

# Stationary Random Processes - Examples

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles,Jr. and B. Shi

# Outline

- 1 Random Phase Oscillator
  - Problem definition
  - First order distribution
    - Uniform random variable  $\Theta$
    - Uniform random variable  $T$
  - Second order distribution
  - Mean and variance
- 2 Stationary Process Examples
  - Examples - A
  - Examples - B
- 3 Cyclo-stationary Process Examples
  - Examples

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# $\sin(t)$ , $A\sin(t)$

- $\sin(t)$ 
  - not random process.
- $x(t) = A\sin(t)$ 
  - can be a **random process** if  $A$  is a **random variable**
  - However,  $x(t)$  is not **stationary**, but it is **cyclostationary**,
  - its statistical properties vary periodically.

<https://dsp.stackexchange.com/questions/32000/why-is-sint-a-stationary-process>

# $A\sin(t + \phi)$

- $x(t) = A\sin(t + \phi)$ 
  - the  $x(t)$  process is **stationary** because of the added **random phase**
  - the random phase  $\phi \in [0, 2\pi]$  is a **uniformly distributed random variable** which is independent of  $A$ .
  - its statistical properties are independent of  $t$ , and hence, the process is **stationary**.

<https://dsp.stackexchange.com/questions/32000/why-is-sint-a-stationary-process>

## Signals in an oscilloscope

When analyzing a signal with an oscilloscope,  
it can be observed that

the signal's **amplitude spectrum**  
does not vary over moving windows

so a sinusoidal wave is sort of **stationary** in frequency.

Additionally, the signal is itself **stationary** in envelope  
(modulus 1 for the analytic version of the signal).

<https://dsp.stackexchange.com/questions/32000/why-is-sint-a-stationary-process>

## Window function (1)

In signal processing and statistics, a **window function** is a mathematical function that is

- zero-valued outside of some chosen interval
- normally symmetric around the middle of the interval
- usually near a maximum in the middle
- usually tapering away from the middle.

[https://en.wikipedia.org/wiki/Window\\_function](https://en.wikipedia.org/wiki/Window_function)



## Window function (2)

when another function or waveform is  
"multiplied" by a **window function**,

the product is also zero-valued outside the interval:  
all that is left is the part where they overlap,  
the "*view through the window*".

[https://en.wikipedia.org/wiki/Window\\_function](https://en.wikipedia.org/wiki/Window_function)

# Envelope

- the **envelope** of an oscillating signal is a smooth curve outlining its extremes.
- the **envelope** thus generalizes the concept of a constant amplitude into an instantaneous amplitude.
- a modulated sine wave varying between an upper envelope and a lower envelope.
- the **envelope function** may be a function of time, space, angle, or indeed of any variable

[https://en.wikipedia.org/wiki/Envelope\\_\(waves\)](https://en.wikipedia.org/wiki/Envelope_(waves))

# Random Variable Definition

## A random variable

a real function over a sample space  $S = \{s_1, s_2, s_3, \dots, s_n\}$

$$s \rightarrow X(s)$$

$$x = X(s)$$

a random variable : a capital letter  $X$

a particular value : a lowercase letter  $x$

a sample space  $S = \{s_1, s_2, s_3, \dots, s_n\}$

an element of  $S$  :  $s$

## Random Variable Example

### Example

$$X(s_1) = x_1 \quad s_1 \longrightarrow x_1$$

$$X(s_2) = x_2 \quad s_2 \longrightarrow x_2$$

...

$$X(s_n) = x_n \quad s_n \longrightarrow x_n$$

$$S = \{s_1, s_2, s_3, \dots, s_n\}$$

$$X = \{x_1, x_2, x_3, \dots, x_n\}$$

a sample space  
a random variable

# Random Process (1)

## A random process

a function of both **time**  $t$  and **outcome**  $\theta$

$$X(t, \theta)$$

assigning a **time function** to every **outcome**  $\theta_i$

$$\theta_i \rightarrow x_i(t)$$

where  $x_i(t) = x(t, \theta_i)$

the family of such **time functions**  
is called a **random process**  
and denoted by  $X(t, \theta)$

