ | $a^{+n} \cdot u(n)$ | (2) | $a^{-1} z^{+1}$ | $a^{-n} \cdot u(n)$ |
| | $* a$ | $a^{+n+1} \cdot u(n)$ | | $/ a$ | $a^{-n-1} \cdot u(n)$ |
| (7) | $a^{+1} z^{+1}$ | $a^{+n} \cdot u(n-1)$ | (8) | $a^{-1} z^{+1}$ | $a^{-n} \cdot u(n-1)$ |
| | $/ a$ | $a^{+n-1} \cdot u(n-1)$ | | $* a$ | $a^{-n+1} \cdot u(n-1)$ |
| (5) | $a^{-1} z^{-1}$ | $a^{+n} \cdot u(-n-1)$ | (6) | $a^{+1} z^{-1}$ | $a^{-n} \cdot u(-n-1)$ |
| | $* a$ | $a^{+n+1} \cdot u(-n-1)$ | | $/ a$ | $a^{-n-1} \cdot u(-n-1)$ |
| (3) | $a^{-1} z^{-1}$ | $a^{+n} \cdot u(-n)$ | (4) | $a^{+1} z^{-1}$ | $a^{-n} \cdot u(-n)$ |
| | $/ a$ | $a^{+n-1} \cdot u(-n)$ | | $* a$ | $a^{-n+1} \cdot u(-n)$ |

| | | | | | |
|-----|-----------------|--------------------------|-----|-----------------|--------------------------|
| (1) | $a^{+1} z^{+1}$ | $a^{+n} \cdot u(n)$ | (2) | $a^{-1} z^{+1}$ | $a^{-n} \cdot u(n)$ |
| | $* z$ | $a^{+n-1} \cdot u(n-1)$ | | $* z$ | $a^{-n+1} \cdot u(n-1)$ |
| (7) | $a^{+1} z^{+1}$ | $a^{+n} \cdot u(n-1)$ | (8) | $a^{-1} z^{+1}$ | $a^{-n} \cdot u(n-1)$ |
| | $/ z$ | $a^{+n+1} \cdot u(n)$ | | $/ z$ | $a^{-n-1} \cdot u(n)$ |
| (5) | $a^{-1} z^{-1}$ | $a^{+n} \cdot u(-n-1)$ | (6) | $a^{+1} z^{-1}$ | $a^{-n} \cdot u(-n-1)$ |
| | $* z$ | $a^{+n-1} \cdot u(-n)$ | | $* z$ | $a^{-n+1} \cdot u(-n)$ |
| (3) | $a^{-1} z^{-1}$ | $a^{+n} \cdot u(-n)$ | (4) | $a^{+1} z^{-1}$ | $a^{-n} \cdot u(-n)$ |
| | $/ z$ | $a^{+n+1} \cdot u(-n-1)$ | | $/ z$ | $a^{-n-1} \cdot u(-n-1)$ |

| | | | |
|---------------------------|-----------|-----|-----|
| Complement Type I | $u(n)$ | (1) | (2) |
| | $u(-n-1)$ | (5) | (6) |
| Complement Type II | $u(-n)$ | (3) | (4) |
| | $u(n-1)$ | (7) | (8) |

| | | | |
|---------------------|--------------------------|---------------------|--------------------------|
| (1) $a^{+1} z^{+1}$ | $a^{+n} \cdot u(n)$ | (2) $a^{-1} z^{+1}$ | $a^{-n} \cdot u(n)$ |
| $* a$ | $a^{+n+1} \cdot u(n)$ | $/ a$ | $a^{-n-1} \cdot u(n)$ |
| (5) $a^{-1} z^{-1}$ | $a^{+n} \cdot u(-n-1)$ | (6) $a^{+1} z^{-1}$ | $a^{-n} \cdot u(-n-1)$ |
| $* a$ | $a^{+n+1} \cdot u(-n-1)$ | $/ a$ | $a^{-n-1} \cdot u(-n-1)$ |
| (3) $a^{-1} z^{-1}$ | $a^{+n} \cdot u(-n)$ | (4) $a^{+1} z^{-1}$ | $a^{-n} \cdot u(-n)$ |
| $/ a$ | $a^{+n-1} \cdot u(-n)$ | $* a$ | $a^{-n+1} \cdot u(-n)$ |
| (7) $a^{+1} z^{+1}$ | $a^{+n} \cdot u(n-1)$ | (8) $a^{-1} z^{+1}$ | $a^{-n} \cdot u(n-1)$ |
| $/ a$ | $a^{+n-1} \cdot u(n-1)$ | $* a$ | $a^{-n+1} \cdot u(n-1)$ |

| | | | |
|---------------------|--------------------------|---------------------|--------------------------|
| (1) $a^{+1} z^{+1}$ | $a^{+n} \cdot u(n)$ | (2) $a^{-1} z^{+1}$ | $a^{-n} \cdot u(n)$ |
| $* z$ | $a^{+n-1} \cdot u(n-1)$ | $* z$ | $a^{-n+1} \cdot u(n-1)$ |
| (5) $a^{-1} z^{-1}$ | $a^{+n} \cdot u(-n-1)$ | (6) $a^{+1} z^{-1}$ | $a^{-n} \cdot u(-n-1)$ |
| $* z$ | $a^{+n-1} \cdot u(-n)$ | $* z$ | $a^{-n+1} \cdot u(-n)$ |
| (3) $a^{-1} z^{-1}$ | $a^{+n} \cdot u(-n)$ | (4) $a^{+1} z^{-1}$ | $a^{-n} \cdot u(-n)$ |
| $/ z$ | $a^{+n+1} \cdot u(-n-1)$ | $/ z$ | $a^{-n-1} \cdot u(-n-1)$ |
| (7) $a^{+1} z^{+1}$ | $a^{+n} \cdot u(n-1)$ | (8) $a^{-1} z^{+1}$ | $a^{-n} \cdot u(n-1)$ |
| $/ z$ | $a^{+n+1} \cdot u(n)$ | $/ z$ | $a^{-n-1} \cdot u(n)$ |

Shifted Geometric Series (1)

by multiplying a or a^{-1}

Positive Exponent

(1) \leftarrow $\frac{1}{1-az} \quad |z| < a^{-1}$ $a^n u(n) \times a$ $\frac{a}{1-az} \quad |z| < a^{-1}$ $a^{n+1} u(n)$

(7) \rightarrow $\frac{az}{1-az} \quad |z| < a^{-1}$ $a^n u(n-1) \times a^{-1}$ $\frac{z}{1-az} \quad |z| < a^{-1}$ $a^{n-1} u(n-1)$

(5) \leftarrow $-\frac{a^2 z^{-1}}{1-a^2 z^{-1}} \quad |z| > a^{-1}$ $a^n u(-n-1) \times a$ $-\frac{z^{-1}}{1-a^2 z^{-1}} \quad |z| > a^{-1}$ $a^{n+1} u(-n-1)$

(3) \rightarrow $-\frac{1}{1-a^2 z^{-1}} \quad |z| > a^{-1}$ $a^n u(-n) \times a^{-1}$ $-\frac{a^2}{1-a^2 z^{-1}} \quad |z| > a^{-1}$ $a^{n-1} u(-n)$

Negative Exponent

(2) \leftarrow $\frac{1}{1-a^2 z} \quad |z| < a$ $a^{-n} u(n) \times a^{-1}$ $\frac{a^2}{1-a^2 z} \quad |z| < a$ $a^{-n-1} u(n)$

(8) \rightarrow $\frac{a^2 z}{1-a^2 z} \quad |z| < a$ $a^{-n} u(n-1) \times a$ $\frac{z}{1-a^2 z} \quad |z| < a$ $a^{-n+1} u(n-1)$

(6) \leftarrow $-\frac{az^{-1}}{1-az^{-1}} \quad |z| > a$ $a^{-n} u(-n-1) \times a^{-1}$ $-\frac{z^{-1}}{1-az^{-1}} \quad |z| > a$ $a^{-n-1} u(-n-1)$

(4) \rightarrow $-\frac{1}{1-az^{-1}} \quad |z| > a$ $a^{-n} u(-n) \times a$ $-\frac{a}{1-az^{-1}} \quad |z| > a$ $a^{-n+1} u(-n)$

Shifted Geometric Series (2)

by multiplying z or z^{-1}

Positive Exponent

(1) \leftarrow $\frac{az}{1-az} \quad |z| < a^{-1}$ $\overset{n \leftarrow n+1}{a^n u(n-1)} z^{-1}$ $\frac{a}{1-az} \quad |z| < a^{-1}$ $a^{n+1} u(n)$

(7) \rightarrow $\frac{1}{1-az} \quad |z| < a^{-1}$ $\overset{n \leftarrow n-1}{a^n u(n)} \times z$ $\frac{z}{1-az} \quad |z| < a^{-1}$ $a^{n-1} u(n-1)$

(5) \leftarrow $-\frac{1}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$ $\overset{n \leftarrow n+1}{a^n u(-n)} \times z^{-1}$ $-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$ $a^{n+1} u(-n-1)$

(3) \rightarrow $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$ $\overset{n \leftarrow n-1}{a^n u(-n-1)} \times z$ $-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$ $a^{n-1} u(-n)$

Negative Exponent

(2) \leftarrow $\frac{a^{-1}z}{1-a^{-1}z} \quad |z| < a$ $\overset{n \leftarrow n+1}{a^{-n} u(n-1)} \times z^{-1}$ $\frac{a^{-1}}{1-a^{-1}z} \quad |z| < a$ $a^{-n-1} u(n)$

(8) \rightarrow $\frac{1}{1-a^{-1}z} \quad |z| < a$ $\overset{n \leftarrow n-1}{a^{-n} u(n)} \times z$ $\frac{z}{1-a^{-1}z} \quad |z| < a$ $a^{-n+1} u(n-1)$

(6) \leftarrow $-\frac{1}{1-a^{-1}z^{-1}} \quad |z| > a$ $\overset{n \leftarrow n+1}{a^{-n} u(-n)} \times z^{-1}$ $-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a$ $a^{-n-1} u(-n-1)$

(4) \rightarrow $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a$ $\overset{n \leftarrow n-1}{a^{-n} u(-n-1)} \times z$ $-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a$ $a^{-n+1} u(-n)$

Shifted Geometric Series (3)

by multiplying a or a^{-1}

Assume $a > 1$

- (1) $a^n u(n)$ $*a \leftarrow$ $a^{n+1} u(n)$
- (2) $a^{-n} u(n)$ $/a \leftarrow$ $a^{-n-1} u(n)$
- (3) $a^n u(-n)$ $/a \rightarrow$ $a^{n-1} u(-n)$
- (4) $a^{-n} u(-n)$ $*a \rightarrow$ $a^{-n+1} u(-n)$
- (5) $a^n u(-n-1)$ $*a \leftarrow$ $a^{n+1} u(-n-1)$
- (6) $a^{-n} u(-n-1)$ $/a \leftarrow$ $a^{-n-1} u(-n-1)$
- (7) $a^n u(n-1)$ $/a \rightarrow$ $a^{n-1} u(n-1)$
- (8) $a^{-n} u(n-1)$ $*a \rightarrow$ $a^{-n+1} u(n-1)$

| | | | | | |
|-----|-----|------|------|---------------|---------------|
| (1) | (2) | $*a$ | $/a$ | \leftarrow | \leftarrow |
| (3) | (4) | $/a$ | $*a$ | \rightarrow | \rightarrow |
| (5) | (6) | $*a$ | $/a$ | \leftarrow | \leftarrow |
| (7) | (8) | $/a$ | $*a$ | \rightarrow | \rightarrow |

| | | | | | |
|-----|-----|------|------|---------------|---------------|
| (1) | (2) | $*a$ | $/a$ | \leftarrow | \leftarrow |
| (5) | (6) | $*a$ | $/a$ | \leftarrow | \leftarrow |
| (3) | (4) | $/a$ | $*a$ | \rightarrow | \rightarrow |
| (7) | (8) | $/a$ | $*a$ | \rightarrow | \rightarrow |

