

Laurent Series and z-Transform - Geometric Series Time Shift A

20180910 Mon

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Causal Signal $a(n) \Rightarrow f(z), X(z)$
($n \geq 0$) ($|z| < 2$) ($|z| > 0.5$)

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^1 + \left(\frac{1}{2}\right)^2 z^2 + \dots = \frac{1}{1 - \left(\frac{z}{2}\right)}$$

$$X(z) = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \dots = \frac{1}{1 - \left(\frac{1}{2z}\right)}$$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1 - \left(\frac{z}{2}\right)} \rightarrow \frac{2}{2 - z} \quad (|z| < 2)$$

$$X(z) = \frac{1}{1 - \left(\frac{1}{2z}\right)} \rightarrow \frac{z}{z - 0.5} \quad (|z| > 0.5)$$

Anti-Causal Signal $a(n) \Rightarrow f(z), X(z)$

$$(n < 0) \quad (|z| > p) \quad (|z| < p^{-1})$$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n < 0)$$

$$f(z) = \left(\frac{1}{2}\right)^{-1} z^{-1} + \left(\frac{1}{2}\right)^{-2} z^{-2} + \left(\frac{1}{2}\right)^{-3} z^{-3} + \dots = \frac{\left(\frac{2}{z}\right)}{1 - \left(\frac{2}{z}\right)}$$

$$X(z) = \left(\frac{1}{2}\right)^{-1} z^1 + \left(\frac{1}{2}\right)^{-2} z^2 + \left(\frac{1}{2}\right)^{-3} z^3 + \dots = \frac{(2z)}{1 - (2z)}$$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n < 0)$$

$$f(z) = \frac{\left(\frac{2}{z}\right)}{1 - \left(\frac{2}{z}\right)} \rightarrow \frac{2}{z - 2} \quad (|z| > 2)$$

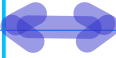

$$X(z) = \frac{(2z)}{1 - (2z)} \rightarrow \frac{z}{0.5 - z} \quad (|z| < 0.5)$$

$$a_n = -\left(\frac{1}{2}\right)^n \quad (n < 0)$$

$$f(z) = \frac{-\left(\frac{2}{z}\right)}{1 - \left(\frac{2}{z}\right)} \rightarrow \frac{2}{2 - z} \quad (|z| > 2)$$

$$X(z) = \frac{-(2z)}{1 - (2z)} \rightarrow \frac{z}{z - 0.5} \quad (|z| < 0.5)$$

Inverse z $z \leftarrow z^{-1}$, $\text{Roc}(z) \leftarrow \text{Roc}(z^{-1})$

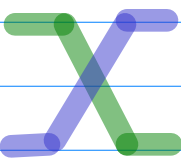
Causal	z^{-1}	anti-Causal
$f(z) = \frac{z}{2-z} \quad (z < 2)$		$f(z^{-1}) = \frac{z}{z-0.5} \quad (z > 0.5)$
$X(z) = \frac{z}{z-0.5} \quad (z > 0.5)$		$X(z^{-1}) = \frac{z}{2-z} \quad (z < 2)$

$$f(z^{-1}) = \frac{z}{2-z^{-1}} \quad (|z^{-1}| < 2) \quad \xrightarrow{\text{purple}} \quad f(z^{-1}) = X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

$$X(z^{-1}) = \frac{z^{-1}}{z^{-1}-0.5} \quad (|z^{-1}| > 0.5) \quad \xrightarrow{\text{green}} \quad X(z^{-1}) = f(z) = \frac{z}{2-z} \quad (|z| < 2)$$

$f(z^{-1}) = X(z)$ Laurent Series (anti-causal signal) with the same formula as causal $X(z)$

$X(z^{-1}) = f(z)$ z-Transform (anti-causal signal) with the same formula as causal $f(z)$

Causal		anti-Causal
$f(z) = \frac{z}{2-z} \quad (z < 2)$		$X(z^{-1}) = \frac{z}{2-z} \quad (z < 2)$
$X(z) = \frac{z}{z-0.5} \quad (z > 0.5)$		$f(z^{-1}) = \frac{z}{z-0.5} \quad (z > 0.5)$

Inverse z $f(z^{-1})$, $\text{Roc}(z^{-1}) \Rightarrow a_n$

Causal

anti-Causal

$$f(z) = \frac{2}{2-z} \quad (|z| < 2)$$

$$f(z^{-1}) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

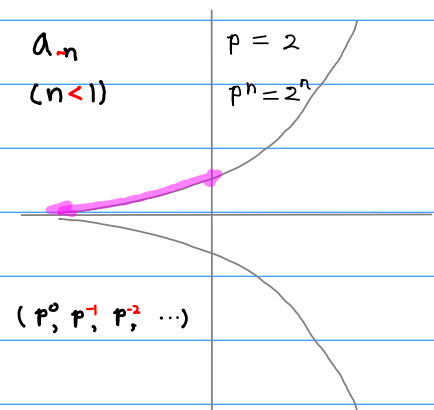
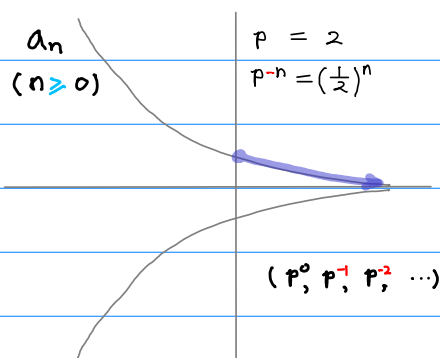
$$X(z^{-1}) = \frac{2}{2-z} \quad (|z| < 2)$$

$$f(z) = \frac{2}{2-z} = \frac{1}{1-(\frac{z}{2})} = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^1 + \left(\frac{1}{2}\right)^2 z^2 + \dots \quad a_n = \left(\frac{1}{2}\right)^n$$

$$= p^0 z^0 + p^1 z^1 + p^2 z^2 + \dots \quad n = 0, 1, 2, \dots$$

$$f(z^{-1}) = \frac{z}{z-0.5} = \frac{1}{1-(\frac{1}{2z})} = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \dots \quad a_{-n} = \left(\frac{1}{2}\right)^n$$

$$= p^0 z^0 + p^1 z^{-1} + p^2 z^{-2} + \dots \quad n = 0, -1, -2, \dots$$



Inverse z $X(z^{-1})$, $\text{Roc}(z^{-1}) \Rightarrow a_n$

Causal

anti-Causal

$$f(z) = \frac{z}{2-z} \quad (|z| < 2)$$

$$f(z^{-1}) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

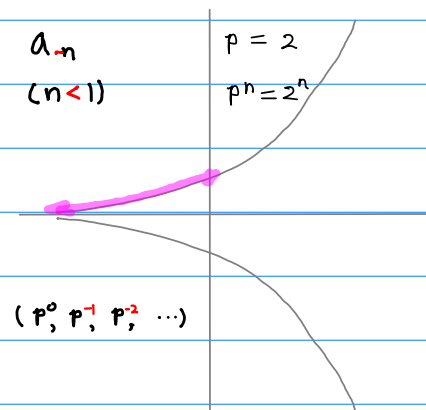
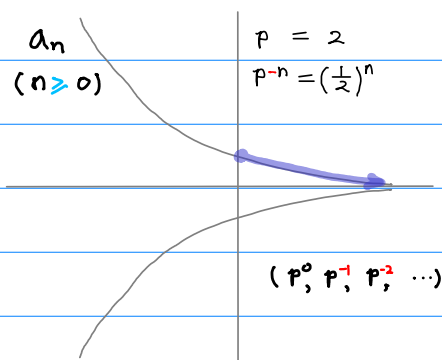
$$X(z^{-1}) = \frac{z}{2-z} \quad (|z| < 2)$$

$$X(z) = \frac{z}{z-0.5} = \frac{1}{1 - (\frac{1}{2z})} = (\frac{1}{2})^0 z^0 + (\frac{1}{2})^1 z^{-1} + (\frac{1}{2})^2 z^{-2} + \dots \quad a_n = (\frac{1}{2})^n$$

$$= p^0 z^0 + p^1 z^{-1} + p^2 z^{-2} + \dots \quad n = 0, 1, 2, \dots$$

$$X(z^{-1}) = \frac{z^{-1}}{z^{-1}-0.5} = \frac{1}{1 - (\frac{z}{2})} = (\frac{1}{2})^0 z^0 + (\frac{1}{2})^1 z^1 + (\frac{1}{2})^2 z^2 + \dots \quad a_{-n} = (\frac{1}{2})^{-n}$$

$$= p^0 z^0 + p^1 z^1 + p^2 z^2 + \dots \quad n = 0, -1, -2, \dots$$





Inverse ROC

$$\text{Roc}(z) \leftarrow \text{Roc}(z^{-1})$$

Causal

anti-Causal

$$f(z) = \frac{z}{2-z} \quad (|z| < 2) \quad \longrightarrow \quad -f(z) = -\frac{z}{2-z} \quad (|z| > 2)$$

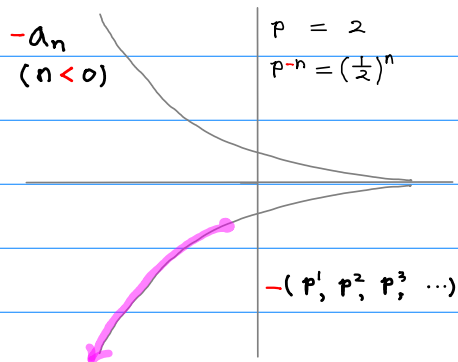
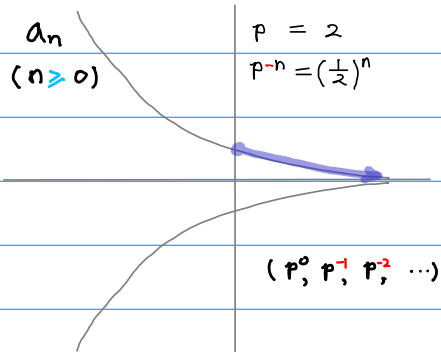
$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5) \quad \longrightarrow \quad -X(z) = -\frac{z}{z-0.5} \quad (|z| < 0.5)$$

$$f(z) = \frac{z}{2-z} = \frac{1}{1 - (\frac{z}{2})} = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^1 + \left(\frac{1}{2}\right)^2 z^2 + \dots \quad a_n = \left(\frac{1}{2}\right)^n$$

$$= p^0 z^0 + p^1 z^1 + p^2 z^2 + \dots \quad n = 0, 1, 2, \dots$$

$$-f(z) = \frac{z}{z-2} = \frac{\left(\frac{z}{2}\right)}{1 - \left(\frac{z}{2}\right)} = \left(\frac{1}{2}\right)^1 z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \left(\frac{1}{2}\right)^3 z^{-3} + \dots \quad a_n = -\left(\frac{1}{2}\right)^n$$

$$= p^1 z^{-1} + p^2 z^{-2} + p^3 z^{-3} + \dots \quad n = -1, -2, -3, \dots$$



Inverse ROC $f(z)$, $\text{ROC}(z^{-1}) \Rightarrow -a_n$

Causal

anti-Causal

$$f(z) = \frac{2}{2-z} \quad (|z| < 2)$$

$$f(z) = \frac{2}{2-z} \quad (|z| > 0.5)$$

$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

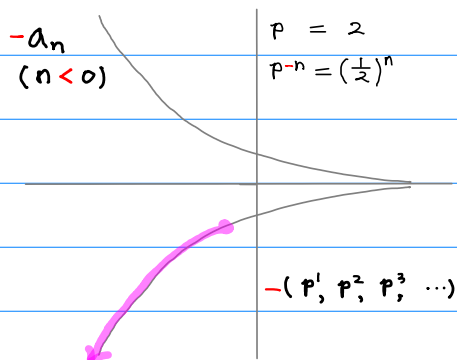
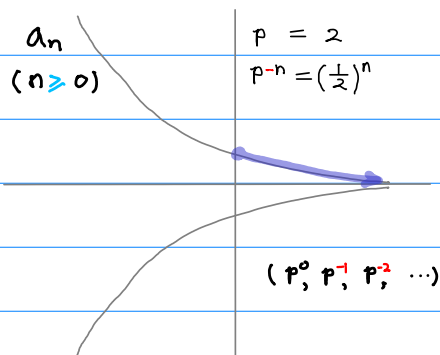
$$X(z) = \frac{z}{z-0.5} \quad (|z| < 2)$$

$$f(z) = \frac{2}{2-z} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^1 + \left(\frac{1}{2}\right)^2 z^2 + \dots \quad \therefore a_n = \left(\frac{1}{2}\right)^n$$

$$= p^0 z^0 + p^1 z^1 + p^2 z^2 + \dots \quad n = 0, 1, 2, \dots$$

$$f(z) = \frac{2}{2-z} = \frac{-\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} = -\left[\left(\frac{2}{z}\right)^1 z^{-1} + \left(\frac{2}{z}\right)^2 z^{-2} + \left(\frac{2}{z}\right)^3 z^{-3} + \dots \right] \quad \therefore -a_n = -\left(\frac{1}{2}\right)^n$$

$$= -\left[p^1 z^{-1} + p^2 z^{-2} + p^3 z^{-3} + \dots \right] \quad n = -1, -2, -3, \dots$$