## Random Process (2)

### A random process

a random process  $X(t, \theta)$   
assigns a time function for a every outcome  $\theta$

$$x(t, \theta) = X(t, \theta)$$

a short notation

$$x(t) = X(t)$$

# Ensemble of time functions

## Time functions

A random process  $X(t, \theta)$  represents  
 a family or ensemble of **time functions**

$$X(t, \theta_1) = x_1(t) \quad \theta_1 \longrightarrow x_1(t) = \cos(\omega t + \theta_1)$$

$$X(t, \theta_2) = x_2(t) \quad \theta_2 \longrightarrow x_2(t) = \cos(\omega t + \theta_2)$$

...

...

$$X(t, \theta_n) = x_n(t) \quad \theta_n \longrightarrow x_n(t) = \cos(\omega t + \theta_n)$$

$S = \{ \theta_1, \theta_2, \theta_3, \dots, \theta_n \}$  a sample space

$X(t) = \{ x_1(t), x_2(t), x_3(t), \dots, x_n(t) \}$  a random process

## A sample function $x(t, \theta)$

A random process  $X(t, \theta)$  represents  
a family or ensemble of **time functions**

$$\theta \rightarrow x(t, \theta) = \cos(\omega t + \theta)$$

$x(t, \theta)$  represents

- a **sample function**
- an ensemble member
- a realization of the process

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>



## Random process $X(t, \theta)$

A random process  $X(t, \theta)$  represents  
a family or ensemble of **time functions**

$$\theta \rightarrow x(t, \theta) = \cos(\omega t + \theta)$$
$$x(t) = X(t, \theta)$$

$X(t, \theta)$  becomes

- a **single time function**  $x(t, \theta)$
- when  $t$  is a variable and  $\theta$  is **fixed** at an **outcome**

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

## Random variables with time

a **random process**  $X(t, s)$  represents a **single time function** when  $t$  is a variable and  $s$  is fixed at an outcome

a random process  $X(t, s)$  represents a **single random variable** when both  $t$  and  $s$  are fixed at a time and an outcome, respectively

$$X_i = X(t_i, s) = X(t_i)$$

*random variable*

$$X(t, s) = X(t)$$

*random process*

## Random phase in $X(t) = \cos(\omega t + \Theta)$

Consider the output of a sinusoidal oscillator that has a **random phase** and an **amplitude** of the form:

$$X(t) = \cos(\omega t + \Theta)$$

where the **random variable**  $\Theta \sim U([0, 2\pi])$

to specify the explicit dependence on the underlying **sample space**  $S$  the oscillator output can be written as

$$x(t, \Theta) = \cos(\omega t + \Theta)$$

## Random variable $X_t(\theta)$

Consider the **random variable**

$$X(t, \theta) = \cos(\omega t + \theta)$$

where the time  $t$  is fixed

In other words,

$$X_t(\theta) = \cos(\omega t + \theta)$$

where  $\theta_0 = \omega t$  is fixed (a *non-random* quantity)

thus the time  $t$  is fixed

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

## Values of a time function

Consider the **random variable** for the fixed time  $t$

$$X_t(\theta) = \cos(\omega t + \theta)$$

if the sample value  $\theta$  as well as the time  $t$  is fixed,  
 then the values of the time function

$$x_1 = x(t_1) = \cos(\omega t_1 + \theta)$$

$$x_2 = x(t_2) = \cos(\omega t_2 + \theta)$$

where  $x$  is the **time function** for a fixed outcome  $\theta$  and  
 let  $x_i$  denotes the value of the time function  $x$  at times  $t_i$   
 (here  $x_i$  is not a sample function)

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$f_X(x)$  of  $X(t) = \cos(\omega t + \Theta)$

- Uniform Random Variable  $\Theta$
- Random Process  $X(t) = \cos(\omega t + \Theta)$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

## first order distribution

To get the **first order distribution** of the **random process**

$$X(t) = \cos(\omega t + \Theta)$$

consider the **first order distribution** of the **random variable**

$$X_t(\Theta) = \cos(\theta_0 + \Theta)$$

where  $\theta_0 = \omega t$  is fixed (a *non-random* quantity)