Shifted Geometric Series (4)

by multiplying z or z^{-1}

Assume $a > 1$

- (1) $a^n u(n)$ $*z \Rightarrow$ $a^{n-1} u(n-1)$
- (2) $a^{-n} u(n)$ $*z \Rightarrow$ $a^{-n+1} u(n-1)$
- (3) $a^n u(-n)$ $/z \Leftarrow$ $a^{n+1} u(-n-1)$
- (4) $a^{-n} u(-n)$ $/z \Leftarrow$ $a^{-n-1} u(-n-1)$
- (5) $a^n u(-n-1)$ $*z \Rightarrow$ $a^{n-1} u(-n)$
- (6) $a^{-n} u(-n-1)$ $*z \Rightarrow$ $a^{-n+1} u(-n)$
- (7) $a^n u(n-1)$ $/z \Leftarrow$ $a^{n+1} u(n)$
- (8) $a^{-n} u(n-1)$ $/z \Leftarrow$ $a^{-n-1} u(n)$

| | | | | | |
|-----|-----|------|------|---------------|---------------|
| (1) | (2) | $*z$ | $*z$ | \Rightarrow | \Rightarrow |
| (3) | (4) | $/z$ | $/z$ | \Leftarrow | \Leftarrow |
| (5) | (6) | $*z$ | $*z$ | \Rightarrow | \Rightarrow |
| (7) | (8) | $/z$ | $/z$ | \Leftarrow | \Leftarrow |

| | | | | | |
|-----|-----|------|------|---------------|---------------|
| (1) | (2) | $*z$ | $*z$ | \Rightarrow | \Rightarrow |
| (5) | (6) | $*z$ | $*z$ | \Rightarrow | \Rightarrow |
| (3) | (4) | $/z$ | $/z$ | \Leftarrow | \Leftarrow |
| (7) | (8) | $/z$ | $/z$ | \Leftarrow | \Leftarrow |

Assume $a > 1$

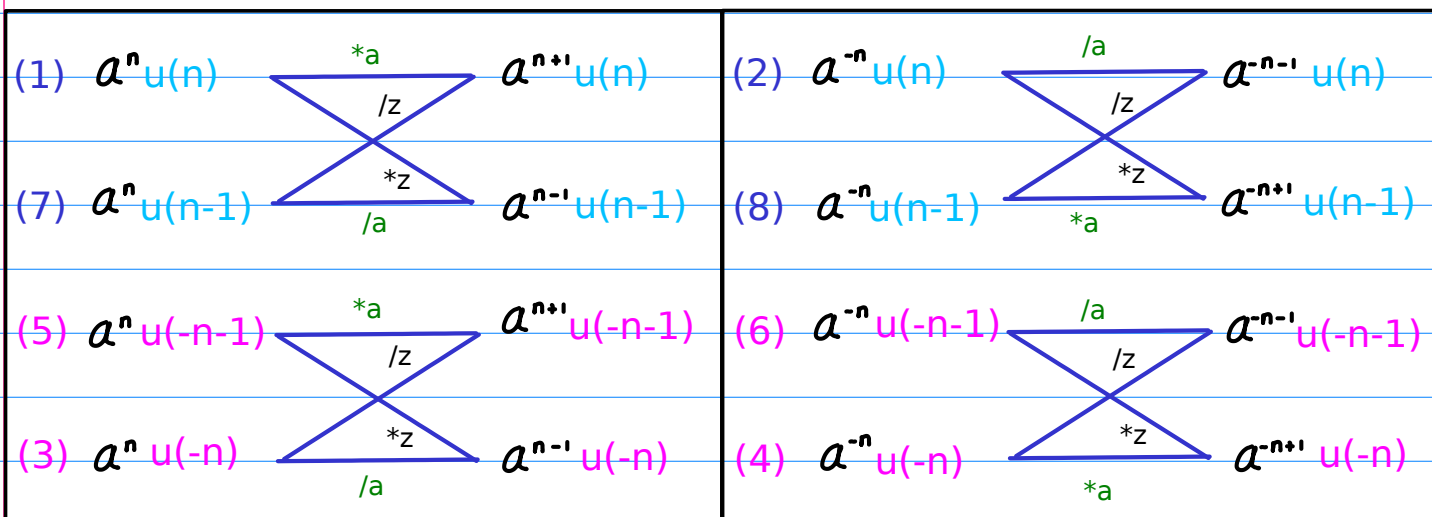
| | | |
|-----|------------------|----------------------------------|
| (1) | $a^n u(n)$ | $*a \leftarrow a^{n+1} u(n)$ |
| (7) | $a^n u(n-1)$ | $/a \rightarrow a^{n-1} u(n-1)$ |
| (5) | $a^n u(-n-1)$ | $*a \leftarrow a^{n+1} u(-n-1)$ |
| (3) | $a^n u(-n)$ | $/a \rightarrow a^{n-1} u(-n)$ |
| (2) | $a^{-n} u(n)$ | $/a \leftarrow a^{-n-1} u(n)$ |
| (8) | $a^{-n} u(n-1)$ | $*a \rightarrow a^{-n+1} u(n-1)$ |
| (6) | $a^{-n} u(-n-1)$ | $/a \leftarrow a^{-n-1} u(-n-1)$ |
| (4) | $a^{-n} u(-n)$ | $*a \rightarrow a^{-n+1} u(-n)$ |

| | |
|---------------|-------------------|
| $2^n u(n)$ | $2^{n+1} u(n)$ |
| $2^n u(n-1)$ | $2^{n-1} u(n-1)$ |
| $2^n u(-n-1)$ | $2^{n+1} u(-n-1)$ |
| $2^n u(-n)$ | $2^{n-1} u(-n)$ |

| | |
|------------------------------|--------------------------------|
| $(\frac{1}{2})^{-n} u(n)$ | $(\frac{1}{2})^{-n-1} u(n)$ |
| $(\frac{1}{2})^{-n} u(n-1)$ | $(\frac{1}{2})^{-n+1} u(n-1)$ |
| $(\frac{1}{2})^{-n} u(-n-1)$ | $(\frac{1}{2})^{-n-1} u(-n-1)$ |
| $(\frac{1}{2})^{-n} u(-n)$ | $(\frac{1}{2})^{-n+1} u(-n)$ |

| | |
|---------------------------|-------------------------------|
| $(\frac{1}{2})^n u(n)$ | $(\frac{1}{2})^{n+1} u(n)$ |
| $(\frac{1}{2})^n u(n-1)$ | $(\frac{1}{2})^{n-1} u(n-1)$ |
| $(\frac{1}{2})^n u(-n-1)$ | $(\frac{1}{2})^{n+1} u(-n-1)$ |
| $(\frac{1}{2})^n u(-n)$ | $(\frac{1}{2})^{n-1} u(-n)$ |

| | |
|------------------|--------------------|
| $2^{-n} u(n)$ | $2^{-n-1} u(n)$ |
| $2^{-n} u(n-1)$ | $2^{-n+1} u(n-1)$ |
| $2^{-n} u(-n-1)$ | $2^{-n-1} u(-n-1)$ |
| $2^{-n} u(-n)$ | $2^{-n+1} u(-n)$ |



(1) *a *z
 (7) /a /z
 (5) *a *z
 (3) /a /z

(2) /a *z
 (8) *a /z
 (6) /a *z
 (4) *a /z

\leftarrow (1) *a = (7) /z \Leftarrow \leftarrow (2) /a = (8) /z \Leftarrow
 \Rightarrow (1) *z = (7) /a \rightarrow \Rightarrow (2) *z = (8) *a \rightarrow
 \leftarrow (5) *a = (3) /z \Leftarrow \leftarrow (6) /a = (4) /z \Leftarrow
 \Rightarrow (5) *z = (3) /a \rightarrow \Rightarrow (6) *z = (4) *a \rightarrow

(1) *a *z
 (5) *a *z
 (3) /z /a
 (7) /z /a

(2) /a *z
 (6) /a *z
 (4) *a /z
 (8) *a /z

| | | | | |
|---------------|-----------------|---------------|-----------------|---------------|
| \leftarrow | (1) *a = (7) /z | \leftarrow | (2) /a = (8) /z | \leftarrow |
| \Rightarrow | (1) *z = (7) /a | \rightarrow | (2) *z = (8) *a | \rightarrow |
| \leftarrow | (5) *a = (3) /z | \leftarrow | (6) /a = (4) /z | \leftarrow |
| \Rightarrow | (5) *z = (3) /a | \rightarrow | (6) *z = (4) *a | \rightarrow |

| | | | |
|--------|----|--------|----|
| (1) *a | *z | (2) /a | *z |
| (5) *a | *z | (6) /a | *z |
| (3) /a | /z | (4) *a | /z |
| (7) /a | /z | (8) *a | /z |

| | | | |
|--------|--------------|--------|--------------|
| (1) *a | \leftarrow | (2) /a | \leftarrow |
| (5) *a | \leftarrow | (6) /a | \leftarrow |
| (3) /z | \leftarrow | (4) /z | \leftarrow |
| (7) /z | \leftarrow | (8) /z | \leftarrow |

| | | | |
|--------|---------------|--------|---------------|
| (1) *z | \Rightarrow | (2) *z | \Rightarrow |
| (5) *z | \Rightarrow | (6) *z | \Rightarrow |
| (3) /a | \rightarrow | (4) *a | \rightarrow |
| (7) /a | \rightarrow | (8) *a | \rightarrow |

| | | | | | |
|---|-----------------|---|---|-----------------|---|
| ← | (1) *a = (7) /z | ↔ | ← | (2) /a = (8) /z | ↔ |
| ⇒ | (1) *z = (7) /a | → | ⇒ | (2) *z = (8) *a | → |
| ↔ | (3) /z = (5) *a | ← | ↔ | (4) /z = (6) /a | ← |
| → | (3) /a = (5) *z | ⇒ | → | (4) *a = (6) *z | ⇒ |

| | | | |
|--------|----|--------|----|
| (1) *a | *z | (2) /a | *z |
| (3) /z | /a | (4) /z | *a |
| (5) *a | *z | (6) /a | *z |
| (7) /z | /a | (8) /z | *a |

| | | | | | |
|---|--------|---|---|--------|---|
| [| (1) *a | ← | [| (2) /a | ← |
| | (3) /z | ↔ | | (4) /z | ↔ |
| | (5) *a | ← | | (6) /a | ← |
| | (7) /z | ↔ | | (8) /z | ↔ |

| | | | | | |
|---|--------|---|---|--------|---|
| [| (1) *z | ⇒ | [| (2) *z | ⇒ |
| | (3) /a | → | | (4) *a | → |
| | (5) *z | ⇒ | | (6) *z | ⇒ |
| | (7) /a | → | | (8) *a | → |

| | | | |
|-----------------------|-----------|-----|-----|
| Complement Type I | $u(n)$ | (1) | (2) |
| | $u(-n-1)$ | (5) | (6) |
| Complement Type II | $u(-n)$ | (3) | (4) |
| | $u(n-1)$ | (7) | (8) |

$$\begin{array}{l} (1) \quad a z \quad \mathbf{a}^n \\ (5) \quad a^{-1} z^{-1} \quad \mathbf{a}^n \end{array}$$

$$\begin{array}{l} (2) \quad a^{-1} z \quad \mathbf{a}^{-n} \\ (6) \quad a z^{-1} \quad \mathbf{a}^{-n} \end{array}$$

$$\begin{array}{l} (3) \quad a^{-1} z^{-1} \quad \mathbf{a}^n \\ (7) \quad a z \quad \mathbf{a}^n \end{array}$$

$$\begin{array}{l} (4) \quad a z^{-1} \quad \mathbf{a}^{-n} \\ (8) \quad a^{-1} z \quad \mathbf{a}^{-n} \end{array}$$

$$\begin{array}{l} (1) \quad a z \quad \mathbf{*a} \\ (5) \quad a^{-1} z^{-1} \quad \mathbf{*a} \end{array}$$

$$\begin{array}{l} (2) \quad a^{-1} z \quad \mathbf{/a} \\ (6) \quad a z^{-1} \quad \mathbf{/a} \end{array}$$

$$\begin{array}{l} (3) \quad a^{-1} z^{-1} \quad \mathbf{/a} \\ (7) \quad a z \quad \mathbf{/a} \end{array}$$

$$\begin{array}{l} (4) \quad a z^{-1} \quad \mathbf{*a} \\ (8) \quad a^{-1} z \quad \mathbf{*a} \end{array}$$

$$\begin{array}{l} (1) \quad a z \quad \mathbf{*z} \\ (5) \quad a^{-1} z^{-1} \quad \mathbf{*z} \end{array}$$

$$\begin{array}{l} (2) \quad a^{-1} z \quad \mathbf{*z} \\ (6) \quad a z^{-1} \quad \mathbf{*z} \end{array}$$

$$\begin{array}{l} (3) \quad a^{-1} z^{-1} \quad \mathbf{/z} \\ (7) \quad a z \quad \mathbf{/z} \end{array}$$

$$\begin{array}{l} (4) \quad a z^{-1} \quad \mathbf{/z} \\ (8) \quad a^{-1} z \quad \mathbf{/z} \end{array}$$