Inverse ROC $X(z)$, $\text{ROC}(z^{-1}) \Rightarrow -a_n$

Causal

anti-Causal

$$f(z) = \frac{z^2}{2-z} \quad (|z| < 2)$$

$$f(z) = \frac{z^2}{2-z} \quad (|z| > 0.5)$$

$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

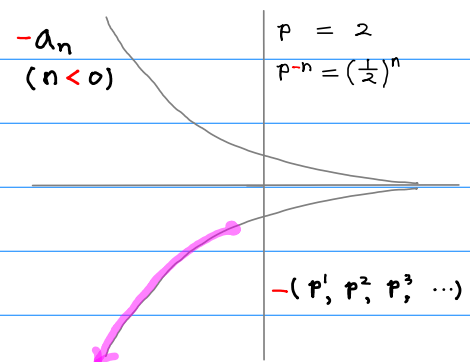
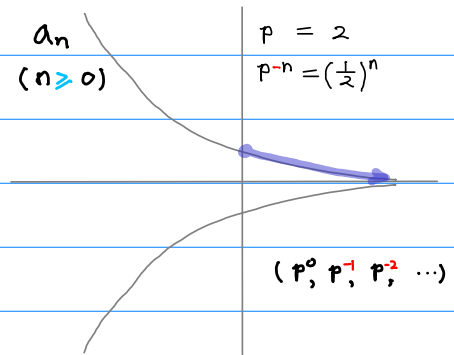
$$X(z) = \frac{z}{z-0.5} \quad (|z| < 2)$$

$$X(z) = \frac{z}{z-0.5} = \frac{1}{1 - (\frac{1}{2z})} = (\frac{1}{2})^0 z^0 + (\frac{1}{2})^1 z^{-1} + (\frac{1}{2})^2 z^{-2} + \dots \quad a_n = (\frac{1}{2})^n$$

$$= p^0 z^0 + p^1 z^{-1} + p^2 z^{-2} + \dots \quad n = 0, 1, 2, \dots$$

$$X(z) = \frac{z}{z-0.5} = \frac{-(2z)}{1-(2z)} = -\left[(2)^1 z^1 + (2)^2 z^2 + (\frac{1}{2})^3 z^3 + \dots \right] \quad a_n = -(\frac{1}{2})^n$$

$$= -\left[p^1 z^1 + p^2 z^2 + p^3 z^3 + \dots \right] \quad n = -1, -2, -3, \dots$$



p=2

causal

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1 - \left(\frac{z}{2}\right)} \quad (|z| < 2)$$

$$X(z) = \frac{1}{1 - \left(\frac{1}{2z}\right)} \quad (|z| > 0.5)$$

$$f(z) = \frac{z}{2 - z} \quad (|z| < 2)$$

$$X(z) = \frac{z}{z - 0.5} \quad (|z| > 0.5)$$

anti-causal

$$a_n = \left(\frac{1}{2}\right)^n \quad (n < 0)$$

$$f(z) = \frac{\left(\frac{2}{z}\right)}{1 - \left(\frac{2}{z}\right)} \quad (|z| > 2)$$

$$X(z) = \frac{(2z)}{1 - (2z)} \quad (|z| < 0.5)$$

$$f(z) = \frac{z}{z - 2} \quad (|z| > 2)$$

$$X(z) = \frac{z}{0.5 - z} \quad (|z| < 0.5)$$

p=1/2

causal

$$a_n = (2)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1 - (2z)} \quad (|z| < 0.5)$$

$$X(z) = \frac{1}{1 - \left(\frac{z}{2}\right)} \quad (|z| > 2)$$

$$f(z) = \frac{0.5}{0.5 - z} \quad (|z| < 0.5)$$

$$X(z) = \frac{z}{z - 2} \quad (|z| > 2)$$

anti-causal

$$a_n = (2)^n \quad (n < 0)$$

$$f(z) = \frac{\left(\frac{1}{2z}\right)}{1 - \left(\frac{1}{2z}\right)} \quad (|z| > 0.5)$$

$$X(z) = \frac{\left(\frac{z}{2}\right)}{1 - \left(\frac{z}{2}\right)} \quad (|z| < 2)$$