$f_X(x)$  can be obtained via the **derivative method**

$$\frac{d}{dx} F_X(x) = f_\Theta(\theta) \cdot \frac{d\theta}{dx}$$

$$f_X(x) = \frac{1}{\pi\sqrt{1-x^2}}, \quad |x| < 1$$



# $f_X(x)$ of $X(t) = \cos(\omega t + \Theta)$

The **first order distribution** of the process  $X(t) = \cos(\omega t + \Theta)$

$$f_X(x) = \frac{1}{\pi\sqrt{1-x^2}}, \quad |x| < 1$$

- dependent only on the set of values  $x$  that the process  $X(t)$  takes
- independent of
  - the particular **sampling instant**  $t$
  - the constant **phase offset**  $\theta_0 = \omega t$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

## random variable $\Theta$

Let  $\Theta$  be a uniform **random variable** on  $[0, 2\pi]$ , then

$$F_{\Theta}(\theta) = \frac{\theta}{2\pi}$$

Let

$$X_t(\Theta) = \cos(\theta_0 + \Theta)$$

be the **random variable** describing  $x$  in terms of  $\Theta$ .

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

$$F_X(x) = F_{\Theta}(\theta_1) - F_{\Theta}(\theta_2)$$

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(\cos(\omega t + \Theta) \leq x) \\ &= P\left(\cos^{-1}(x) \leq \omega t + \Theta \leq 2\pi - \cos^{-1}(x)\right) \\ &= P\left(\cos^{-1}(x) - \omega t \leq \Theta \leq 2\pi - \cos^{-1}(x) - \omega t\right) \\ &= P\left(\Theta \leq 2\pi - \cos^{-1}(x) - \omega t\right) - P\left(\Theta \leq \cos^{-1}(x) - \omega t\right) \\ &= F_{\Theta}\left(2\pi - \cos^{-1}(x) - \omega t\right) - F_{\Theta}\left(\cos^{-1}(x) - \omega t\right) \\ &= F_{\Theta}(\theta_1) - F_{\Theta}(\theta_2) \end{aligned}$$

Random variable  $X$ , a particular value  $x$

Random variable  $\Theta$ , particular values  $\theta_1$  and  $\theta_2$

<https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation>

# Chain rule

The chain rule

$$\frac{d}{dx} F_X(x) = \frac{d}{d\theta} F_\Theta(\theta) \cdot \frac{d\theta}{dx}$$

Random variable  $X$ , a particular value  $x$

Random variable  $\Theta$ , a particular value  $\theta$

$$\frac{d}{d\theta} F_\Theta(\theta) = f_\Theta(\theta) \qquad \frac{d}{d\theta} \left( \frac{\theta}{2\pi} \right) = \frac{1}{2\pi}$$

$$\frac{d}{dx} F_X(x) = \frac{d}{d\theta} F_\Theta(\theta) \cdot \frac{d\theta}{dx} = f_\Theta(\theta) \cdot \frac{d\theta}{dx} = \frac{1}{2\pi} \frac{d\theta}{dx}$$

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## derivative of $F_X(x)$

Differentiating both sides, we get:

$$\begin{aligned} \frac{d}{dx} F_X(x) &= \frac{d}{dx} \left\{ F_\Theta \left( 2\pi - \cos^{-1}(x) - \omega t \right) - F_\Theta \left( \cos^{-1}(x) - \omega t \right) \right\} \\ &= \frac{d}{d\theta} F_\Theta \left( 2\pi - \cos^{-1}(x) - \omega t \right) \frac{d}{dx} \left( 2\pi - \cos^{-1}(x) - \omega t \right) \\ &\quad - \frac{d}{d\theta} F_\Theta \left( \cos^{-1}(x) - \omega t \right) \frac{d}{dx} \left( \cos^{-1}(x) - \omega t \right) \end{aligned}$$

note

$$\begin{aligned} \theta_1 &= 2\pi - \cos^{-1}(x) - \omega t & \frac{d\theta_1}{dx} &= -\frac{d}{dx} \cos^{-1}(x) \\ \theta_2 &= \cos^{-1}(x) - \omega t & \frac{d\theta_2}{dx} &= +\frac{d}{dx} \cos^{-1}(x) \end{aligned}$$