Geometric Series Combinations

(1)

| | | |
|----------|---|----------------|
| unit | $\frac{1}{1-az}$ $ z < a^{-1}$ | $a^n u(n)$ |
| non-unit | $-\frac{a^nz^{-1}}{1-a^nz^{-1}}$ $ z > a^{-1}$ | $-a^n u(-n-1)$ |

(2)

| | | |
|----------|--|----------------------------|
| unit | $\frac{1}{1-a^{-1}z}$ $ z < a$ | $(\frac{1}{a})^n u(n)$ |
| non-unit | $-\frac{a^nz^{-1}}{1-a^nz^{-1}}$ $ z > a$ | $-(\frac{1}{a})^n u(-n-1)$ |

(5)

(3)

| | | |
|----------|--|--------------|
| unit | $-\frac{1}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$ | $-a^n u(-n)$ |
| non-unit | $\frac{az}{1-az}$ $ z < a^{-1}$ | $a^n u(n-1)$ |

(6)

(4)

| | | |
|----------|------------------------------------|--------------------------|
| unit | $-\frac{1}{1-a^nz^{-1}}$ $ z > a$ | $-(\frac{1}{a})^n u(-n)$ |
| non-unit | $\frac{a^nz}{1-a^nz}$ $ z < a$ | $(\frac{1}{a})^n u(n-1)$ |

(7)

(8)

| | | Positive Exponent | Negative Exponent |
|--------------------|-----------|-------------------|-------------------|
| Complement Type I | $u(n)$ | (1) | (2) |
| | $u(-n-1)$ | (5) | (6) |
| Complement Type II | $u(-n)$ | (3) | (4) |
| | $u(n-1)$ | (7) | (8) |

Shifted Combinations (I) by scaling $*a$ / a

(1) $*a$

| | |
|---|--------------------|
| $\frac{a}{1-az}$ $ z < a^{-1}$ | $a^{n+1} u(n)$ |
| $-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$ | $-a^{n+1} u(-n-1)$ |

(2) / a

| | |
|--|--------------------------------|
| $\frac{a^{-1}}{1-a^{-1}z}$ $ z < a$ | $(\frac{1}{a})^{n+1} u(n)$ |
| $-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $ z > a$ | $-(\frac{1}{a})^{n+1} u(-n-1)$ |

(5) $*a$

(3) / a

| | |
|---|------------------|
| $-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$ | $-a^{n-1} u(-n)$ |
| $\frac{z}{1-az}$ $ z < a^{-1}$ | $a^{n-1} u(n-1)$ |

(6) / a

(4) $*a$

| | |
|----------------------------------|------------------------------|
| $-\frac{a}{1-az^{-1}}$ $ z > a$ | $-(\frac{1}{a})^{n-1} u(-n)$ |
| $\frac{z}{1-a^{-1}z}$ $ z < a$ | $(\frac{1}{a})^{n-1} u(n-1)$ |

(7) / a

(8) $*a$

| | Positive Exponent | Negative Exponent | | |
|--------------------|-------------------|-------------------|----------------|-----------------|
| Complement Type I | (1) $u(n)$ | (2) $u(n)$ | $*a$ $u(n)$ | / a $u(n)$ |
| | (5) $u(-n-1)$ | (6) $u(-n-1)$ | $*a$ $u(-n-1)$ | / a $u(-n-1)$ |
| Complement Type II | (3) $u(-n)$ | (4) $u(-n)$ | / a $u(-n)$ | $*a$ $u(-n)$ |
| | (7) $u(n-1)$ | (8) $u(n-1)$ | / a $u(n-1)$ | $*a$ $u(n-1)$ |

Shifted Combinations (II) by scaling $*z$ $/z$

(1) $*z$

| | |
|--|------------------|
| $\frac{z}{1-az}$ $ z < a^{-1}$ | $a^{n-1} u(n-1)$ |
| $-\frac{a}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$ | $-a^{n-1} u(-n)$ |

(2) $*z$

| | |
|----------------------------------|------------------------------|
| $\frac{z}{1-a^{-1}z}$ $ z < a$ | $(\frac{1}{a})^{n-1} u(n-1)$ |
| $-\frac{a}{1-az^{-1}}$ $ z > a$ | $-(\frac{1}{a})^{n-1} u(-n)$ |

(5) $*z$

(3) $/z$

| | |
|---|--------------------|
| $-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$ | $-a^{n+1} u(-n-1)$ |
| $\frac{a}{1-az}$ $ z < a^{-1}$ | $a^{n+1} u(n)$ |

(6) $*z$

(4) $/z$

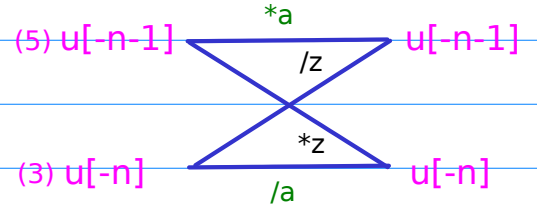
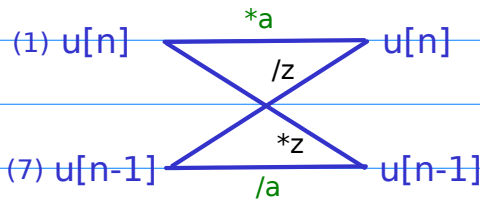
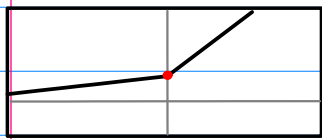
| | |
|---------------------------------------|--------------------------------|
| $-\frac{z^{-1}}{1-az^{-1}}$ $ z > a$ | $-(\frac{1}{a})^{n+1} u(-n-1)$ |
| $\frac{a^{-1}}{1-a^{-1}z}$ $ z < a$ | $(\frac{1}{a})^{n+1} u(n)$ |

(7) $/z$

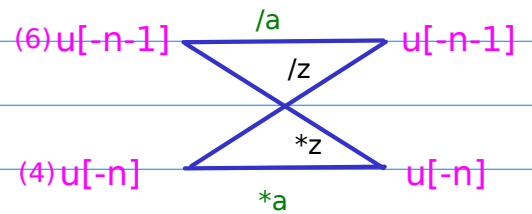
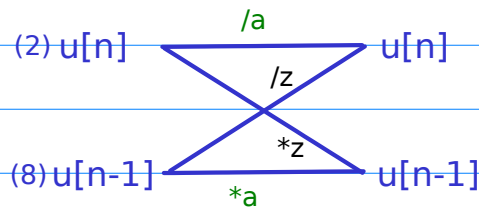
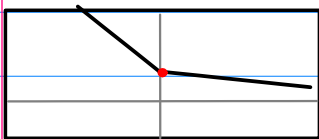
(8) $/z$

| | Positive Exponent | Negative Exponent | | |
|--------------------|-------------------|-------------------|----------------|----------------|
| Complement Type I | (1) $u(n)$ | (2) $u(n)$ | $*z$ $u(n-1)$ | $*z$ $u(n-1)$ |
| | (5) $u(-n-1)$ | (6) $u(-n-1)$ | $*z$ $u(-n)$ | $*z$ $u(-n)$ |
| Complement Type II | (3) $u(-n)$ | (4) $u(-n)$ | $/z$ $u(-n-1)$ | $/z$ $u(-n-1)$ |
| | (7) $u(n-1)$ | (8) $u(n-1)$ | $/z$ $u(n)$ | $/z$ $u(n)$ |

a^n



a^{-n}



| | | | |
|--------------------|-----------|-----|-----|
| Causal | $u(n)$ | (1) | (2) |
| | $u(n-1)$ | (7) | (8) |
| Anti-Causal | $u(-n-1)$ | (5) | (6) |
| | $u(-n)$ | (3) | (4) |

| Positive Exponent | Negative Exponent |
|--------------------------|--------------------------|
| (1) | (2) |
| (3) | (4) |
| (5) | (6) |
| (7) | (8) |

$$(1) a^n u(n) \xrightarrow[*a]{/z} a^{n+1} u(n)$$

$$(7) a^n u(n-1) \xrightarrow[*z]{/a} a^{n-1} u(n-1)$$

$$(5) a^n u(-n-1) \xrightarrow[*a]{/z} a^{n+1} u(-n-1)$$

$$(3) a^n u(-n) \xrightarrow[*z]{/a} a^{n-1} u(-n)$$

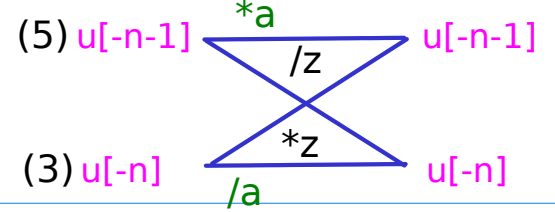
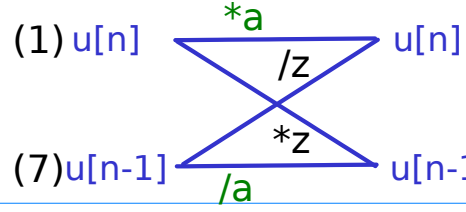
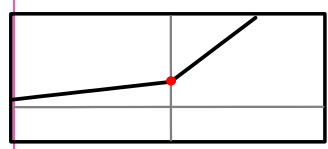
$$(2) a^{-n} u(n) \xrightarrow{/a} a^{-n-1} u(n)$$

$$(8) a^{-n} u(n-1) \xrightarrow[*a]{/z} a^{-n+1} u(n-1)$$

$$(6) a^{-n} u(-n-1) \xrightarrow{/a} a^{-n-1} u(-n-1)$$

$$(4) a^{-n} u(-n) \xrightarrow[*z]{/a} a^{-n+1} u(-n)$$

a^n



(1) $*a$

| | |
|--------------------------------------|-------------------------------------|
| $\frac{1}{1-az} \quad z < a^{-1}$ | $\frac{a}{1-az} \quad z < a^{-1}$ |
| $\frac{az}{1-az} \quad z < a^{-1}$ | $\frac{z}{1-az} \quad z < a^{-1}$ |

(7) $/a$

(1) $*a$

| | |
|--|--|
| $a^n u(n)$ (a^0, a^1, a^2, \dots) | $a^{n+1} u(n)$ (a^1, a^2, a^3, \dots) |
| $a^n u(n-1)$ (a^1, a^2, a^3, \dots) | $a^{n-1} u(n-1)$ (a^0, a^1, a^2, \dots) |

(7) $/a$

(1) $*z$

| | |
|--------------------------------------|-------------------------------------|
| $\frac{1}{1-az} \quad z < a^{-1}$ | $\frac{z}{1-az} \quad z < a^{-1}$ |
| $\frac{az}{1-az} \quad z < a^{-1}$ | $\frac{a}{1-az} \quad z < a^{-1}$ |

(7) $/z$

(1) $*z$

| | |
|--|--|
| $a^n u(n)$ (a^0, a^1, a^2, \dots) | $a^{n-1} u(n-1)$ (a^0, a^1, a^2, \dots) |
| $a^n u(n-1)$ (a^1, a^2, a^3, \dots) | $a^{n+1} u(n)$ (a^1, a^2, a^3, \dots) |

(7) $/z$

(5) $*a$

| | |
|---|---|
| $-\frac{a'z^{-1}}{1-a'z^{-1}} \quad z > a^{-1}$ | $-\frac{z^{-1}}{1-a'z^{-1}} \quad z > a^{-1}$ |
| $-\frac{1}{1-a'z^{-1}} \quad z > a^{-1}$ | $-\frac{a'}{1-a'z^{-1}} \quad z > a^{-1}$ |

(3) $/a$

(5) $*a$

| | |
|--|--|
| $-a^n u(-n-1)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$ | $-a^{n+1} u(-n-1)$ $-(\dots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$ |
| $-a^n u(-n)$ $-(\dots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$ | $-a^{n-1} u(-n)$ $-(\dots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$ |

(3) $/a$

(5) $*z$

| | |
|---|---|
| $-\frac{a'z^{-1}}{1-a'z^{-1}} \quad z > a^{-1}$ | $-\frac{a'}{1-a'z^{-1}} \quad z > a^{-1}$ |
| $-\frac{1}{1-a'z^{-1}} \quad z > a^{-1}$ | $-\frac{z^{-1}}{1-a'z^{-1}} \quad z > a^{-1}$ |