$$f(z) = \frac{0.5}{z - 0.5} \quad (|z| > 0.5)$$

$$X(z) = \frac{z}{2 - z} \quad (|z| < 2)$$

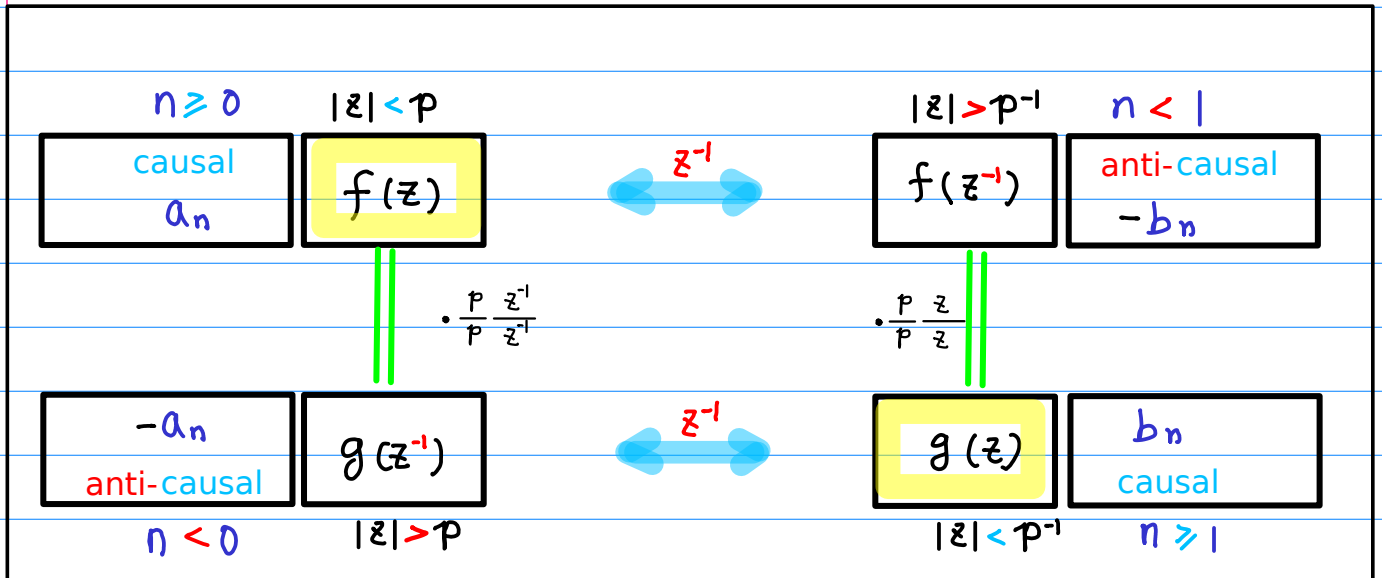


$f(z)$	$f(z^{-1})$
$g(z^{-1})$	$g(z)$

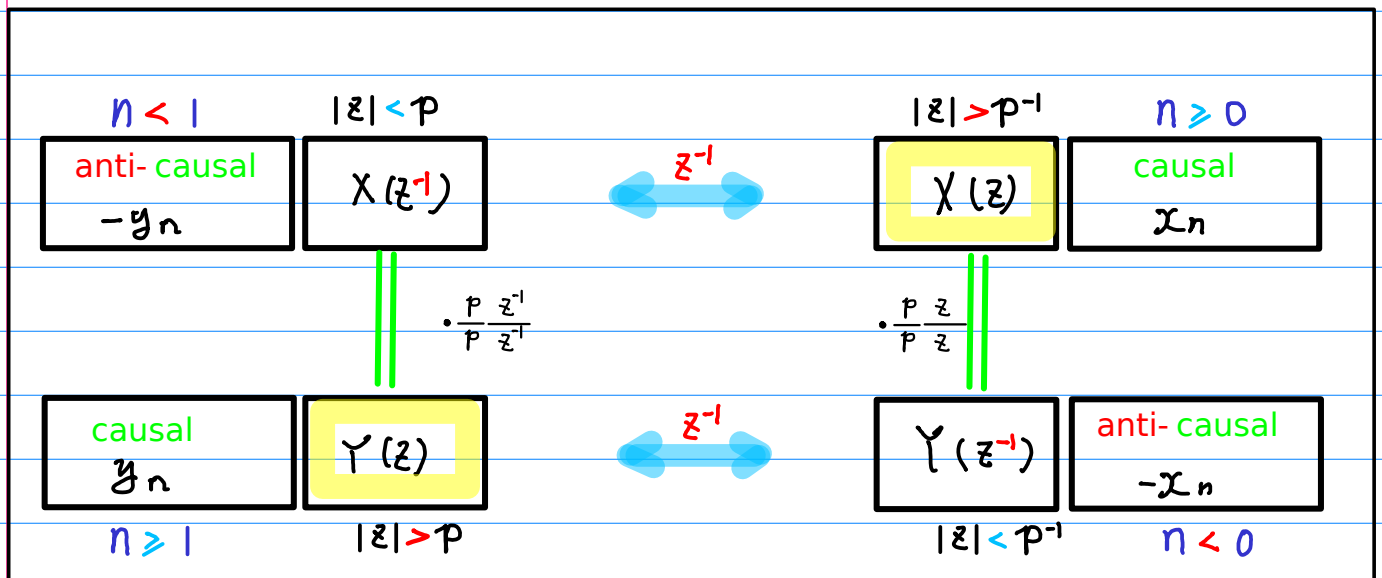
$X(z^{-1})$	$X(z)$
$Y(z)$	$Y(z^{-1})$

1

Laurent Series $a_n \leftrightarrow f(z)$ $b_n \leftrightarrow g(z)$



Z-Transform $X(z) \leftrightarrow x_n$ $Y(z) \leftrightarrow y_n$

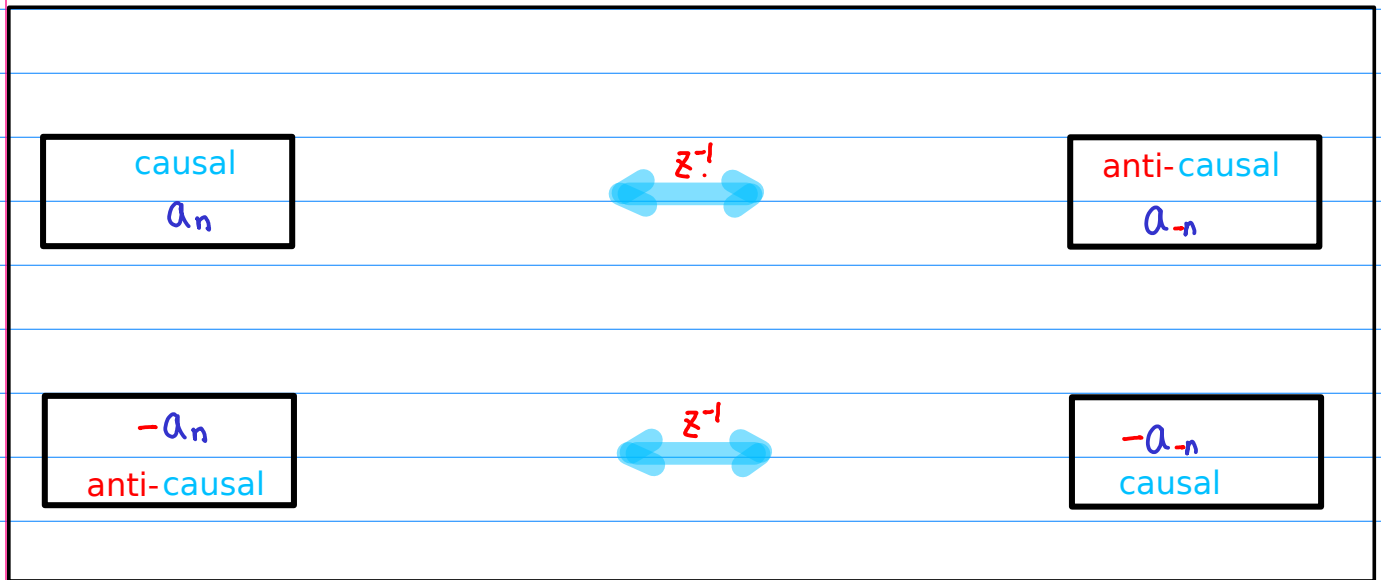


$f(z)$	$f(z^{-1})$
$g(z^{-1})$	$g(z)$

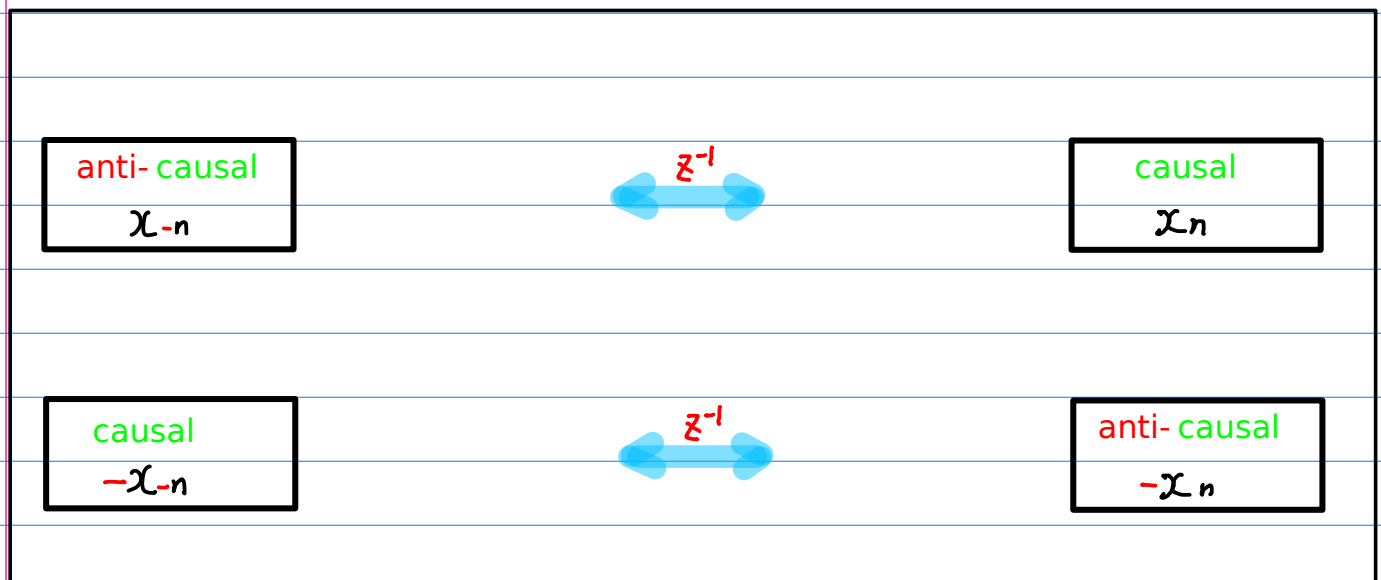
$X(z^{-1})$	$X(z)$
$Y(z)$	$Y(z^{-1})$

2

Laurent Series $a_n \leftrightarrow f(z)$ $-a_{-n} = b_n \leftrightarrow g(z)$

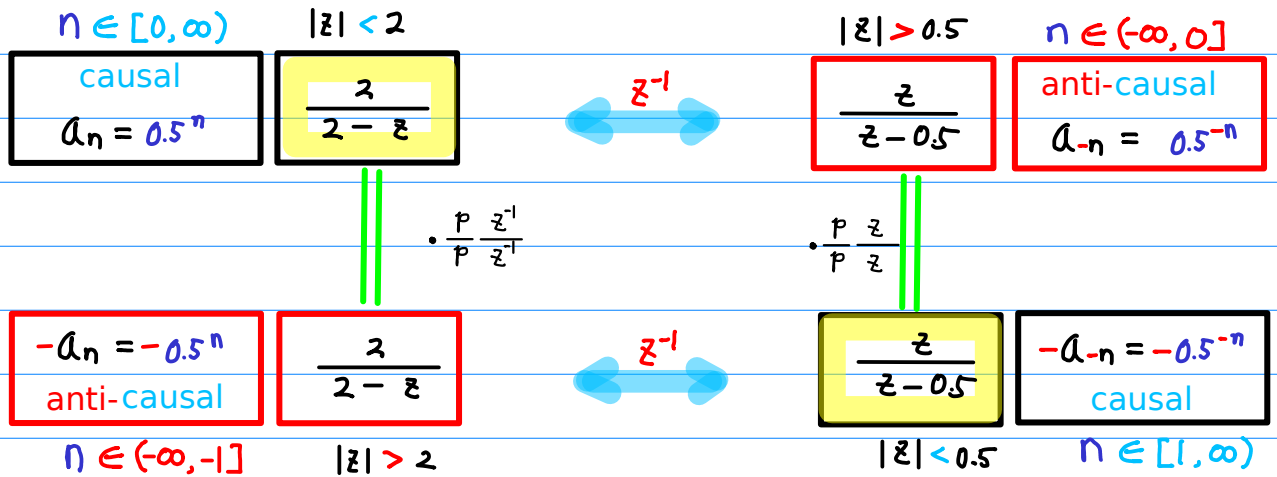


z-Transform $X(z) \leftrightarrow x_n$ $Y(z) \leftrightarrow y_n = -x_{-n}$

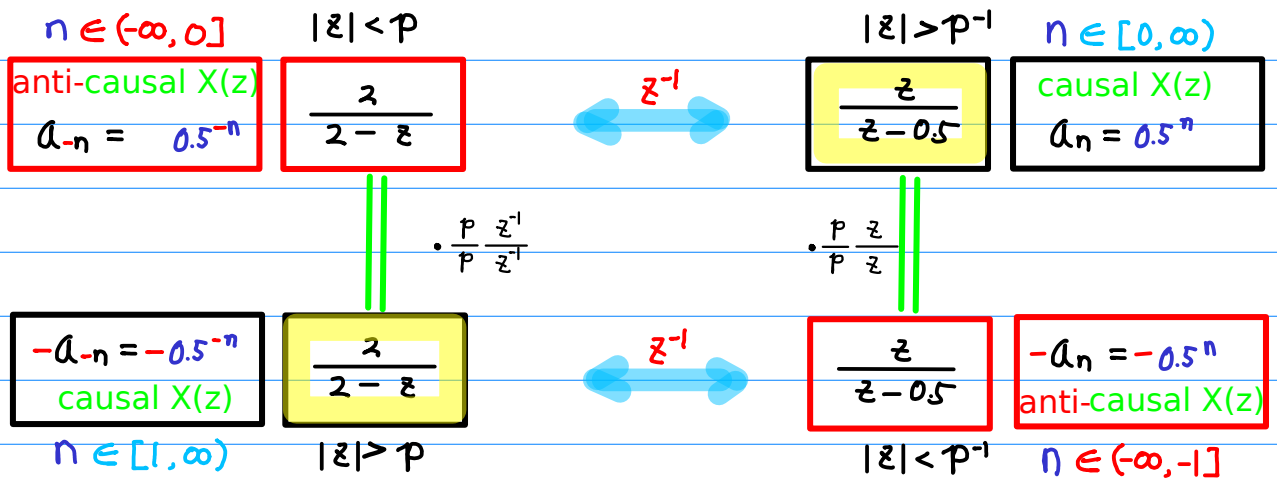


$$f(z), X(z) = \frac{z}{2-z}, \frac{z}{z-0.5}$$

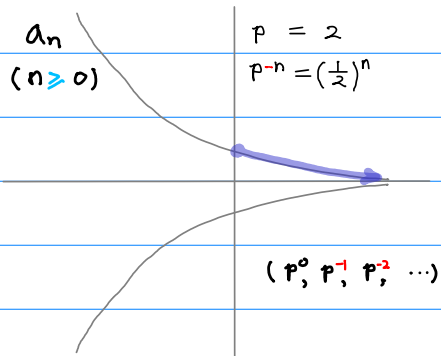
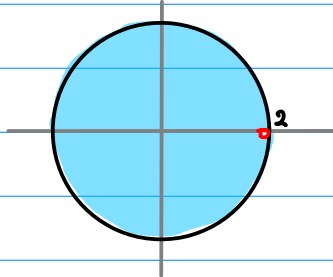
f(z) Laurent Series



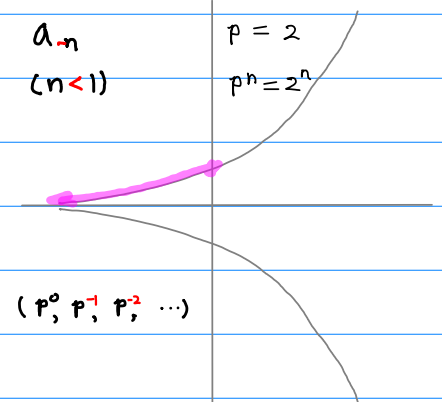
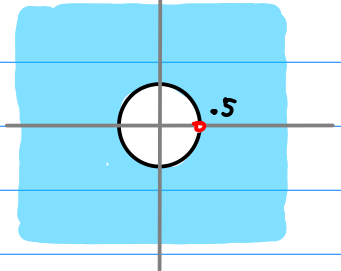
X(z) z-Transform



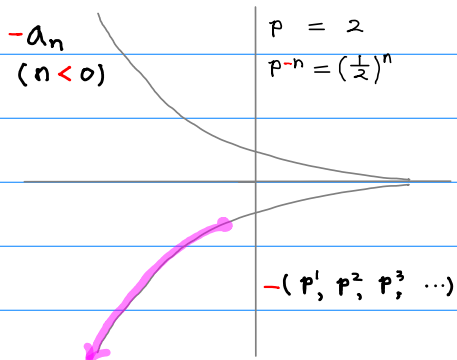
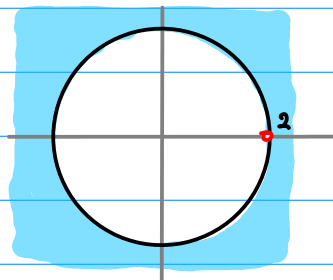
$n \in [0, \infty)$ $|z| < 2$
 causal
 $a_n = 0.5^n$ $\frac{z}{2-z}$