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$f_X(x)$  of  $X(t) = \cos(\omega t + \Theta)$

$$X_t(\Theta) = \cos(\omega t + \Theta)$$

$$\cos^{-1}(x) \leq \omega t + \Theta \leq 2\pi - \cos^{-1}(x)$$

$$F_X(x) = F_\Theta(2\pi - \cos^{-1}(x) - \omega t) - F_\Theta(\cos^{-1}(x) - \omega t)$$

using the chain rule

$$\frac{d}{dx} F_X(x) = \frac{d}{d\theta} F_\Theta(\theta) \frac{d\theta}{dx} = f_\Theta(\theta) \frac{d\theta}{dx} = \frac{1}{2\pi} \frac{d\theta}{dx}$$

$$f_X(x) = f_\Theta(2\pi - \cos^{-1}(x) - \omega t) \frac{d}{dx} (-\cos^{-1}(x)) - f_\Theta(\cos^{-1}(x) - \omega t) \frac{d}{dx} (\cos^{-1}(x))$$

<https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation>

# $f_X(x)$ of $X(t) = \cos(\omega t + \Theta)$

$$f_X(x) = f_\Theta \left( 2\pi - \cos^{-1}(x) - \omega t \right) \frac{d}{dx} \left( -\cos^{-1}(x) \right) \\ - f_\Theta \left( \cos^{-1}(x) - \omega t \right) \frac{d}{dx} \left( \cos^{-1}(x) \right)$$

Now, since  $f_\Theta(\theta) = \frac{1}{2\pi}$  and  $\frac{d}{dx} \cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$ , we have:

$$f_X(x) = \frac{1}{2\pi} \left( \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} \right) \\ = \frac{1}{\pi\sqrt{1-x^2}}$$

<https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation>

$$f_X(x) \text{ of } X(t) = \cos(\omega t + \Theta) \quad (10)$$

Consider the output of a sinusoidal oscillator that has a random phase and an amplitude of the form:

$$X(t) = \cos(\omega t + \Theta)$$

where  $\Theta$  is a uniform random variable on  $[0, 2\pi]$  then the **first order pdf** of  $X(t)$  is

$$f_X(x) = \frac{1}{\pi\sqrt{1-x^2}}, \quad x \in (-1, 1)$$

Note that the probability is unaffected by angular velocity and initial phase  $(\omega, \theta_0)$ , which is, intuitively, expected.

<https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation>



$f_X(x)$  of  $X = \cos(\omega T + \phi)$  (1)

- Uniform Random Variable  $T$
- Random Variable  $X = \cos(\omega T + \phi)$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

## $f_X(x)$ of $X = \cos(\omega T + \phi)$ (2)

Let  $T$  be a uniform **random variable** on  $[0, \frac{2\pi}{\omega}]$  that describes time. Then

$$F_T(t) = \frac{\omega}{2\pi} \cdot t = ft$$

where  $f$  is the oscillation's frequency. Now, let:

$$X = \cos(\omega T + \phi)$$

be the **random variable** describing  $x$  in terms of  $T$ .  
 it is not a time function

$$X(t) \neq \cos(\omega T + \phi)$$

# $f_X(x)$ of $X = \cos(\omega T + \phi)$ (3)

$$\begin{aligned}
 F_X(x) &= P(X \leq x) \\
 &= P(\cos(\omega T + \phi) \leq x) \\
 &= P\left(\cos^{-1}(x) \leq \omega T + \phi \leq 2\pi - \cos^{-1}(x)\right) \\
 &= P\left(\frac{\cos^{-1}(x) - \phi}{\omega} \leq T \leq \frac{2\pi - \cos^{-1}(x) - \phi}{\omega}\right) \\
 &= P\left(T \leq \frac{2\pi - \cos^{-1}(x) - \phi}{\omega}\right) - P\left(T \leq \frac{\cos^{-1}(x) - \phi}{\omega}\right) \\
 &= F_T\left(\frac{2\pi - \cos^{-1}(x) - \phi}{\omega}\right) - F_T\left(\frac{\cos^{-1}(x) - \phi}{\omega}\right) \\
 &= F_T(t_1) - F_T(t_2)
 \end{aligned}$$