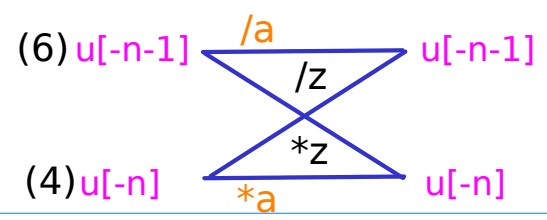
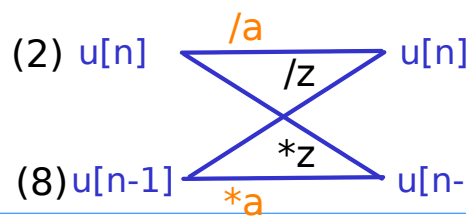
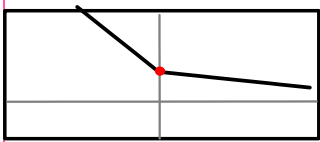
(3) $/z$

(5) $*z$

| | |
|---|--|
| $-a^n u(-n-1)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$ | $-a^{n-1} u(-n)$ $-(\dots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$ |
| $-a^n u(-n)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$ | $-a^{n+1} u(-n-1)$ $-(\dots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$ |

(3) $/z$

a^{-n}



(2) $/a$

| | |
|---|--|
| $\frac{1}{1-a^{-1}z} \quad z < a$ | $\frac{a^{-1}}{1-a^{-1}z} \quad z < a$ |
| $\frac{a^{-1}z}{1-a^{-1}z} \quad z < a$ | $\frac{z}{1-a^{-1}z} \quad z < a$ |

(2) $/a$

| | |
|--|--|
| $(\frac{1}{a})^n u(n)$ $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$ | $(\frac{1}{a})^{n+1} u(n)$ $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$ |
| $(\frac{1}{a})^n u(n-1)$ $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$ | $(\frac{1}{a})^{n-1} u(n-1)$ $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$ |

(8) $*a$

(8) $*a$

(2) $*z$

| | |
|---|--|
| $\frac{1}{1-a^{-1}z} \quad z < a$ | $\frac{z}{1-a^{-1}z} \quad z < a$ |
| $\frac{a^{-1}z}{1-a^{-1}z} \quad z < a$ | $\frac{a^{-1}}{1-a^{-1}z} \quad z < a$ |

(2) $*z$

| | |
|--|--|
| $(\frac{1}{a})^n u(n)$ $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$ | $(\frac{1}{a})^{n-1} u(n-1)$ $(\frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^3}, \dots)$ |
| $(\frac{1}{a})^n u(n-1)$ $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$ | $(\frac{1}{a})^{n+1} u(n)$ $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$ |

(8) $/z$

(8) $/z$

(6) $/a$

| | |
|--|---|
| $-\frac{az^{-1}}{1-az^{-1}} \quad z > a$ | $-\frac{z^{-1}}{1-az^{-1}} \quad z > a$ |
| $-\frac{1}{1-az^{-1}} \quad z > a$ | $-\frac{a}{1-az^{-1}} \quad z > a$ |

(6) $/a$

| | |
|---|--|
| $-(\frac{1}{a})^n u(-n-1)$ $-(\dots, a^3, a^2, a^1)$ | $-(\frac{1}{a})^{n+1} u(-n-1)$ $-(\dots, a^2, a^1, a^0)$ |
| $-(\frac{1}{a})^n u(-n)$ $-(\dots, a^2, a^1, a^0)$ | $-(\frac{1}{a})^{n-1} u(-n)$ $-(\dots, a^1, a^0, a^{-1})$ |

(4) $*a$

(4) $*a$

(6) $*z$

| | |
|--|---|
| $-\frac{az^{-1}}{1-az^{-1}} \quad z > a$ | $-\frac{a}{1-az^{-1}} \quad z > a$ |
| $-\frac{1}{1-az^{-1}} \quad z > a$ | $-\frac{z^{-1}}{1-az^{-1}} \quad z > a$ |

(6) $*z$

| | |
|---|---|
| $-(\frac{1}{a})^n u(-n-1)$ $-(\dots, a^3, a^2, a^1)$ | $-(\frac{1}{a})^{n-1} u(-n)$ $-(\dots, a^3, a^2, a^1)$ |
| $-(\frac{1}{a})^n u(-n)$ $-(\dots, a^2, a^1, a^0)$ | $-(\frac{1}{a})^{n+1} u(-n-1)$ $-(\dots, a^2, a^1, a^0)$ |

(4) $/z$

(4) $/z$

Scale by **a**

1. Geometric Series

(1)

***a**

(2)

/a

| | | | | |
|----------|---|---|--|---|
| | $\frac{1}{1-az} \quad z < a^{-1}$ | $\frac{a}{1-az} \quad z < a^{-1}$ | $\frac{1}{1-a^{-1}z} \quad z < a$ | $\frac{a^{-1}}{1-a^{-1}z} \quad z < a$ |
| Comp.ROC | $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad z > a^{-1}$ | $-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad z > a^{-1}$ | $-\frac{az^{-1}}{1-az^{-1}} \quad z > a$ | $-\frac{z^{-1}}{1-az^{-1}} \quad z > a$ |

(5)

***a**

(6)

/a

(3)

/a

(4)

***a**

| | | | | |
|----------|--|---|---|--------------------------------------|
| | $-\frac{1}{1-a^{-1}z^{-1}} \quad z > a^{-1}$ | $-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad z > a^{-1}$ | $-\frac{1}{1-az^{-1}} \quad z > a$ | $-\frac{a}{1-az^{-1}} \quad z > a$ |
| Comp.ROC | $\frac{az}{1-az} \quad z < a^{-1}$ | $\frac{z}{1-az} \quad z < a^{-1}$ | $\frac{a^{-1}z}{1-a^{-1}z} \quad z < a$ | $\frac{z}{1-a^{-1}z} \quad z < a$ |

(7)

/a

(8)

***a**

| | | | | | | | |
|-----|---------------|---|-------------------|-----|------------------|---|--------------------|
| (1) | $a^n u(n)$ | $\begin{array}{c} \xrightarrow{*a} \\ /z \\ \xrightarrow{*z} \end{array}$ | $a^{n+1} u(n)$ | (2) | $a^{-n} u(n)$ | $\begin{array}{c} \xrightarrow{/a} \\ /z \\ \xrightarrow{*z} \end{array}$ | $a^{-n-1} u(n)$ |
| (7) | $a^n u(n-1)$ | $\begin{array}{c} \xrightarrow{*z} \\ /a \end{array}$ | $a^{n-1} u(n-1)$ | (8) | $a^{-n} u(n-1)$ | $\begin{array}{c} \xrightarrow{*a} \\ *z \end{array}$ | $a^{-n+1} u(n-1)$ |
| (5) | $a^n u(-n-1)$ | $\begin{array}{c} \xrightarrow{*a} \\ /z \\ \xrightarrow{*z} \end{array}$ | $a^{n+1} u(-n-1)$ | (6) | $a^{-n} u(-n-1)$ | $\begin{array}{c} \xrightarrow{/a} \\ /z \\ \xrightarrow{*z} \end{array}$ | $a^{-n-1} u(-n-1)$ |
| (3) | $a^n u(-n)$ | $\begin{array}{c} \xrightarrow{*z} \\ /a \end{array}$ | $a^{n-1} u(-n)$ | (4) | $a^{-n} u(-n)$ | $\begin{array}{c} \xrightarrow{*a} \\ *z \end{array}$ | $a^{-n+1} u(-n)$ |

Scale by **a**

2. Sequences

(1)

***a**

| |
|-----------------|
| $a^n u(n)$ |
| $- a^n u(-n-1)$ |

| |
|---------------------|
| $a^{n+1} u(n)$ |
| $- a^{n+1} u(-n-1)$ |

(2)

/a

| |
|----------------------------|
| $(\frac{1}{a})^n u(n)$ |
| $-(\frac{1}{a})^n u(-n-1)$ |

| |
|--------------------------------|
| $(\frac{1}{a})^{n+1} u(n)$ |
| $-(\frac{1}{a})^{n+1} u(-n-1)$ |

Comp.ROC

(5)

***a**

| |
|---------------|
| $- a^n u(-n)$ |
| $a^n u(n-1)$ |

| |
|-------------------|
| $- a^{n-1} u(-n)$ |
| $a^{n-1} u(n-1)$ |

(6)

/a

| |
|--------------------------|
| $-(\frac{1}{a})^n u(-n)$ |
| $(\frac{1}{a})^n u(n-1)$ |

| |
|------------------------------|
| $-(\frac{1}{a})^{n-1} u(-n)$ |
| $(\frac{1}{a})^{n-1} u(n-1)$ |

Comp.ROC

(3)

/a

(4)

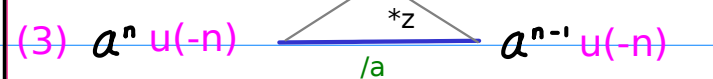
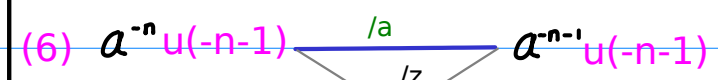
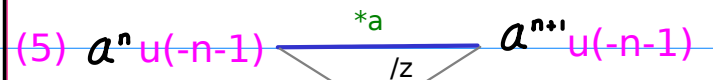
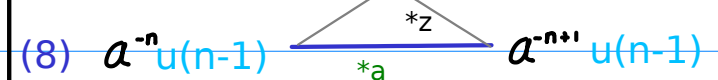
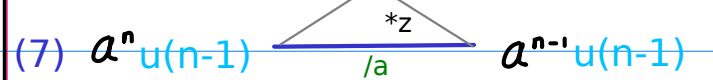
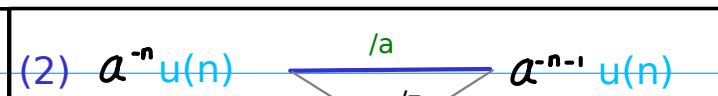
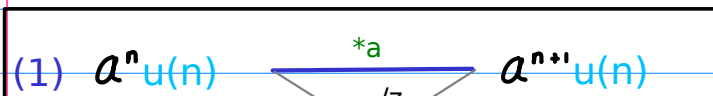
***a**

(7)

/a

(8)

***a**



Scale by **a**

3. Sequence values

(1)

***a**

| | |
|---|---|
| (a^0, a^1, a^2, \dots) | (a^1, a^2, a^3, \dots) |
| $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$ | $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$ |

(2)

/a

| | |
|--|--|
| $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$ | $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$ |
| $-(\dots, a^3, a^2, a^1)$ | $-(\dots, a^2, a^1, a^0)$ |

Comp.ROC

(5)

***a**

| | |
|---|---|
| $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$ | $-(\dots, \frac{1}{a^3}, \frac{1}{a^4}, \frac{1}{a^5})$ |
| (a^1, a^2, a^3, \dots) | (a^0, a^1, a^2, \dots) |

(6)

/a

| | |
|--|--|
| $-(\dots, a^2, a^1, a^0)$ | $-(\dots, a^3, a^2, a^1)$ |
| $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$ | $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$ |

Comp.ROC

(3)

/a

(7)

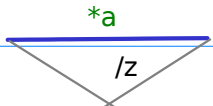
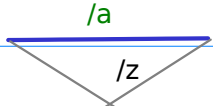
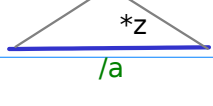
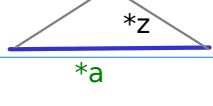
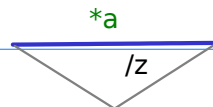
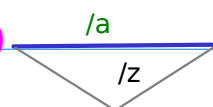
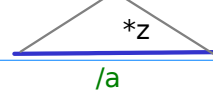
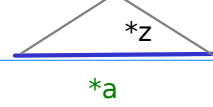
/a

(4)