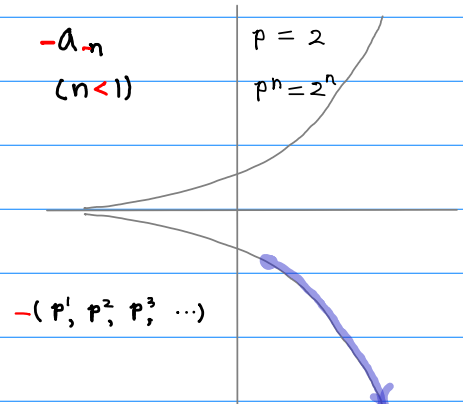
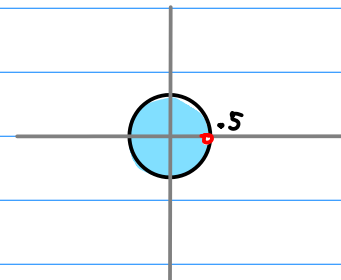
$|z| > 0.5$ $n \in (-\infty, 0]$
 $\frac{z}{z-0.5}$ anti-causal
 $a_{-n} = 0.5^{-n}$



$-a_n = -0.5^n$
 anti-causal
 $n \in (-\infty, -1]$ $|z| > 2$



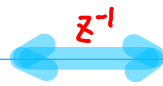
$\frac{z}{z-0.5}$ $-a_{-n} = -0.5^{-n}$
 causal
 $|z| < 0.5$ $n \in [1, \infty)$



$n < 1$ $|z| < 2$

anti-causal
 $x_{-n} = 0.5^{-n}$

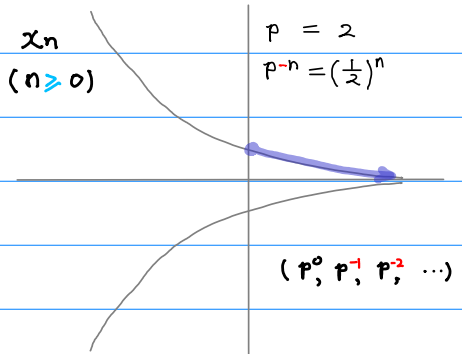
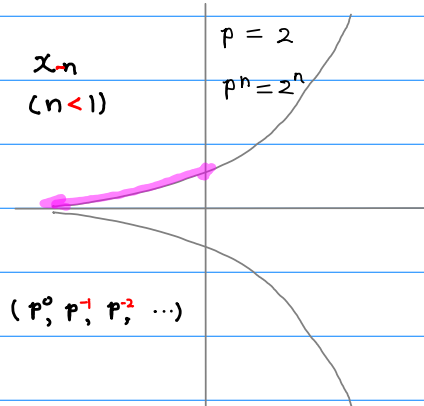
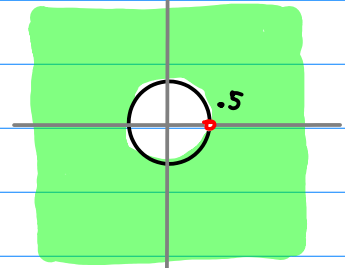
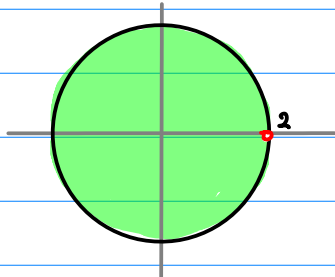
$-\frac{0.5}{1 - 0.5z}$



$|z| > 0.5$ $n \geq 0$

causal
 $x_n = 0.5^n$

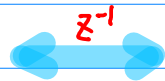
$\frac{0.5}{1 - 0.5z^{-1}}$



$n \geq 1$ $|z| > 2$

causal
 $-x_{-n} = -0.5^{-n}$

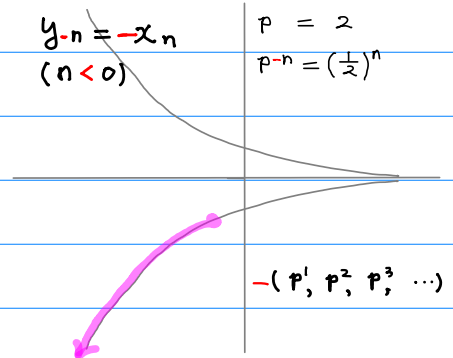
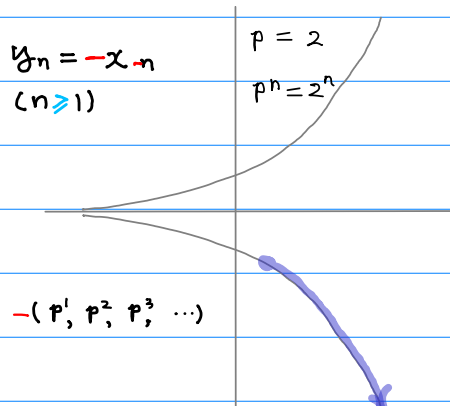
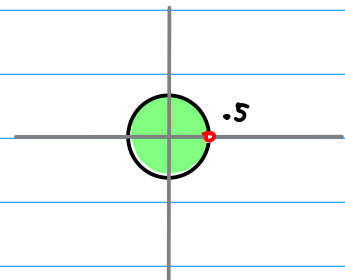
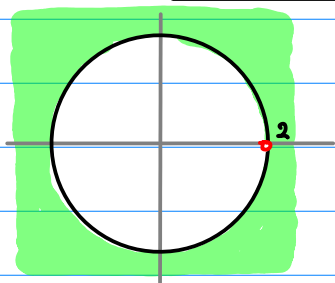
$\frac{z^{-1}}{1 - 2z^{-1}}$



$|z| < 0.5$ $n < 0$

anti-causal
 $-x_n = -0.5^n$

$\frac{z}{1 - 2z}$



2 formulas (A)

$$\frac{1}{z-p}$$

$$\frac{1}{z^{-1}-p}$$

2 representations each

$$\frac{1}{z-p}$$

$$-\frac{p^{-1}}{1-p^{-1}z}$$

$$\frac{z^{-1}}{1-pz^{-1}}$$

$$\frac{1}{z^{-1}-p}$$

$$-\frac{p^{-1}}{1-p^{-1}z^{-1}}$$

$$\frac{z}{1-pz}$$

2 formulas (B)

$$\frac{p}{z-p}$$

$$\frac{p}{z^{-1}-p}$$

2 representations each

$$\frac{p}{z-p}$$

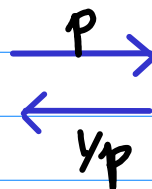
$$-\frac{1}{1-p^{-1}z}$$

$$\frac{pz^{-1}}{1-pz^{-1}}$$

$$\frac{p}{z^{-1}-p}$$

$$-\frac{1}{1-p^{-1}z^{-1}}$$

$$\frac{pz}{1-pz}$$



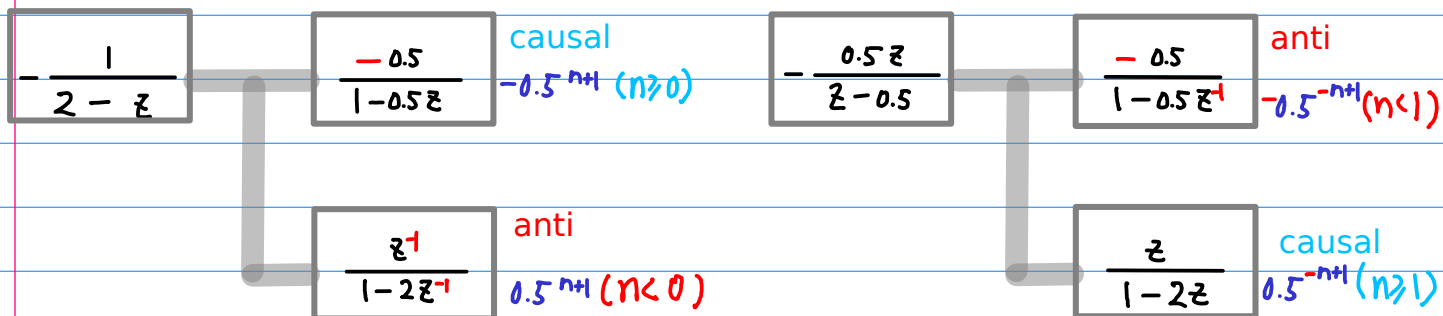
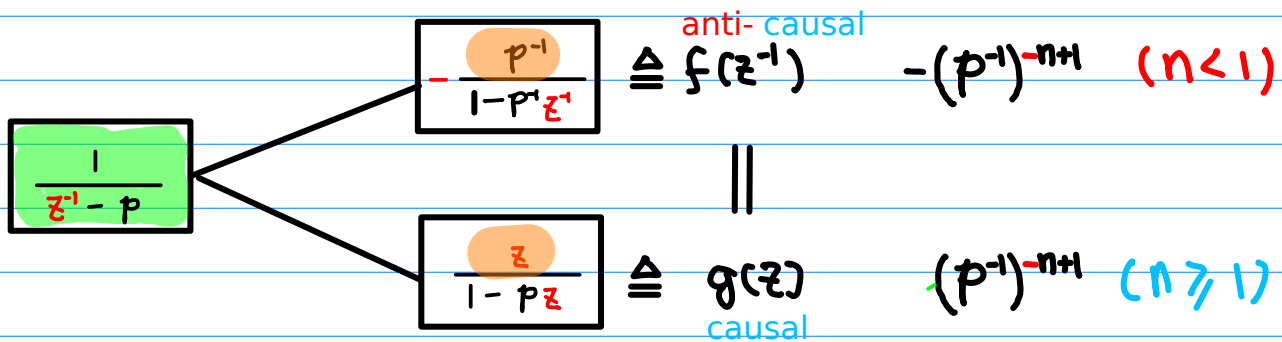
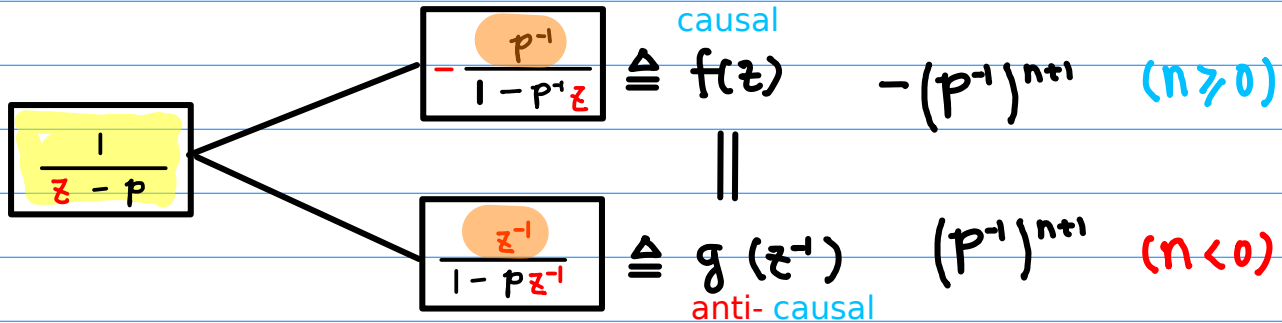
Ⓐ $f(z) \rightarrow a_n$

2 formulas Ⓐ

$$\frac{1}{z-p}$$

$$\frac{1}{z^{-1}-p}$$

2 representations each



Ⓐ $f(z) \rightarrow a_n$ method 1

$$p^{-1} \boxed{-\frac{p^{-1}}{1-p^{-1}z}} \rightarrow -(p^{-1}) \rightarrow -(p^{-1})^{n+1} \rightarrow -(p^{-1})^{n+1}$$

$n=0, 1, 2, \dots$
 $-(p^1 z^0 + p^2 z^1 + p^3 z^2 + \dots)$

$$p \boxed{\frac{z^{-1}}{1-pz^{-1}}} \rightarrow (p) \rightarrow (p)^{-n-1} \rightarrow (p^{-1})^{n+1}$$