Random variable  $X$ , a particular value  $x$

Random variable  $T$ , particular values  $t_1$  and  $t_2$

<https://math.stackexchange.com/questions/3456122/probability-density-function->

$$f_X(x) \text{ of } X(t) = \cos(\omega T + \phi) \quad (4)$$

The chain rule

$$\frac{d}{dt} F_T(t) = \frac{d}{d\theta} F_\Theta(\theta) \cdot \frac{d\theta}{dt}$$

Random variable  $T$ , a particular value  $t$

Random variable  $\Theta$ , a particular value  $\theta$

$$\frac{d}{dt} F_T(t) = f_T(t) \qquad \frac{d}{dt} \left( \frac{\omega}{2\pi} \cdot t \right) = \frac{\omega}{2\pi}$$

$$\frac{d}{dt} F_T(t) = \frac{d}{dt} F_T(t) \cdot \frac{d\theta}{dt} = f_T(t) \cdot \frac{d\theta}{dt} = \frac{\omega}{2\pi} \frac{d\theta}{dt}$$

<https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation>

$$f_X(x) \text{ of } X(t) = \cos(\omega T + \phi) \quad (5)$$

Differentiating both sides, we get:

$$\begin{aligned} \frac{d}{dx} F_X(x) &= \frac{d}{dx} \left\{ F_T \left( \frac{2\pi - \cos^{-1}(x) - \phi}{\omega} \right) - F_T \left( \frac{\cos^{-1}(x) - \phi}{\omega} \right) \right\} \\ &= \frac{d}{dt} F_T \left( \frac{2\pi - \cos^{-1}(x) - \phi}{\omega} \right) \frac{d}{dx} \left( \frac{\pi - \cos^{-1}(x) - \phi}{\omega} \right) \\ &\quad - \frac{d}{dt} F_T \left( \frac{\cos^{-1}(x) - \phi}{\omega} \right) \frac{d}{dx} \left( \frac{\cos^{-1}(x) - \phi}{\omega} \right) \end{aligned}$$

note

$$\begin{aligned} t_1 &= \frac{2\pi - \cos^{-1}(x) - \phi}{\omega} & \frac{dt_1}{dx} &= \frac{-\cos^{-1}(x)}{\omega} \\ t_2 &= \frac{\cos^{-1}(x) - \phi}{\omega} & \frac{dt_2}{dx} &= \frac{+\cos^{-1}(x)}{\omega} \end{aligned}$$

<https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation>

$f_X(x)$  of  $X(t) = \cos(\omega T + \phi)$  (6)

$$X(t) = \cos(\omega T + \phi)$$

$$\cos^{-1}(x) \leq \omega T + \phi \leq 2\pi - \cos^{-1}(x)$$

$$F_X(x) = F_T\left(\frac{2\pi - \cos^{-1}(x) - \phi}{\omega}\right) - F_T\left(\frac{\cos^{-1}(x) - \phi}{\omega}\right)$$

using the chain rule

$$\frac{d}{dt} F_T(t) = \frac{d}{dt} F_T(t) \cdot \frac{dt}{dx} = f_T(t) \cdot \frac{dt}{dx} = \frac{\omega}{2\pi} \frac{dt}{dx}$$

$$f_X(x) = f_T\left(\frac{2\pi - \cos^{-1}(x) - \phi}{\omega}\right) \frac{d}{dx} \left(-\frac{\cos^{-1}(x)}{\omega}\right) - f_T\left(\frac{\cos^{-1}(x) - \phi}{\omega}\right) \frac{d}{dx} \left(\frac{\cos^{-1}(x)}{\omega}\right)$$