***a**

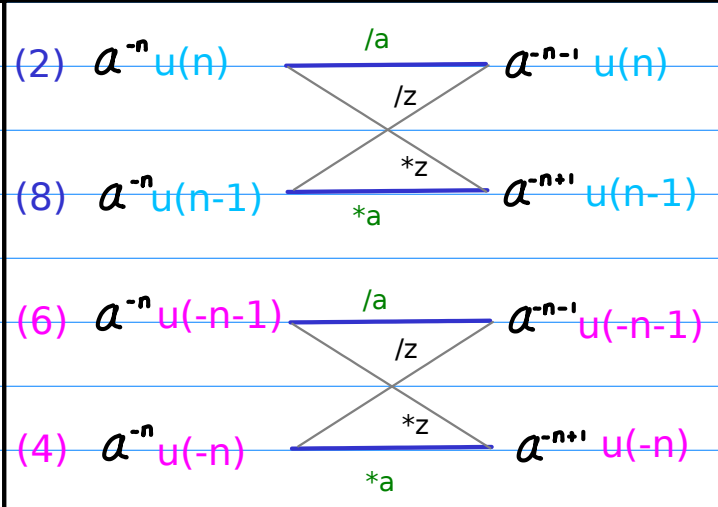
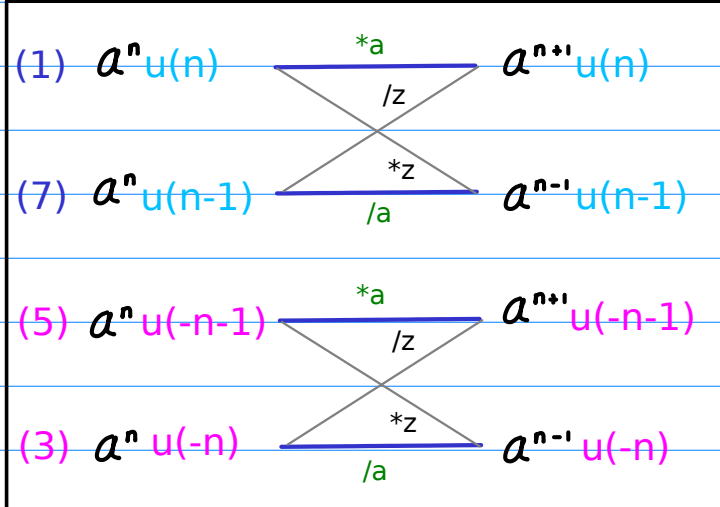
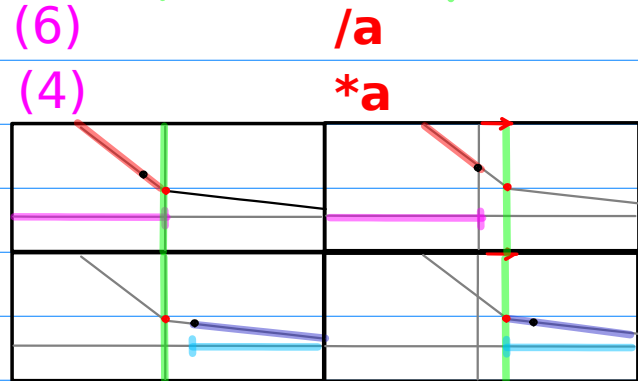
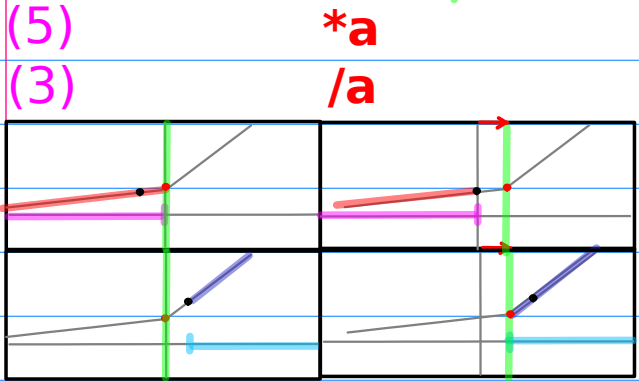
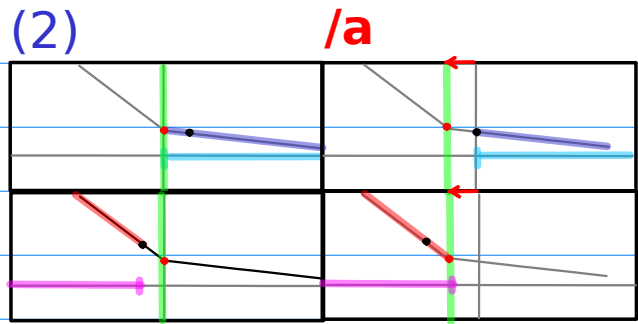
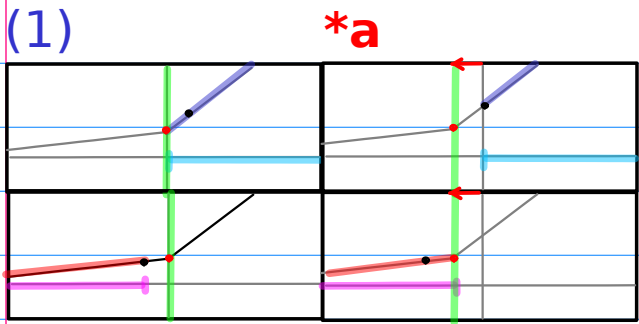
(8)

***a**

| | | | | | |
|-------------------|---|-------------------|----------------------|---|--------------------|
| (1) $a^n u(n)$ |  | $a^{n+1} u(n)$ | (2) $a^{-n} u(n)$ |  | $a^{-n-1} u(n)$ |
| (7) $a^n u(n-1)$ |  | $a^{n-1} u(n-1)$ | (8) $a^{-n} u(n-1)$ |  | $a^{-n+1} u(n-1)$ |
| (5) $a^n u(-n-1)$ |  | $a^{n+1} u(-n-1)$ | (6) $a^{-n} u(-n-1)$ |  | $a^{-n-1} u(-n-1)$ |
| (3) $a^n u(-n)$ |  | $a^{n-1} u(-n)$ | (4) $a^{-n} u(-n)$ |  | $a^{-n+1} u(-n)$ |

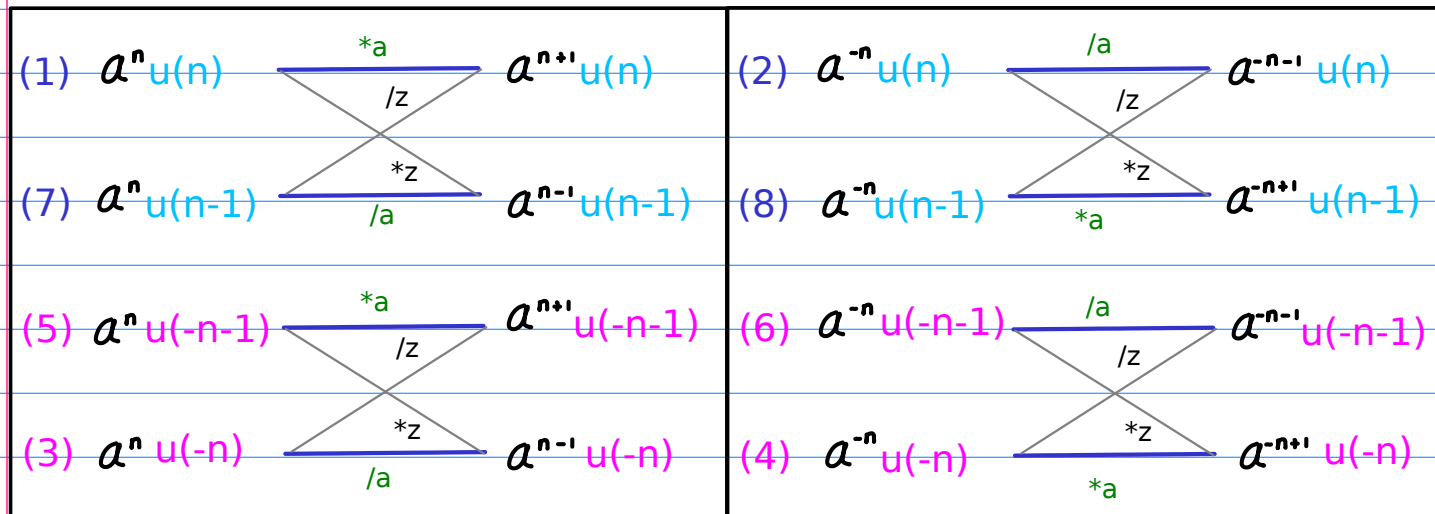
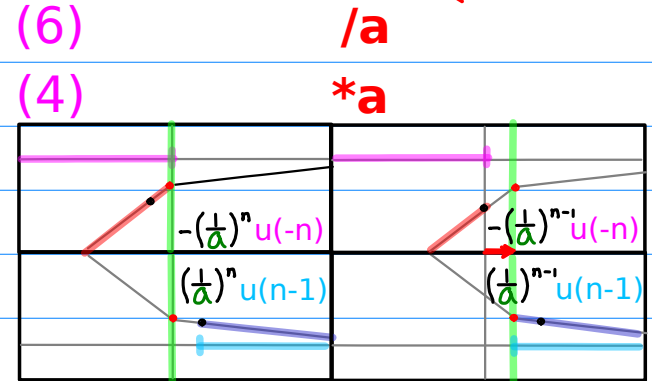
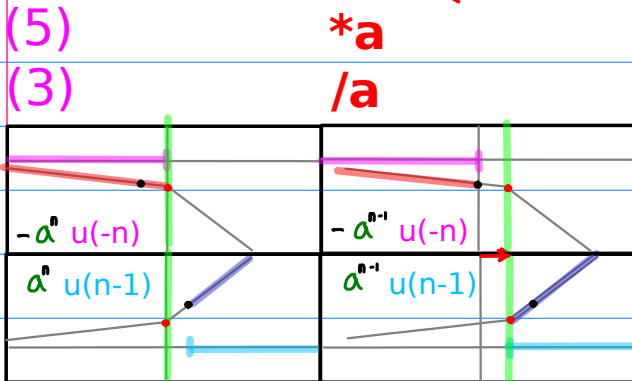
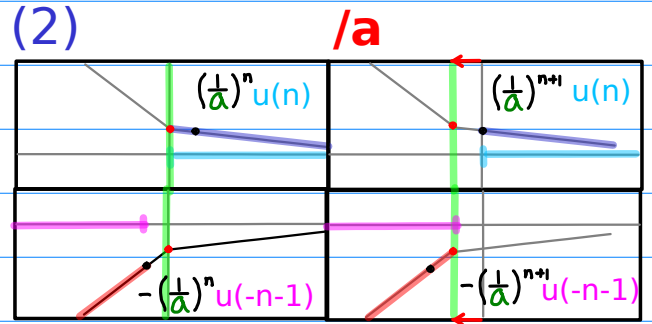
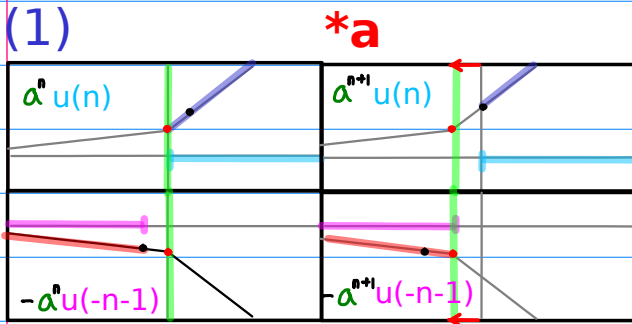
Scale by **a**