$n=-1, -2, -3, \dots$
 $(p^0 z^1 + p^1 z^2 + p^2 z^3 + \dots)$

$$p^{-1} \boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} \rightarrow -(p^{-1}) \rightarrow -(p^{-1})^{-n+1} \rightarrow -(p^{-1})^{-n+1}$$

$n=0, -1, -2, \dots$
 $-(p^1 z^0 + p^2 z^{-1} + p^3 z^{-2} + \dots)$

$$p \boxed{\frac{z}{1-pz}} \rightarrow (p) \rightarrow (p)^{n-1} \rightarrow (p^{-1})^{-n+1}$$

$n=1, 2, 3, \dots$
 $(p^0 z^1 + p^1 z^2 + p^2 z^3 + \dots)$

$$\textcircled{A} f(z) \rightarrow a_n$$

method 2

$$p^{-1} \boxed{\frac{p^{-1}}{1 - p^{-1}z}} \quad \begin{matrix} -(p^{-1}) \\ z \end{matrix} \rightarrow n = 0, 1, 2, \dots \rightarrow \begin{matrix} -(p^{-1})^{n+1} \\ (p)^{-n-1} \end{matrix}$$

$$p \boxed{\frac{z^{-1}}{1 - pz^{-1}}} \quad \begin{matrix} (p) \\ z^{-1}/z^{-1} \end{matrix} \rightarrow n = -1, -2, -3, \dots \rightarrow \begin{matrix} (p^{-1})^{n+1} \\ (p)^{-n-1} \end{matrix}$$

$$p \boxed{\frac{p^{-1}}{1 - p^{-1}z^{-1}}} \quad \begin{matrix} -(p^{-1}) \\ z^{-1} \end{matrix} \rightarrow n = 0, -1, -2, \dots \rightarrow \begin{matrix} -(p^{-1})^{-n+1} \\ (p)^{n-1} \end{matrix}$$

$$p^{-1} \boxed{\frac{z}{1 - pz}} \quad \begin{matrix} (p) \\ z/z \end{matrix} \rightarrow n = 1, 2, 3, \dots \rightarrow \begin{matrix} (p^{-1})^{-n+1} \\ (p)^{n-1} \end{matrix}$$

$$\textcircled{B} f(z) \rightarrow a_n$$

2 formulas

$$\frac{p}{z-p}$$

$$\frac{p}{z^{-1}-p}$$

2 representations each

$$\frac{p}{z-p} \begin{cases} \frac{1}{1-p^{-1}z} \triangleq f'(z) \quad \text{causal} \quad -(p^{-1})^n \quad (n \geq 0) \\ \frac{pz^{-1}}{1-pz^{-1}} \triangleq g'(z^{-1}) \quad \text{anti-causal} \quad (p^{-1})^n \quad (n < 0) \end{cases}$$

$$\frac{p}{z^{-1}-p} \begin{cases} \frac{1}{1-p^{-1}z^{-1}} \triangleq f'(z^{-1}) \quad \text{anti-causal} \quad -(p^{-1})^{-n} \quad (n < 1) \\ \frac{pz}{1-pz} \triangleq g'(z) \quad \text{causal} \quad (p^{-1})^{-n} \quad (n \geq 1) \end{cases}$$

$$\frac{z}{2-z} \quad \frac{1}{1-0.5z} \quad \text{causal} \quad 0.5^n \quad (n \geq 0)$$

$$\frac{z}{z-0.5} \quad \frac{1}{1-0.5z^{-1}} \quad \text{anti} \quad 0.5^{-n} \quad (n < 1)$$

$$\frac{-2z^{-1}}{1-2z^{-1}} \quad \text{anti} \quad -0.5^n \quad (n < 0)$$

$$\frac{-2z}{1-2z} \quad \text{causal} \quad -0.5^{-n} \quad (n \geq 1)$$

ⓑ $f(z) \rightarrow \hat{a}_n$ method 1

$$p^{-1} \boxed{\frac{1}{1 - p^{-1}z}} \rightarrow -(p^{-1}) \rightarrow -(p^{-1})^n \rightarrow -(p^{-1})^n$$

$n = 0, 1, 2, \dots$
 $-(p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \dots)$

$$p \boxed{\frac{pz^{-1}}{1 - pz^{-1}}} \rightarrow (p) \rightarrow (p)^{-n} \rightarrow (p^{-1})^n$$

$n = -1, -2, -3, \dots$
 $(p^1 z^1 + p^2 z^2 + p^3 z^3 + \dots)$

$$p \boxed{\frac{1}{1 - p^1 z^{-1}}} \rightarrow -(p^1) \rightarrow -(p^1)^{-n} \rightarrow -(p^{-1})^{-n}$$

$n = 0, -1, -2, \dots$
 $-(p^0 z^0 + p^1 z^1 + p^2 z^2 + \dots)$

$$p^{-1} \boxed{\frac{pz}{1 - pz}} \rightarrow (p) \rightarrow (p)^n \rightarrow (p^{-1})^{-n}$$

$n = 1, 2, 3, \dots$
 $(p^1 z^1 + p^2 z^2 + p^3 z^3 + \dots)$

ⓑ $f(z) \rightarrow a_n$ method 2

$$p^{-1} \quad \boxed{\frac{1}{1 - p^{-1}z}} \quad \begin{matrix} -(p^{-1}) \\ z \end{matrix} \rightarrow n = 0, 1, 2, \dots \rightarrow \begin{matrix} -p^{-n} \\ (p)^{-n} \end{matrix}$$

$$p \quad \boxed{\frac{pz^{-1}}{1 - pz^{-1}}} \quad \begin{matrix} (p) \\ z^{-1}/z^{-1} \end{matrix} \rightarrow n = -1, -2, -3, \dots \rightarrow \begin{matrix} p^{-n} \\ (p)^{-n} \end{matrix}$$

$$p \quad \boxed{\frac{1}{1 - p^{-1}z^{-1}}} \quad \begin{matrix} -(p^{-1}) \\ z^{-1} \end{matrix} \rightarrow n = 0, -1, -2, \dots \rightarrow \begin{matrix} -(p^{-1})^{-n} \\ (p)^n \end{matrix}$$

$$p^{-1} \quad \boxed{\frac{pz}{1 - pz}} \quad \begin{matrix} (p) \\ z/z \end{matrix} \rightarrow n = 1, 2, 3, \dots \rightarrow \begin{matrix} (p^{-1})^{-n} \\ (p)^n \end{matrix}$$

2 formulas

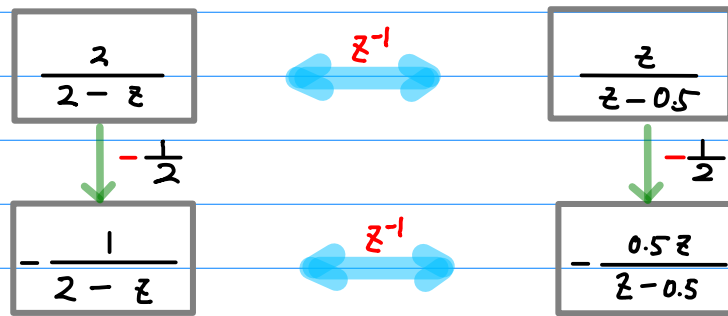
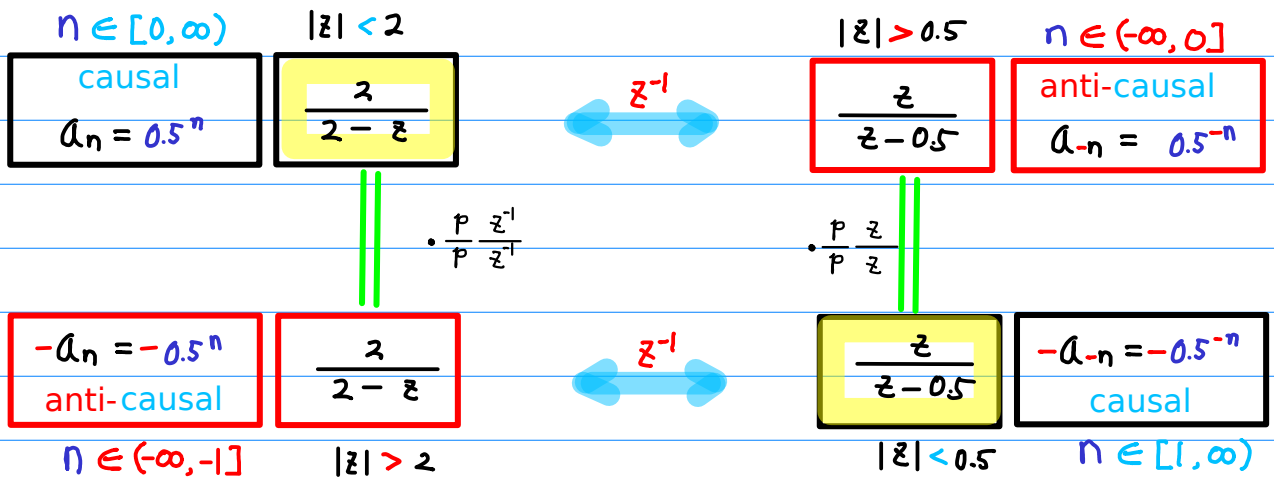
$$\frac{p}{z - p}$$

$$\frac{p}{z^{-1} - p}$$

2 representations each

$$\frac{p}{z - p} \begin{cases} \cong \frac{1}{1 - p^{-1}z} \triangleq \overset{\text{causal}}{f'(z)} = \overset{\text{anti-causal}}{\chi'(z^{-1})} \\ \cong \frac{pz^{-1}}{1 - pz^{-1}} \triangleq \underset{\text{causal}}{\gamma'(z)} = \underset{\text{anti-causal}}{g'(z^{-1})} \end{cases}$$

$$\frac{p}{z^{-1} - p} \begin{cases} \cong \frac{1}{1 - p^{-1}z^{-1}} \triangleq \overset{\text{causal}}{\chi'(z)} = \overset{\text{anti-causal}}{f'(z^{-1})} \\ \cong \frac{pz}{1 - pz} \triangleq \underset{\text{causal}}{g'(z)} = \underset{\text{anti-causal}}{\gamma'(z^{-1})} \end{cases}$$