<https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation>

$$f_X(x) \text{ of } X = \cos(\omega T + \phi) \quad (7)$$

Differentiating both sides, we get:

$$\begin{aligned} f_X(x) &= f_T \left( \frac{2\pi - \cos^{-1}(x) - \phi}{\omega} \right) \frac{d}{dx} \left( -\frac{\cos^{-1}(x)}{\omega} \right) \\ &\quad - f_T \left( \frac{\cos^{-1}(x) - \phi}{\omega} \right) \frac{d}{dx} \left( \frac{\cos^{-1}(x)}{\omega} \right) \end{aligned}$$

Now, since  $f_T(t) = \frac{\omega}{2\pi}$  and  $\frac{d}{dx} \cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$ , we have:

$$\begin{aligned} f_X(x) &= \frac{1}{2\pi} \left( \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} \right) \\ &= \frac{1}{\pi\sqrt{1-x^2}} \end{aligned}$$

<https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation>

$f_X(x)$  of  $X = \cos(\omega T + \phi)$  (8)

$$f_X(x) = \frac{1}{\pi\sqrt{1^2 - x^2}}, \quad x \in (-1, 1)$$

the probability is unaffected by angular velocity ( $\omega$ ) and initial phase ( $\phi$ ), which is, intuitively, expected.

<https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation>



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## Second order distribution (1)

to get the **second-order distribution**

use the **conditional distribution**  $f_{X(t_1)|X(t_2)}(x_1|x_2)$

as in :

$$f_{X(t_1),X(t_2)}(x_1,x_2) = f_{X(t_2)}(x_2)f_{X(t_1)|X(t_2)}(x_1|x_2)$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

## Second order distribution (2)

$$x_2 = x(t_2) = \cos(\omega t_2 + \theta)$$

This can happen only when :

$$(\omega t_2 + \theta) = \cos^{-1}(x_2)$$

$$(\omega t_2 + \theta) = 2\pi - \cos^{-1}(x_2)$$

$$\theta = \cos^{-1}(x_2) - \omega t_2$$

$$\theta = 2\pi - \cos^{-1}(x_2) - \omega t_2$$

where  $0 \leq \cos^{-1}(x_2) \leq \pi$  and  $0 \leq \theta \leq 2\pi$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

## Second order distribution (3)

given that  $x_2 = x(t_2) = \cos(\omega t_2 + \theta)$ :  
 find  $\theta$ ,

$$\theta = \begin{cases} + (\cos^{-1}(x_2) - \omega t_2) \\ - (\cos^{-1}(x_2) + \omega t_2) \end{cases}$$

then  $x_1 = x(t_1) = \cos(\omega t_1 + \theta)$  have two values

$$x(t_1) = \begin{cases} \cos(\omega t_1 + (\cos^{-1}(x(t_2)) - \omega t_2)) = x_{11} \\ \cos(\omega t_1 - (\cos^{-1}(x(t_2)) + \omega t_2)) = x_{12} \end{cases}$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

## Second order distribution (4)

given that  $x_2 = x(t_2) = \cos(\omega t_2 + \theta)$

find  $\theta$ , then  $x_1 = x(t_1) = \cos(\omega t_1 + \theta)$

has only two values with an equal probability 0.5

$$x(t_1) = \begin{cases} \cos(\omega t_1 + (\cos^{-1}(x(t_2)) - \omega t_2)) = x_{11} \\ \cos(\omega t_1 - (\cos^{-1}(x(t_2)) + \omega t_2)) = x_{12} \end{cases}$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

## Second order distribution (5)

the **conditional distribution** of  $x(t_1) = x_1$  given that  $x(t_2) = x_2$ :

$$\begin{aligned} f_{X(t_1)|X(t_2)}(x_1|x_2) &= \left( \frac{1}{2} \delta(x_1 - x_{11}) + \frac{1}{2} \delta(x_1 - x_{12}) \right) \\ &= \frac{1}{2} \delta \left( x_1 - \cos \left[ \omega t_1 + \left( \cos^{-1}(x_2) - \omega t_2 \right) \right] \right) \\ &\quad + \frac{1}{2} \delta \left( x_1 - \cos \left[ \omega t_1 - \left( \cos^{-1}(x_2) + \omega t_2 \right) \right] \right) \end{aligned}$$

$$\begin{aligned} f_{X(t_1)|X(t_2)}(x(t_1)|x(t_2)) &= \left( \frac{1}{2} \delta(x(t_1) - x_{11}) + \frac{1}{2} \delta(x(t_1) - x_{12}) \right) \\ &= \frac{1}{2} \delta \left( x(t_1) - \cos \left[ \omega t_1 + \left( \cos^{-1}(x(t_2)) - \omega t_2 \right) \right] \right) \\ &\quad + \frac{1}{2} \delta \left( x(t_1) - \cos \left[ \omega t_1 - \left( \cos^{-1}(x(t_2)) + \omega t_2 \right) \right] \right) \end{aligned}$$