4. Graphs



Scale by **a**

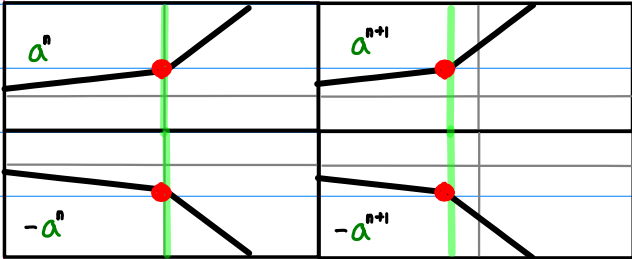
5. Graphs - signs



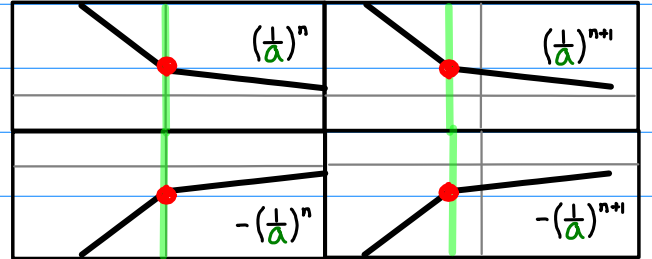
Scale by **a**

6. Graphs - Exponents

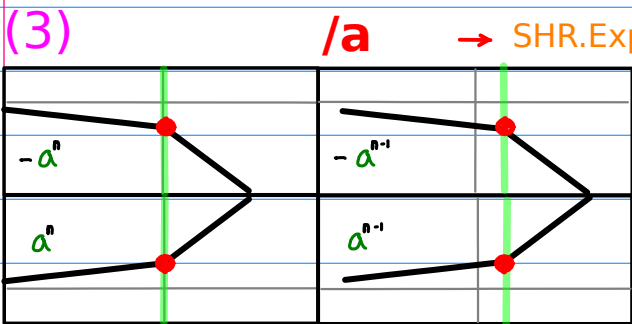
(1) ***a** ← SHL.Exp



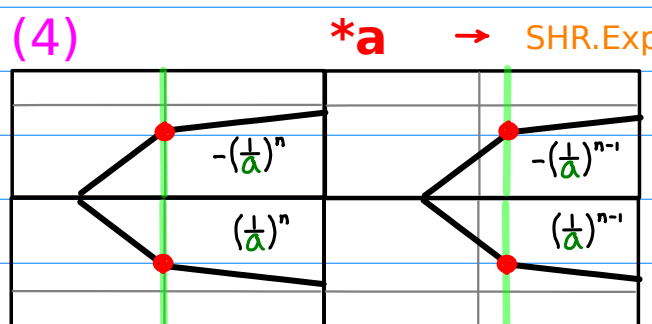
(2) **/a** ← SHL.Exp



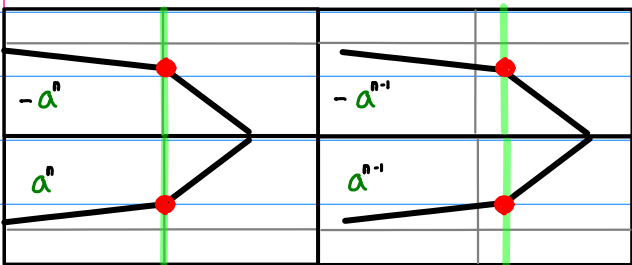
(5) ***a** ← SHL.Exp



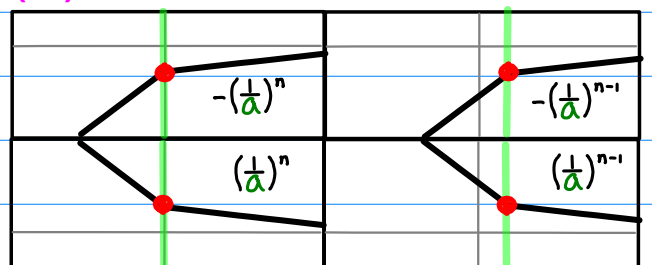
(6) **/a** ← SHL.Exp



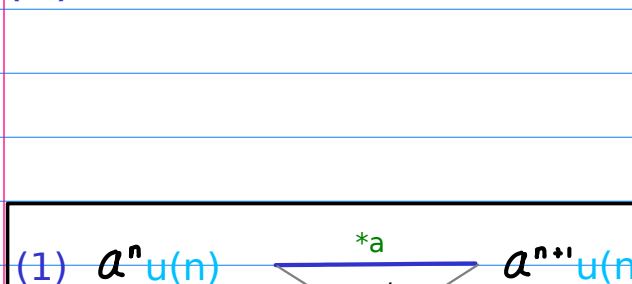
(3) **/a** → SHR.Exp



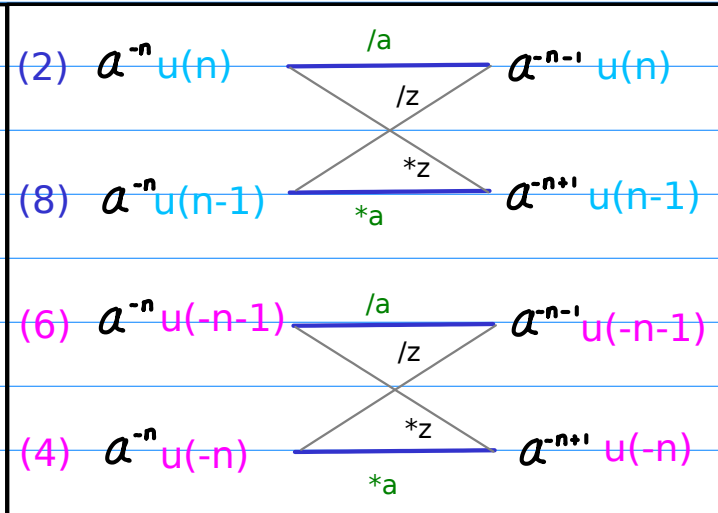
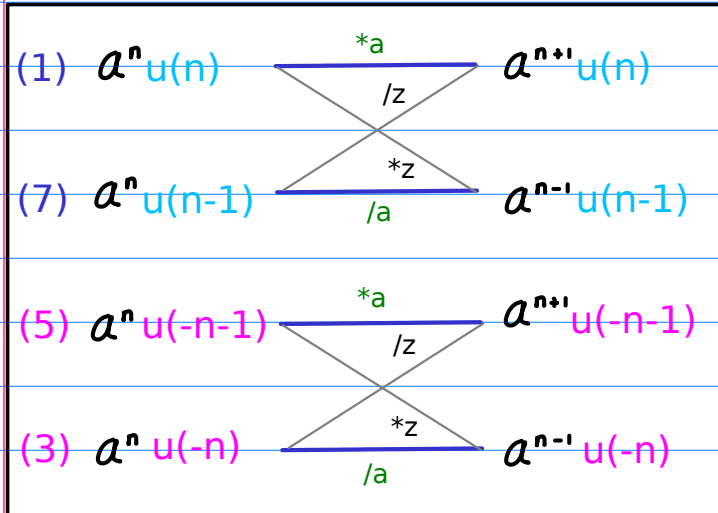
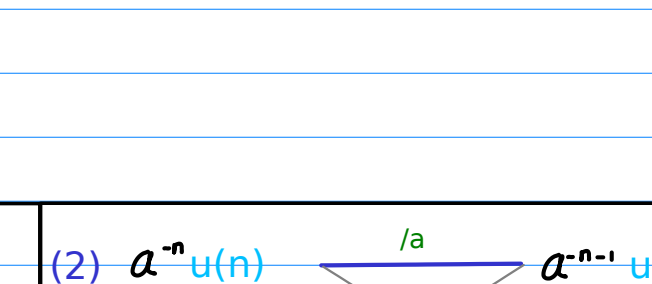
(4) ***a** → SHR.Exp



(7) **/a** → SHR.Exp

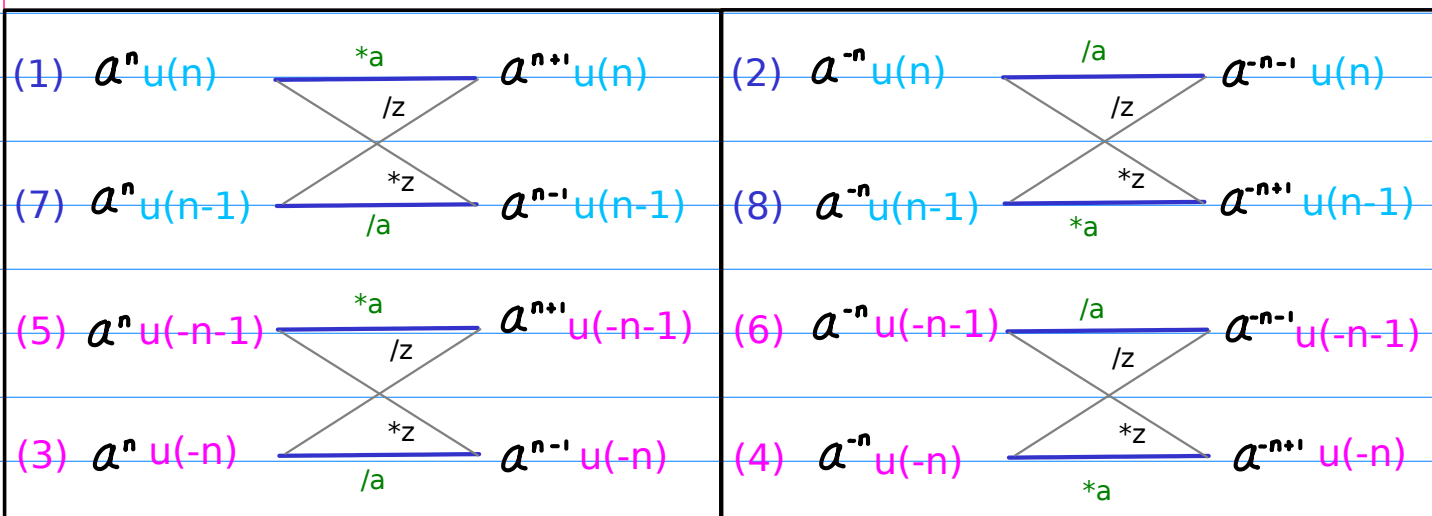
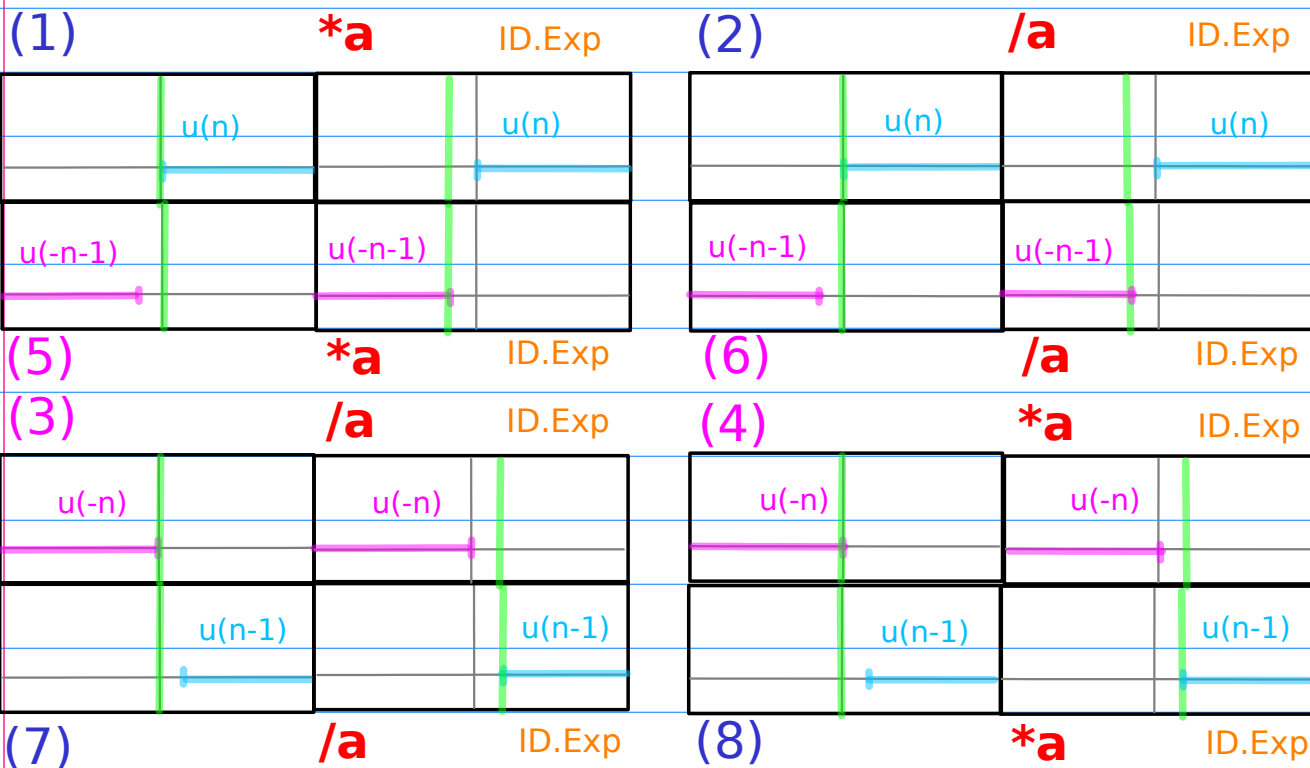