$$0.5^{-n+1} = \left(\frac{1}{2}\right)^{-n+1}$$

$$= 2^{+n-1}$$

$$\frac{z}{2-z}$$

$$\frac{1}{1-0.5z}$$

causal
 $0.5^n (n \geq 0)$

$$\frac{z}{z-0.5}$$

$$\frac{1}{1-0.5z^{-1}}$$

anti
 $0.5^{-n} (n < 1)$

$$\frac{-2z^{-1}}{1-2z^{-1}}$$

anti
 $-0.5^n (n < 0)$

$$\frac{-2z}{1-2z}$$

causal
 $-0.5^{-n} (n \geq 1)$

$$\frac{1}{2-z}$$

$$\frac{-0.5}{1-0.5z}$$

causal
 $-0.5^{n+1} (n \geq 0)$

$$-\frac{0.5z}{z-0.5}$$

$$\frac{-0.5}{1-0.5z^{-1}}$$

anti
 $-0.5^{-n+1} (n < 1)$

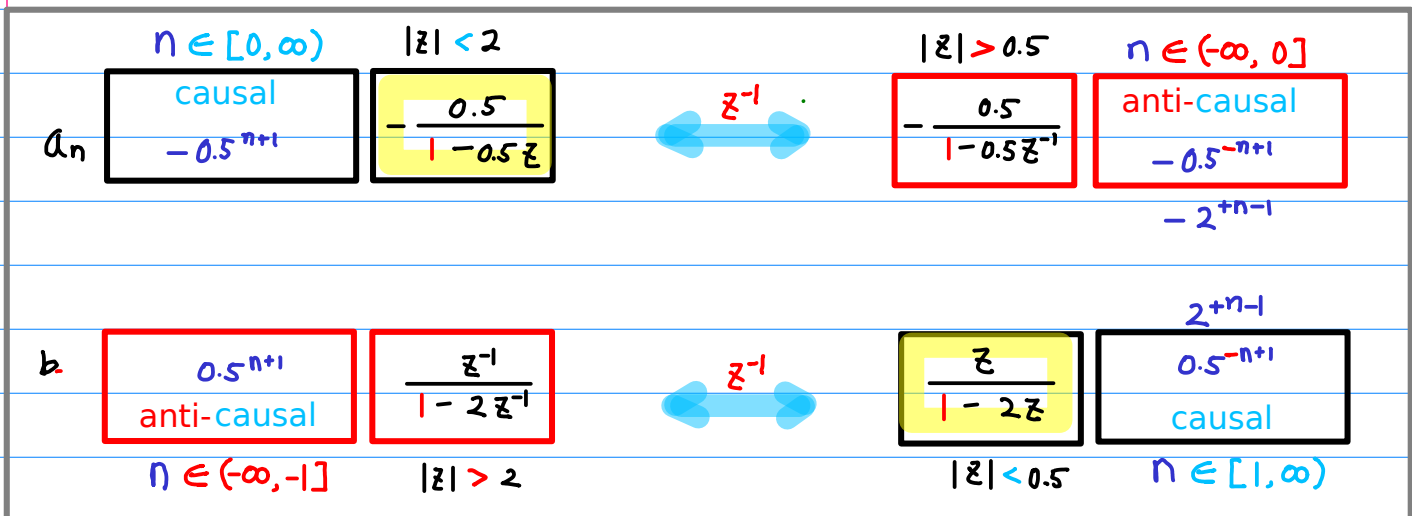
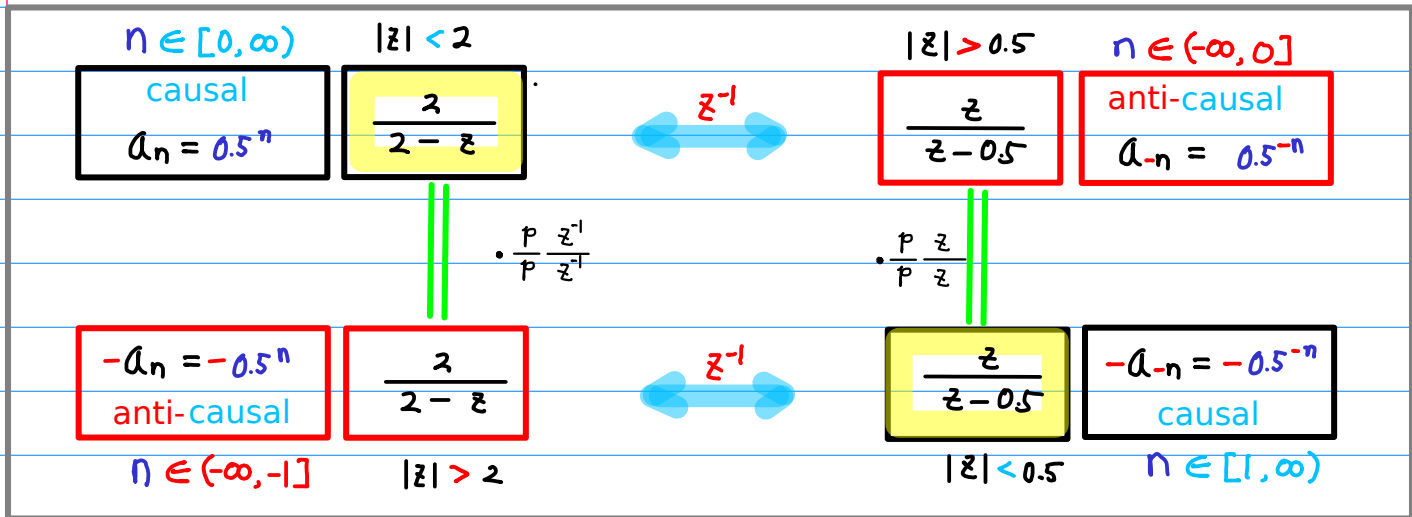
$$\frac{z^{-1}}{1-2z^{-1}}$$

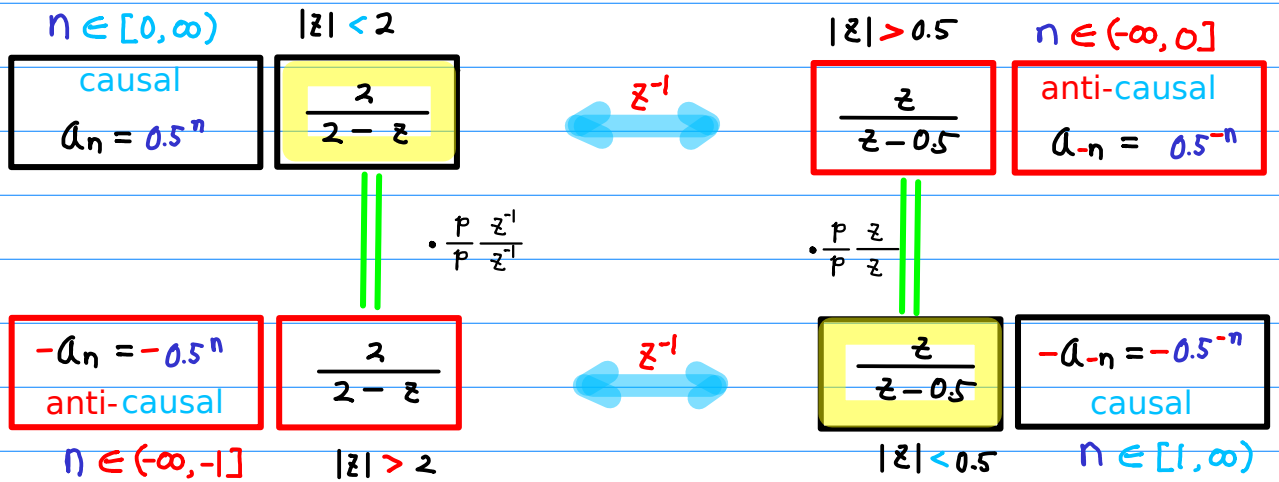
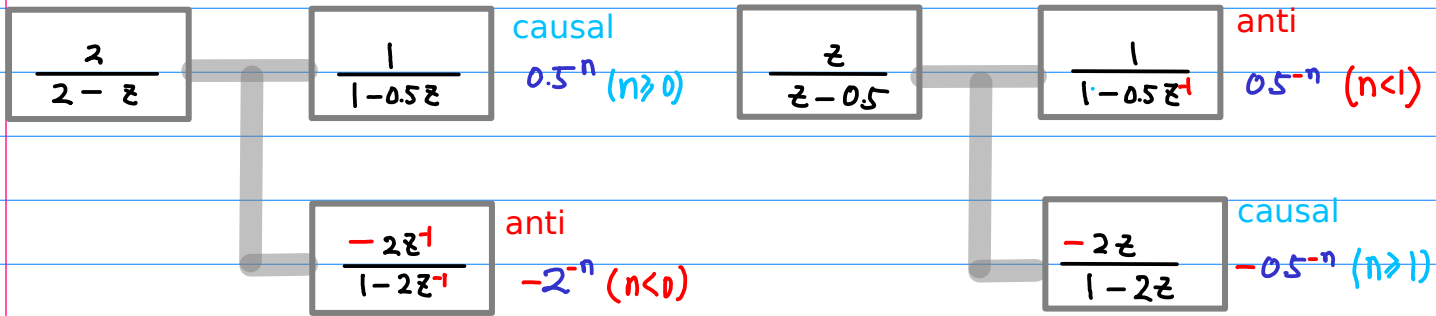
anti
 $0.5^{n+1} (n < 0)$

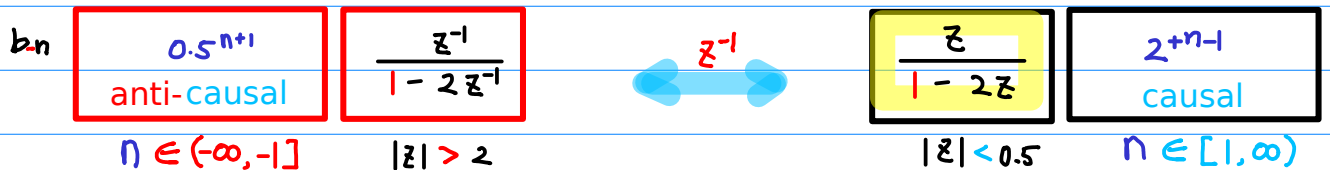
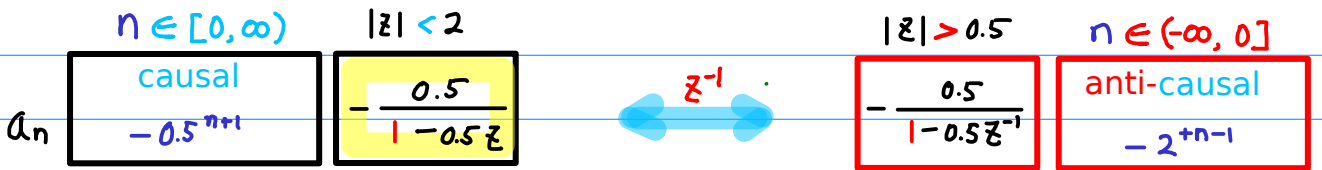
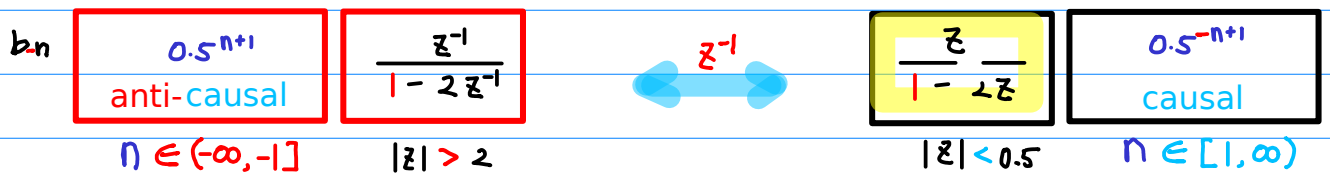
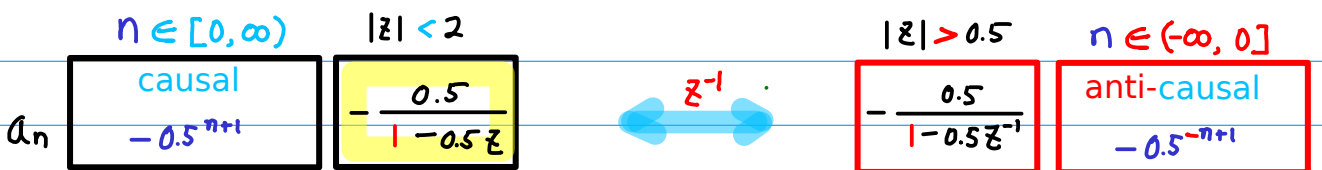
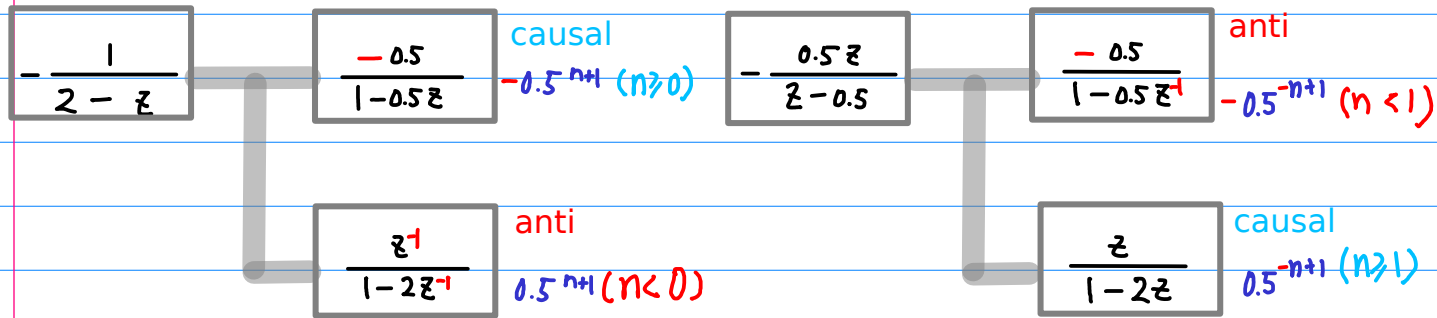
$$\frac{z}{1-2z}$$

causal
 $0.5^{-n+1} (n \geq 1)$

$$\frac{z}{2-z}, \frac{z}{z-0.5} \quad \text{vs.} \quad -\frac{0.5}{1-0.5z}, \frac{z}{1-2z}$$