(8) ***a** → SHR.Exp



Scale by **a**

7. Graphs - Ranges



Scale by z

1. Geometric Series

(1)

$*z$

| | |
|---|---|
| $\frac{1}{1-az}$ $ z < a^{-1}$ | $\frac{z}{1-az}$ $ z < a^{-1}$ |
| $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$ | $-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$ |

(2)

$*z$

| | |
|--|--|
| $\frac{1}{1-a^{-1}z}$ $ z < a$ | $\frac{z}{1-a^{-1}z}$ $ z < a$ |
| $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $ z > a$ | $-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $ z > a$ |

(5)

$*z$

| | |
|--|---|
| $-\frac{1}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$ | $-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$ |
| $\frac{az}{1-az}$ $ z < a^{-1}$ | $\frac{a}{1-az}$ $ z < a^{-1}$ |

(6)

$*z$

| | |
|---------------------------------------|---------------------------------------|
| $-\frac{1}{1-az^{-1}}$ $ z > a$ | $-\frac{z^{-1}}{1-az^{-1}}$ $ z > a$ |
| $\frac{a^{-1}z}{1-a^{-1}z}$ $ z < a$ | $\frac{a^{-1}}{1-a^{-1}z}$ $ z < a$ |

(3)

$/z$

(7)

$/z$

(4)

$/z$

(8)

$/z$

| | | | | | |
|-------------------|------|-------------------|----------------------|------|--------------------|
| (1) $a^n u(n)$ | $*a$ | $a^{n+1} u(n)$ | (2) $a^{-n} u(n)$ | $/a$ | $a^{-n-1} u(n)$ |
| (7) $a^n u(n-1)$ | $/z$ | $a^{n-1} u(n-1)$ | (8) $a^{-n} u(n-1)$ | $*z$ | $a^{-n+1} u(n-1)$ |
| (5) $a^n u(-n-1)$ | $*a$ | $a^{n+1} u(-n-1)$ | (6) $a^{-n} u(-n-1)$ | $/a$ | $a^{-n-1} u(-n-1)$ |
| (3) $a^n u(-n)$ | $*z$ | $a^{n-1} u(-n)$ | (4) $a^{-n} u(-n)$ | $*z$ | $a^{-n+1} u(-n)$ |

Scale by z

2. Sequences

(1)

$*z$

| | |
|-----------------|-------------------|
| $a^n u(n)$ | $a^{n-1} u(n-1)$ |
| $- a^n u(-n-1)$ | $- a^{n-1} u(-n)$ |

(2)

$*z$

| | |
|----------------------------|------------------------------|
| $(\frac{1}{a})^n u(n)$ | $(\frac{1}{a})^{n-1} u(n-1)$ |
| $-(\frac{1}{a})^n u(-n-1)$ | $-(\frac{1}{a})^{n-1} u(-n)$ |

Comp.ROC

(5)

$*z$

| | |
|---------------|---------------------|
| $- a^n u(-n)$ | $- a^{n+1} u(-n-1)$ |
| $a^n u(n-1)$ | $a^{n+1} u(n)$ |

(6)

$*z$

| | |
|--------------------------|--------------------------------|
| $-(\frac{1}{a})^n u(-n)$ | $-(\frac{1}{a})^{n+1} u(-n-1)$ |
| $(\frac{1}{a})^n u(n-1)$ | $(\frac{1}{a})^{n+1} u(n)$ |

Comp.ROC

(3)

$/z$

(7)

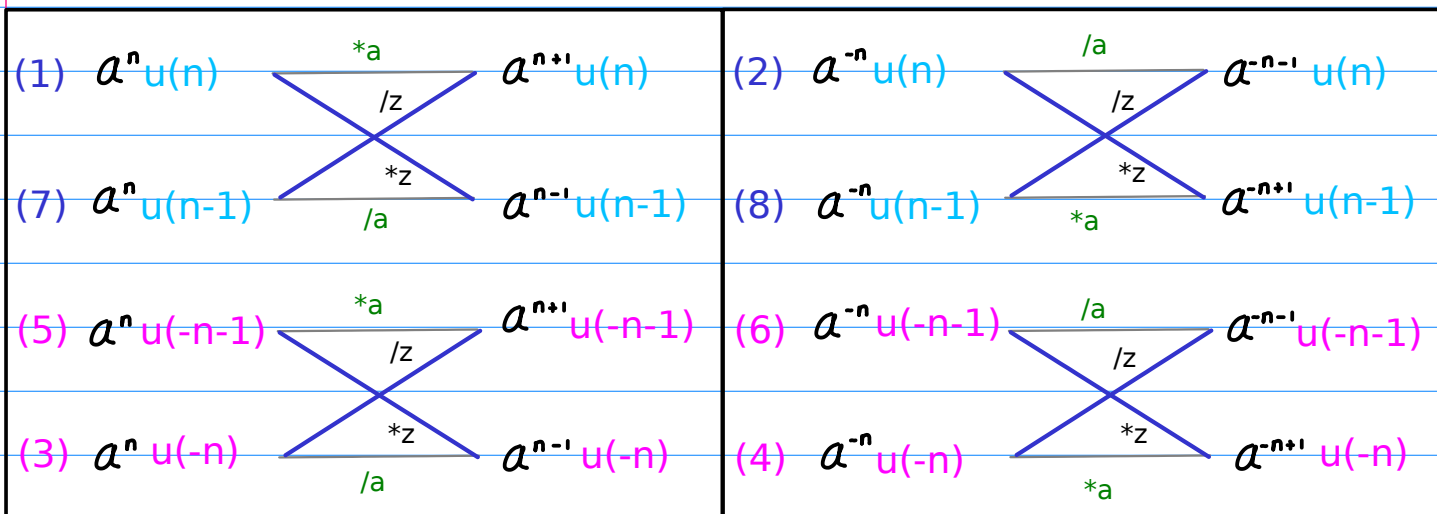
$/z$

(4)

$/z$

(8)

$/z$



Scale by z

3. Sequence values

(1)

$*z$

| | |
|---|---|
| (a^0, a^1, a^2, \dots) | (a^0, a^1, a^2, \dots) |
| $-(\dots, \frac{1}{a^2}, \frac{1}{a^2}, \frac{1}{a^1})$ | $-(\dots, \frac{1}{a^2}, \frac{1}{a^2}, \frac{1}{a^1})$ |

(2)

$*z$

| | |
|--|--|
| $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$ | $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$ |
| $-(\dots, a^3, a^2, a^1)$ | $-(\dots, a^3, a^2, a^1)$ |

Comp.ROC

(5)

$*z$

| | |
|---|---|
| $-(\dots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$ | $-(\dots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$ |
| (a^1, a^2, a^3, \dots) | (a^1, a^2, a^3, \dots) |

(6)

$*z$

| | |
|--|--|
| $-(\dots, a^2, a^1, a^0)$ | $-(\dots, a^2, a^1, a^0)$ |
| $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$ | $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$ |

Comp.ROC

(3)

$/z$

(7)

$/z$

(4)

$/z$

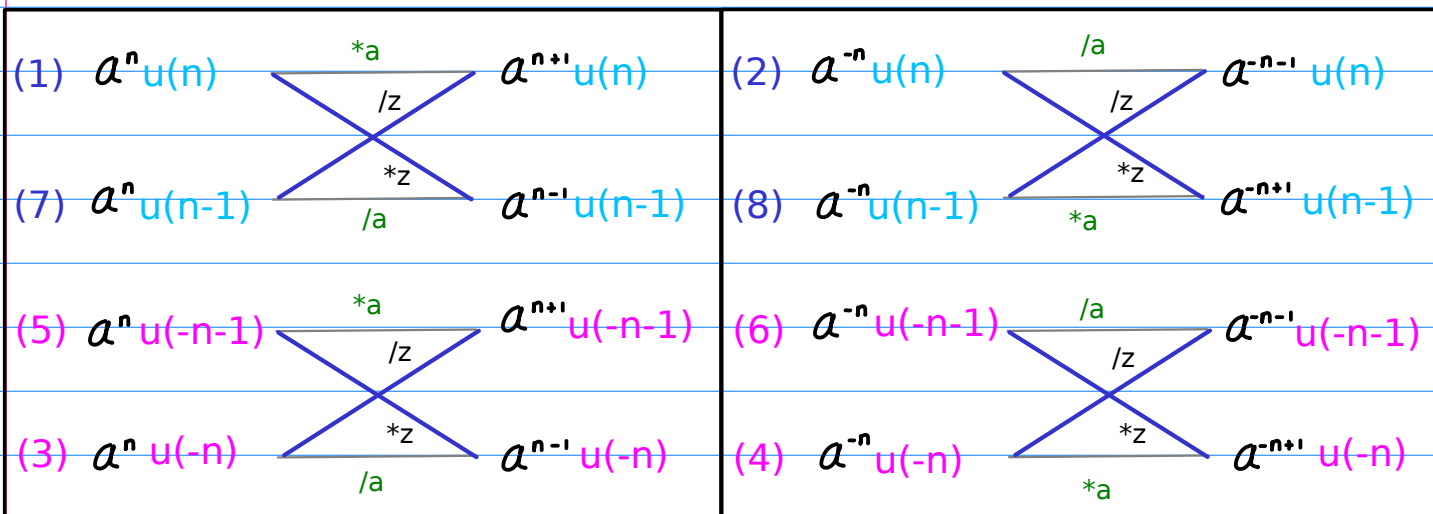
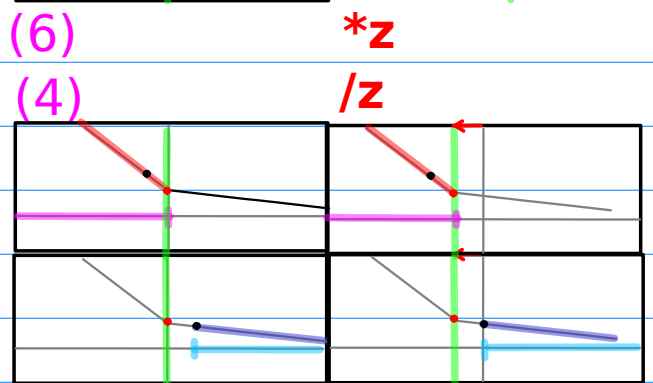
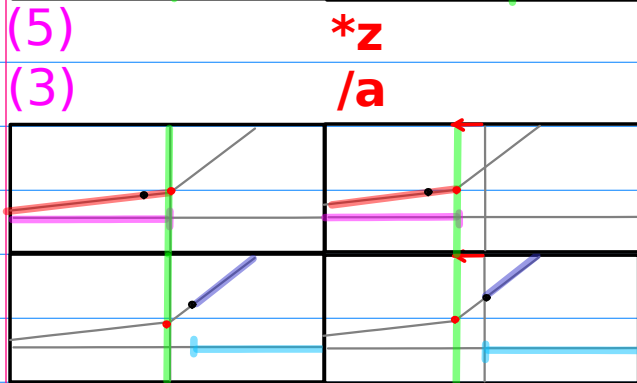
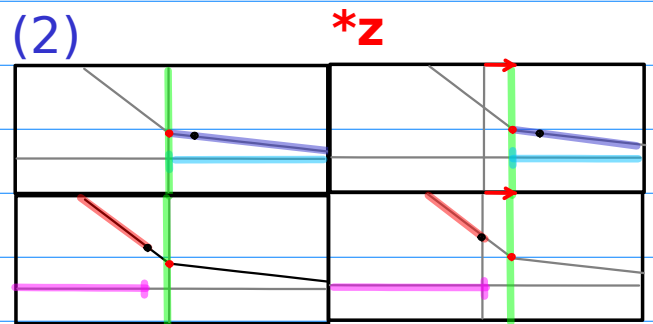
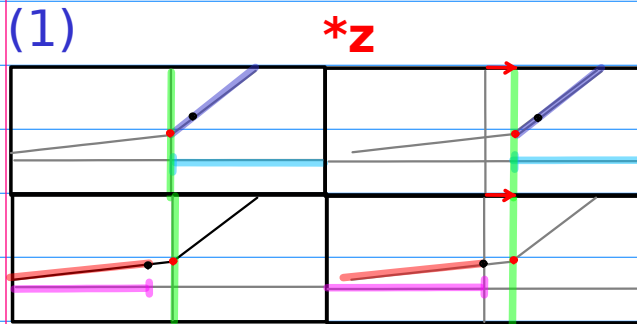
(8)

$/z$

| | | | | | |
|-------------------|------|-------------------|----------------------|------|--------------------|
| (1) $a^n u(n)$ | $*a$ | $a^{n+1} u(n)$ | (2) $a^{-n} u(n)$ | $/a$ | $a^{-n-1} u(n)$ |
| (7) $a^n u(n-1)$ | $/z$ | $a^{n-1} u(n-1)$ | (8) $a^{-n} u(n-1)$ | $/z$ | $a^{-n+1} u(n-1)$ |
| (5) $a^n u(-n-1)$ | $*z$ | $a^{n+1} u(-n-1)$ | (6) $a^{-n} u(-n-1)$ | $*a$ | $a^{-n-1} u(-n-1)$ |
| (3) $a^n u(-n)$ | $/a$ | $a^{n-1} u(-n)$ | (4) $a^{-n} u(-n)$ | $/z$ | $a^{-n+1} u(-n)$ |

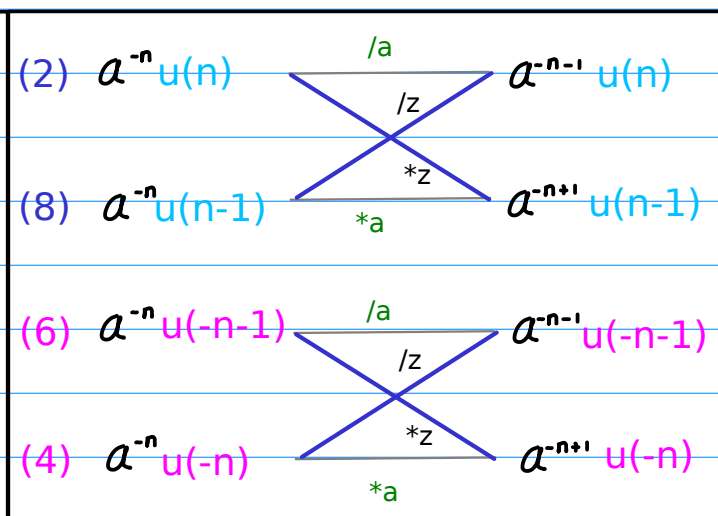
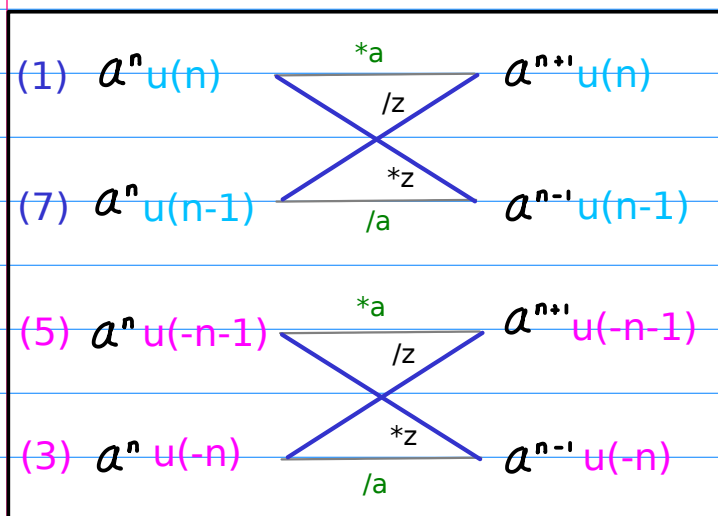
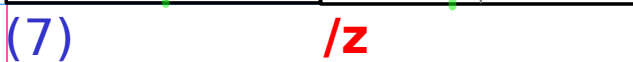
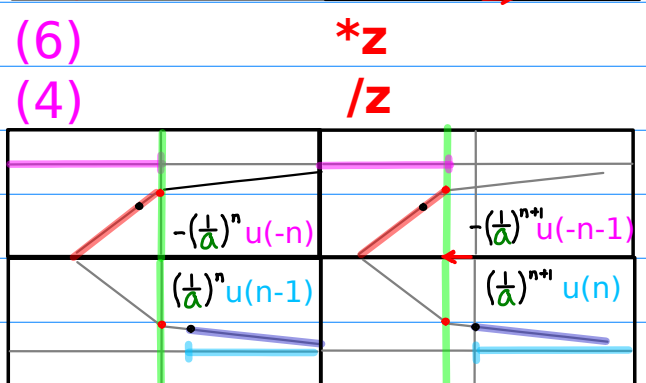
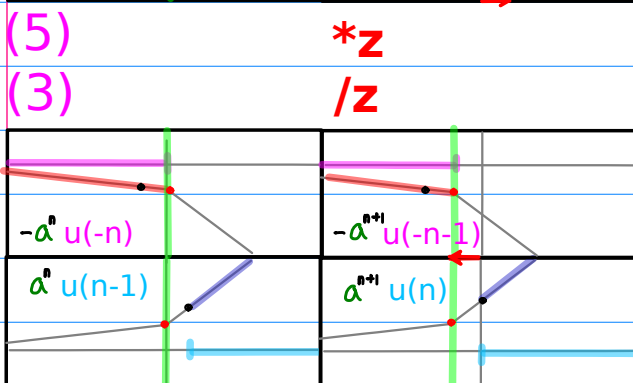
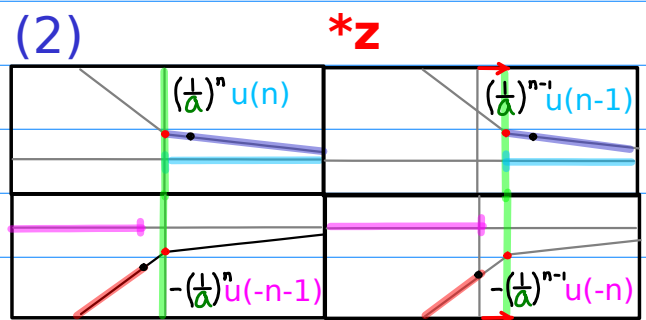
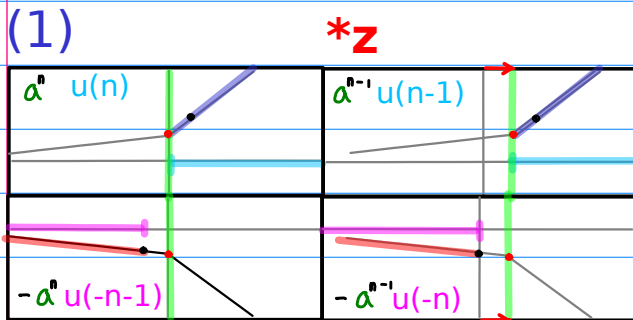
Scale by z

4. Graphs



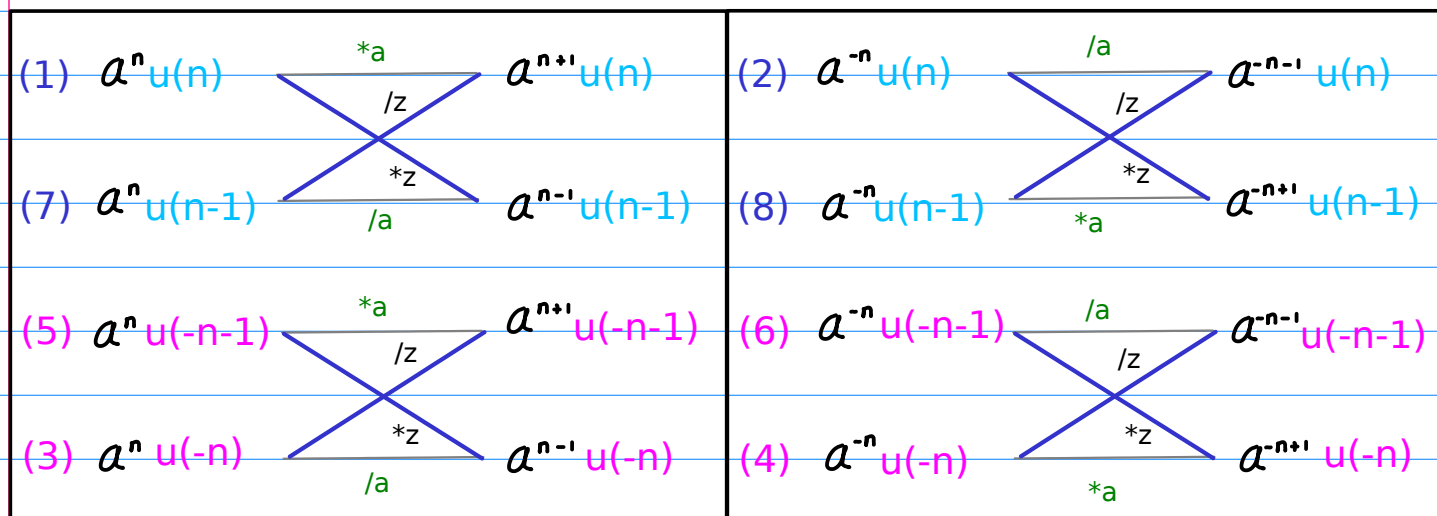
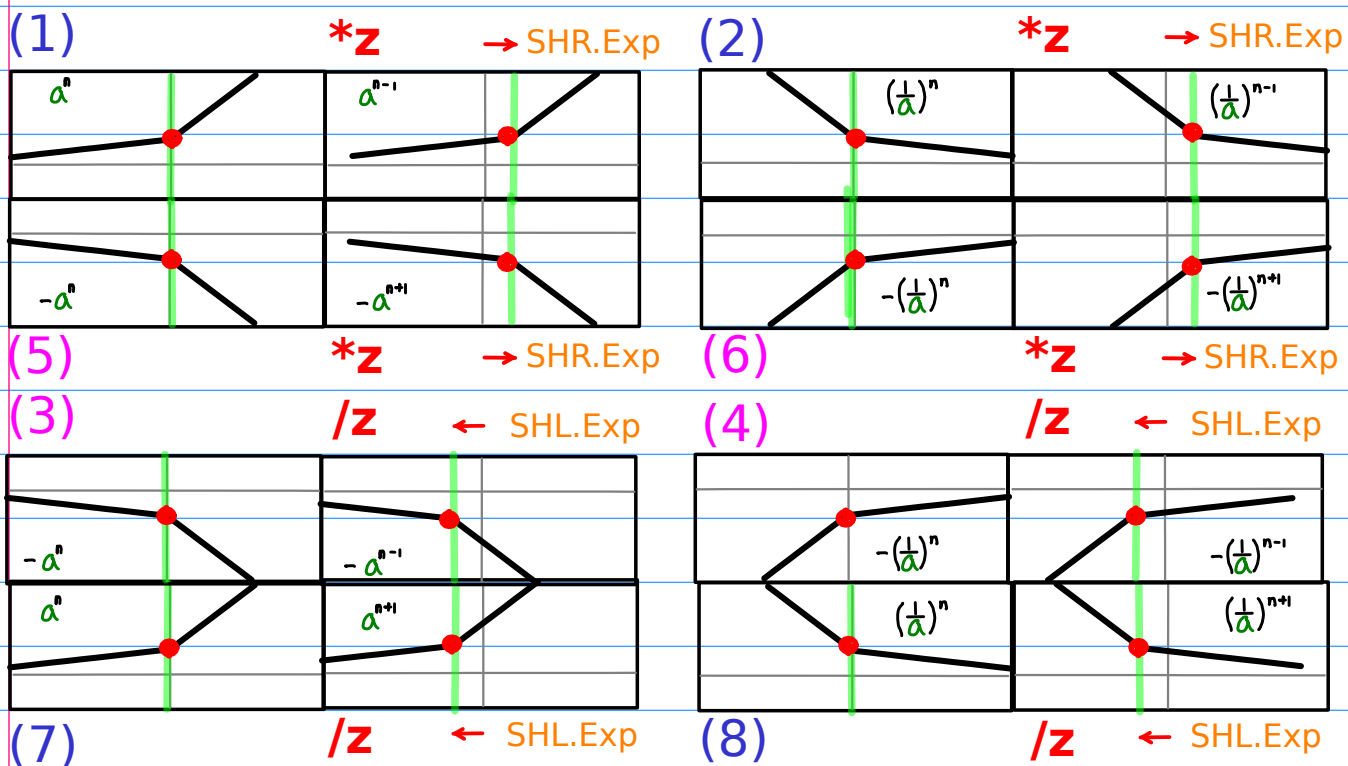
Scale by z

5. Graphs - signs



Scale by z

6. Graphs - Exponents



Scale by z

7. Graphs - Ranges