Time Shift

$$P=2$$

- ① $(n \geq 0) \quad a_n = \left(\frac{1}{2}\right)^n \quad f(z) = \frac{2}{2-z} \quad X(z) = \frac{z}{z-0.5}$
- ③ $(n < 0) \quad a_n = \left(\frac{1}{2}\right)^n \quad f(z) = -\frac{2}{2-z} \quad X(z) = -\frac{z}{z-0.5}$
- ⑤ $(n \geq 1) \quad a_{n-1} = \left(\frac{1}{2}\right)^{n-1} \quad f(z) = \frac{2z}{2-z} \quad X(z) = \frac{1}{z-0.5}$
- ⑦ $(n < 1) \quad a_{n-1} = \left(\frac{1}{2}\right)^{n-1} \quad f(z) = -\frac{2z}{2-z} \quad X(z) = -\frac{1}{z-0.5}$
- ⑨ $(n \geq -1) \quad a_{n+1} = \left(\frac{1}{2}\right)^{n+1} \quad f(z) = \frac{2}{(2-z)z} \quad X(z) = \frac{z^2}{z-0.5}$
- ⑪ $(n < -1) \quad a_{n+1} = \left(\frac{1}{2}\right)^{n+1} \quad f(z) = -\frac{2}{(2-z)z} \quad X(z) = -\frac{z^2}{z-0.5}$

Time Shift

$$P = \frac{1}{2}$$

- ② $(n \geq 0) \quad a_n = (2)^n \quad f(z) = \frac{0.5}{0.5-z} \quad X(z) = \frac{z}{z-2}$
- ④ $(n < 0) \quad a_n = (2)^n \quad f(z) = -\frac{0.5}{0.5-z} \quad X(z) = -\frac{z}{z-2}$
- ⑥ $(n \geq 1) \quad a_{n-1} = (2)^{n-1} \quad f(z) = \frac{0.5z}{0.5-z} \quad X(z) = \frac{1}{z-2}$
- ⑧ $(n < 1) \quad a_{n-1} = (2)^{n-1} \quad f(z) = -\frac{0.5z}{0.5-z} \quad X(z) = -\frac{1}{z-2}$
- ⑩ $(n \geq -1) \quad a_{n+1} = (2)^{n+1} \quad f(z) = \frac{0.5}{(0.5-z)z} \quad X(z) = \frac{z^2}{z-2}$
- ⑫ $(n < -1) \quad a_{n+1} = (2)^{n+1} \quad f(z) = -\frac{0.5}{(0.5-z)z} \quad X(z) = -\frac{z^2}{z-2}$

Time Shift

$$2 \leftrightarrow \frac{1}{2}$$

- | | | | |
|---|--|--------------------------------|-----------------------------|
| ① | $(n \geq 0) \quad a_n = \left(\frac{1}{2}\right)^n$ | $f(z) = \frac{2}{2-z}$ | $X(z) = \frac{z}{z-0.5}$ |
| ② | $(n \geq 0) \quad a_n = (2)^n$ | $f(z) = \frac{0.5}{0.5-z}$ | $X(z) = \frac{z}{z-2}$ |
| ③ | $(n < 0) \quad a_n = \left(\frac{1}{2}\right)^n$ | $f(z) = -\frac{2}{2-z}$ | $X(z) = -\frac{z}{z-0.5}$ |
| ④ | $(n < 0) \quad a_n = (2)^n$ | $f(z) = -\frac{0.5}{0.5-z}$ | $X(z) = -\frac{z}{z-2}$ |
| ⑤ | $(n \geq 1) \quad a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ | $f(z) = \frac{2z}{2-z}$ | $X(z) = \frac{1}{z-0.5}$ |
| ⑥ | $(n \geq 1) \quad a_{n-1} = (2)^{n-1}$ | $f(z) = \frac{0.5z}{0.5-z}$ | $X(z) = \frac{1}{z-2}$ |
| ⑦ | $(n < 1) \quad a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ | $f(z) = -\frac{2z}{2-z}$ | $X(z) = -\frac{1}{z-0.5}$ |
| ⑧ | $(n < 1) \quad a_{n-1} = (2)^{n-1}$ | $f(z) = -\frac{0.5z}{0.5-z}$ | $X(z) = -\frac{1}{z-2}$ |
| ⑨ | $(n \geq -1) \quad a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ | $f(z) = \frac{2}{(2-z)z}$ | $X(z) = \frac{z^2}{z-0.5}$ |
| ⑩ | $(n \geq -1) \quad a_{n+1} = (2)^{n+1}$ | $f(z) = \frac{0.5}{(0.5-z)z}$ | $X(z) = \frac{z^2}{z-2}$ |
| ⑪ | $(n < -1) \quad a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ | $f(z) = -\frac{2}{(2-z)z}$ | $X(z) = -\frac{z^2}{z-0.5}$ |
| ⑫ | $(n < -1) \quad a_{n+1} = (2)^{n+1}$ | $f(z) = -\frac{0.5}{(0.5-z)z}$ | $X(z) = -\frac{z^2}{z-2}$ |

Shift to the right

Shift to the right \rightarrow

delete a_0

$\times z$

$\times z^{-1}$

① $(n \geq 0) \quad a_n = \left(\frac{1}{2}\right)^n$ $f(z) = \frac{2}{2-z}$ $X(z) = \frac{z}{z-0.5}$

⑤ $(n \geq 1) \quad a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = \frac{2z}{2-z}$ $X(z) = \frac{1}{z-0.5}$

② $(n \geq 0) \quad a_n = (2)^n$ $f(z) = \frac{0.5}{0.5-z}$ $X(z) = \frac{z}{z-2}$

⑥ $(n \geq 1) \quad a_{n-1} = (2)^{n-1}$ $f(z) = \frac{0.5z}{0.5-z}$ $X(z) = \frac{1}{z-2}$

Shift to the right \rightarrow

insert a_0

$\times z$

$\times z^{-1}$

③ $(n < 0) \quad a_n = \left(\frac{1}{2}\right)^n$ $f(z) = -\frac{2}{2-z}$ $X(z) = -\frac{z}{z-0.5}$

⑦ $(n < 1) \quad a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = -\frac{2z}{2-z}$ $X(z) = -\frac{1}{z-0.5}$

④ $(n < 0) \quad a_n = (2)^n$ $f(z) = -\frac{0.5}{0.5-z}$ $X(z) = -\frac{z}{z-2}$

⑧ $(n < 1) \quad a_{n-1} = (2)^{n-1}$ $f(z) = -\frac{0.5z}{0.5-z}$ $X(z) = -\frac{1}{z-2}$

Shift to the left

Shift to the left ←

$\times z^{-1}$

$\times z$

delete a_0

① $(n \geq 0) \quad a_n = \left(\frac{1}{2}\right)^n \quad f(z) = \frac{2}{2-z} \quad X(z) = \frac{z}{z-0.5}$

⑨ $(n \geq -1) \quad a_{n+1} = \left(\frac{1}{2}\right)^{n+1} \quad f(z) = \frac{2}{(2-z)z} \quad X(z) = \frac{z}{z-0.5}$

② $(n \geq 0) \quad a_n = (2)^n \quad f(z) = \frac{0.5}{0.5-z} \quad X(z) = \frac{z}{z-2}$

⑩ $(n \geq -1) \quad a_{n+1} = (2)^{n+1} \quad f(z) = \frac{0.5}{(0.5-z)z} \quad X(z) = \frac{z}{z-2}$

Shift to the left ←

$\times z^{-1}$

$\times z$

insert a_0

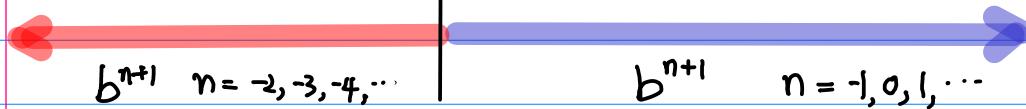
③ $(n < 0) \quad a_n = \left(\frac{1}{2}\right)^n \quad f(z) = -\frac{2}{2-z} \quad X(z) = -\frac{z}{z-0.5}$

⑪ $(n < -1) \quad a_{n+1} = \left(\frac{1}{2}\right)^{n+1} \quad f(z) = -\frac{2}{(2-z)z} \quad X(z) = -\frac{z}{z-0.5}$

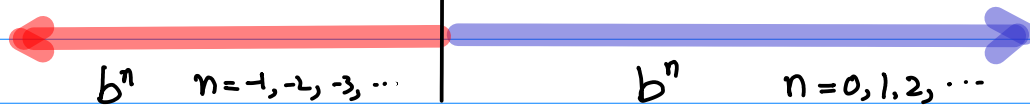
④ $(n < 0) \quad a_n = (2)^n \quad f(z) = -\frac{0.5}{0.5-z} \quad X(z) = -\frac{z}{z-2}$

⑫ $(n < -1) \quad a_{n+1} = (2)^{n+1} \quad f(z) = -\frac{0.5}{(0.5-z)z} \quad X(z) = -\frac{z}{z-2}$

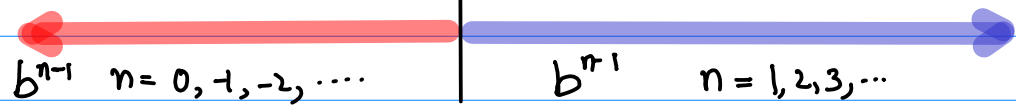
$n = -4$	$n = -3$	$n = -2$	$n = -1$	$n = 0$	$n = 1$	$n = 2$	
b^3	b^2	b^{-1}	b^0	b^1	b^2	b^3	



$n = -3$	$n = -2$	$n = -1$	$n = 0$	$n = 1$	$n = 2$	$n = 3$	
b^3	b^2	b^{-1}	b^0	b^1	b^2	b^3	



$n = -3$	$n = -2$	$n = -1$	$n = 0$	$n = 1$	$n = 2$	$n = 3$		
	b^3	b^2	b^{-1}	b^0	b^1	b^2	b^3	



$$1 \leftrightarrow \frac{1}{z}$$

- | | | | |
|---|--|----------------------------|-------------------------|
| ① | $(n \geq 0) \quad a_n = (1)^n$ | $f(z) = \frac{1}{1-z}$ | $X(z) = \frac{z}{z-1}$ |
| ② | $(n \geq 0) \quad a_n = (1^{-1})^n$ | $f(z) = \frac{1}{1-z}$ | $X(z) = \frac{z}{z-1}$ |
| ③ | $(n < 0) \quad a_n = (1)^n$ | $f(z) = -\frac{1}{1-z}$ | $X(z) = -\frac{z}{z-1}$ |
| ④ | $(n < 0) \quad a_n = (1^{-1})^n$ | $f(z) = -\frac{1}{1-z}$ | $X(z) = -\frac{z}{z-1}$ |
| ⑤ | $(n \geq 1) \quad a_{n-1} = (1)^{n-1}$ | $f(z) = \frac{z}{1-z}$ | $X(z) = \frac{1}{z-1}$ |
| ⑥ | $(n \geq 1) \quad a_{n-1} = (1^{-1})^{n-1}$ | $f(z) = \frac{z}{1-z}$ | $X(z) = \frac{1}{z-1}$ |
| ⑦ | $(n < 1) \quad a_{n-1} = (1)^{n-1}$ | $f(z) = -\frac{z}{1-z}$ | $X(z) = -\frac{1}{z-1}$ |
| ⑧ | $(n < 1) \quad a_{n-1} = (1^{-1})^{n-1}$ | $f(z) = -\frac{z}{1-z}$ | $X(z) = -\frac{1}{z-1}$ |
| ⑨ | $(n \geq -1) \quad a_{n+1} = (1)^{n+1}$ | $f(z) = \frac{1}{(1-z)z}$ | $X(z) = \frac{z}{z-1}$ |
| ⑩ | $(n \geq -1) \quad a_{n+1} = (1^{-1})^{n+1}$ | $f(z) = \frac{1}{(1-z)z}$ | $X(z) = \frac{z}{z-1}$ |
| ⑪ | $(n < -1) \quad a_{n+1} = (1)^{n+1}$ | $f(z) = -\frac{1}{(1-z)z}$ | $X(z) = -\frac{z}{z-1}$ |
| ⑫ | $(n < -1) \quad a_{n+1} = (1^{-1})^{n+1}$ | $f(z) = -\frac{1}{(1-z)z}$ | $X(z) = -\frac{z}{z-1}$ |

$$① \quad (n \geq 0) \quad a_n = (1)^n \quad f(z) = \frac{1}{1-z} \quad X(z) = \frac{z}{z-1}$$

$$② \quad (n \geq 0) \quad a_n = (1^{-1})^n \quad f(z) = \frac{1}{1-z} \quad X(z) = \frac{z}{z-1}$$

Shift to the right \rightarrow
delete a_0

$\times z$

$\times z^{-1}$

$$⑤ \quad (n \geq 1) \quad a_{n-1} = (1)^{n-1} \quad f(z) = \frac{z}{1-z} \quad X(z) = \frac{1}{z-1}$$

$$⑥ \quad (n \geq 1) \quad a_{n-1} = (1^{-1})^{n-1} \quad f(z) = \frac{z}{1-z} \quad X(z) = \frac{1}{z-1}$$

$$③ \quad (n < 0) \quad a_n = (1)^n \quad f(z) = -\frac{1}{1-z} \quad X(z) = -\frac{z}{z-1}$$

$$④ \quad (n < 0) \quad a_n = (1^{-1})^n \quad f(z) = -\frac{1}{1-z} \quad X(z) = -\frac{z}{z-1}$$

Shift to the right \rightarrow
insert a_0

$\times z$

$\times z^{-1}$

$$⑦ \quad (n < 1) \quad a_{n-1} = (1)^{n-1} \quad f(z) = -\frac{z}{1-z} \quad X(z) = -\frac{1}{z-1}$$

$$⑧ \quad (n < 1) \quad a_{n-1} = (1^{-1})^{n-1} \quad f(z) = -\frac{z}{1-z} \quad X(z) = -\frac{1}{z-1}$$

$$\textcircled{1} \quad (n \geq 0) \quad a_n = (1)^n \quad f(z) = \frac{1}{1-z} \quad X(z) = \frac{z}{z-1}$$

$$\textcircled{2} \quad (n \geq 0) \quad a_n = (1^{-1})^n \quad f(z) = \frac{1}{1-z} \quad X(z) = \frac{z}{z-1}$$

Shift to the left ←
delete a_0

$\times z^{-1}$

$\times z$

$$\textcircled{9} \quad (n \geq -1) \quad a_{n+1} = (1)^{n+1} \quad f(z) = \frac{1}{(1-z)z} \quad X(z) = \frac{z}{z-1}$$

$$\textcircled{10} \quad (n \geq -1) \quad a_{n+1} = (1^{-1})^{n+1} \quad f(z) = \frac{1}{(1-z)z} \quad X(z) = \frac{z}{z-1}$$

$$\textcircled{3} \quad (n < 0) \quad a_n = (1)^n \quad f(z) = -\frac{1}{1-z} \quad X(z) = -\frac{z}{z-1}$$

$$\textcircled{4} \quad (n < 0) \quad a_n = (1^{-1})^n \quad f(z) = -\frac{1}{1-z} \quad X(z) = -\frac{z}{z-1}$$

Shift to the left ←
insert a_0

$\times z^{-1}$

$\times z$

$$\textcircled{11} \quad (n < -1) \quad a_{n+1} = (1)^{n+1} \quad f(z) = -\frac{1}{(1-z)z} \quad X(z) = -\frac{z}{z-1}$$

$$\textcircled{12} \quad (n < -1) \quad a_{n+1} = (1^{-1})^{n+1} \quad f(z) = -\frac{1}{(1-z)z} \quad X(z) = -\frac{z}{z-1}$$

Causality

$$f(z) \quad (|z| < p) \quad \leftrightarrow \quad a_n \quad (n \geq 0) \quad - (p^{-1}, p^{-2}, p^{-3}, \dots)$$

$$X(z^{-1}) \quad (|z| < p) \quad \leftrightarrow \quad x_{-n} \quad (n < 1) \quad - (p^{-1}, p^{-2}, p^{-3}, \dots)$$

$$f(z^{-1}) \quad (|z| > p^{-1}) \quad \leftrightarrow \quad a_{-n} \quad (n < 1) \quad - (p^{-1}, p^{-2}, p^{-3}, \dots)$$

$$X(z) \quad (|z| > p^{-1}) \quad \leftrightarrow \quad x_n \quad (n \geq 0) \quad - (p^{-1}, p^{-2}, p^{-3}, \dots)$$

$$f(z) \quad (|z| > p) \quad \leftrightarrow \quad -a_n \quad (n < 0) \quad (p^0, p^1, p^2, \dots)$$

$$X(z^{-1}) \quad (|z| > p) \quad \leftrightarrow \quad -x_{-n} \quad (n \geq 1) \quad (p^0, p^1, p^2, \dots)$$

$$f(z^{-1}) \quad (|z| < p^{-1}) \quad \leftrightarrow \quad -a_{-n} \quad (n \geq 1) \quad (p^0, p^1, p^2, \dots)$$

$$X(z) \quad (|z| < p^{-1}) \quad \leftrightarrow \quad -x_n \quad (n < 0) \quad (p^0, p^1, p^2, \dots)$$

$$\begin{array}{|c|} \hline f(z) \\ \hline g(z^{-1}) \\ \hline \end{array} \quad \begin{array}{|c|} \hline f(z^{-1}) \\ \hline g(z) \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline X(z^{-1}) \\ \hline Y(z) \\ \hline \end{array} \quad \begin{array}{|c|} \hline X(z) \\ \hline Y(z^{-1}) \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline a_n \quad a_{-n} \\ \hline b_{-n} \quad b_n \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline x_{-n} \quad x_n \\ \hline y_n \quad y_{-n} \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline f(z) \quad f(z^{-1}) \\ \hline f(z) \quad f(z^{-1}) \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline X(z^{-1}) \quad X(z) \\ \hline X(z^{-1}) \quad X(z) \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline -(p^1, p^2, p^3, \dots) \quad -(p^1, p^2, p^3, \dots) \\ \hline (p^0, p^1, p^2, \dots) \quad (p^0, p^1, p^2, \dots) \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline -(p^1, p^2, p^3, \dots) \quad -(p^1, p^2, p^3, \dots) \\ \hline (p^0, p^1, p^2, \dots) \quad (p^0, p^1, p^2, \dots) \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline -\frac{p^{-1}}{1-p^{-1}z} \\ \hline \frac{z^{-1}}{1-pz^{-1}} \\ \hline \end{array} \quad \begin{array}{|c|} \hline -\frac{p^{-1}}{1-p^{-1}z^{-1}} \\ \hline \frac{z}{1-pz} \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline -\frac{p^{-1}}{1-p^{-1}z} \\ \hline \frac{z^{-1}}{1-pz^{-1}} \\ \hline \end{array} \quad \begin{array}{|c|} \hline -\frac{p^{-1}}{1-p^{-1}z^{-1}} \\ \hline \frac{z}{1-pz} \\ \hline \end{array}$$

$$\begin{matrix} f(z) & g(z) \\ f(z) & g(z) \end{matrix}$$

$$\begin{matrix} Y(z) & X(z) \\ Y(z) & X(z) \end{matrix}$$

$$\begin{matrix} a_n & a_{-n} \\ -a_n & -a_{-n} \end{matrix}$$

$$\begin{matrix} x_n & x_n \\ -x_n & -x_n \end{matrix}$$

$$\begin{matrix} |z| < p & |z| > p^{-1} \\ |z| > p & |z| < p^{-1} \end{matrix}$$

$$\begin{matrix} |z| < p & |z| > p^{-1} \\ |z| > p & |z| < p^{-1} \end{matrix}$$

$$\begin{matrix} [0, \infty) & (-\infty, 0] \\ (-\infty, -1] & [1, \infty) \end{matrix}$$

$$\begin{matrix} (-\infty, 0] & [0, \infty) \\ [1, \infty) & (-\infty, -1] \end{matrix}$$

$$\begin{matrix} -(p^{-1}, p^{-2}, p^{-3}, \dots) \\ (p^0, p^1, p^2, \dots) \end{matrix}$$

$$\begin{matrix} -(p^{-1}, p^{-2}, p^{-3}, \dots) \\ (p^0, p^1, p^2, \dots) \end{matrix}$$

$$\begin{array}{l} a_n \ a_{-n} \\ -a_n \ -a_{-n} \end{array}$$

$$\begin{array}{l} 2^{-n} \ 2^n \\ -2^{-n} \ -2^n \end{array}$$

$$a_n = -2^{-n}$$

$$\begin{array}{l} x_n \ x_{-n} \\ -x_n \ -x_{-n} \end{array}$$

$$\begin{array}{l} 2^{-n} \ 2^n \\ -2^{-n} \ -2^n \end{array}$$

$$x_n = -2^n$$

$$\begin{array}{l} -(p^0, p^1, p^2, \dots) \ -(p^0, p^1, p^2, \dots) \\ (p^0, p^1, p^2, \dots) \ (p^0, p^1, p^2, \dots) \end{array}$$

$$\begin{array}{l} -(2^0, 2^1, 2^2, \dots) \ -(2^0, 2^1, 2^2, \dots) \\ (2^0, 2^1, 2^2, \dots) \ (2^0, 2^1, 2^2, \dots) \end{array}$$

$$\begin{array}{l} -\frac{p^{-1}}{1-p^{-1}z} \quad -\frac{p^1}{1-p^1z^{-1}} \\ \frac{z^{-1}}{1-pz^{-1}} \quad \frac{z}{1-pz} \end{array}$$

$$\begin{array}{l} \frac{2^{-1}}{1-2^{-1}z} \quad \frac{2^1}{1-2^1z^{-1}} \\ -\frac{z^{-1}}{1-2z^{-1}} \quad -\frac{z}{1-2z} \end{array}$$

$$\begin{array}{l} \frac{(\frac{1}{2})}{1-(\frac{z}{2})} \quad \frac{(\frac{1}{2})}{1-(\frac{1}{2z})} \\ -\frac{(\frac{1}{z})}{1-(\frac{2}{z})} \quad -\frac{z}{1-2z} \end{array}$$

$$\begin{array}{l} |z| < p \quad |z| > p^{-1} \\ |z| > p \quad |z| < p^{-1} \end{array}$$

$$\begin{array}{l} |z| < 2 \quad |z| > 2^{-1} \\ |z| > 2 \quad |z| < 2^{-1} \end{array}$$

$$\begin{array}{l} [0, \infty) \ (-\infty, 0] \\ (-\infty, -1] \ [1, \infty) \end{array}$$

$$\begin{array}{l} [0, \infty) \ (-\infty, 0] \\ (-\infty, -1] \ [1, \infty) \end{array}$$