# First order distribution $f_X(x)$ (1)

the first order distribution of  $x(t_2) = x_2 = \cos(\omega t_2 + \theta)$ :

$$f_{X(t_2)}(x_2) = \frac{1}{2\pi\sqrt{1-x_2^2}}$$

$$f_{X(t_2)}(x(t_2)) = \frac{1}{2\pi\sqrt{1-x^2(t_2)}}$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

## First order distribution $f_X(x)$ (2)

the **first order distribution**  $f_X(x)$  of  $X(t, \theta) = \cos(\omega t + \theta)$

- dependent only on the set of values  $x$  ( $-1 \leq x \leq 1$ ) that the process  $X(t, \theta)$  takes
- independent of
  - the particular sampling instant  $t$
  - the constant phase offset  $\theta_0 = \omega t$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>



## Second order distribution (7)

The **second order pdf** of the process  $X(t) = \cos(\omega t + \Theta)$

$$\begin{aligned} f_{X(t_1), X(t_2)}(x_1, x_2) &= f_{X(t_1)}(x_1) f_{X(t_2)|X(t_1)}(x_2|x_1) \\ &= f_{X(t_1)}(x_1) \left( \frac{1}{2} \delta(x_2 - x_{21}) + \frac{1}{2} \delta(x_2 - x_{22}) \right) \end{aligned}$$

$$\begin{aligned} f_{X(t_1), X(t_2)}(x_1, x_2) &= f_{X(t_2)}(x_2) f_{X(t_1)|X(t_2)}(x_1|x_2) \\ &= f_{X(t_1)}(x_2) \left( \frac{1}{2} \delta(x_1 - x_{11}) + \frac{1}{2} \delta(x_1 - x_{12}) \right) \end{aligned}$$

where  $x(t_1) = x_1$  and  $x(t_2) = x_2$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

## Second order distribution (8)

$$\begin{aligned}
 f_{X(t_1), X(t_2)}(x_1, x_2) &= f_{X(t_2)}(x_2) f_{X(t_1)|X(t_2)}(x_1|x_2) \\
 &= \left\{ \frac{1}{2\pi\sqrt{1-x_2^2}} \right\} \delta(x_1 - \cos[\omega t_1 + (\cos^{-1}(x_2) - \omega t_2)]) \\
 &+ \left\{ \frac{1}{2\pi\sqrt{1-x_2^2}} \right\} \delta(x_1 - \cos[\omega t_1 - (\cos^{-1}(x_2) + \omega t_2)])
 \end{aligned}$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

## Second order distribution (9)

$$\begin{aligned}
 f_{X(t_1), X(t_2)}(x(t_1), x(t_2)) &= f_{X(t_2)}(x(t_2))f_{X(t_1)|X(t_2)}(x(t_1)|x(t_2)) \\
 &= \left\{ \frac{1}{2\pi\sqrt{1-x^2(t_2)}} \right\} \delta(x(t_1) - \cos[\omega t_1 + (\cos^{-1}(x(t_2)) - \omega t_2)]) \\
 &+ \left\{ \frac{1}{2\pi\sqrt{1-x^2(t_2)}} \right\} \delta(x(t_1) - \cos[\omega t_1 - (\cos^{-1}(x(t_2)) + \omega t_2)])
 \end{aligned}$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

## Second order distribution (10)

The **second order pdf** can thus be written as

$$\begin{aligned} f_{X(t_1), X(t_2)}(x_1, x_2) &= f_{X(t_2)}(x_2) f_{X(t_2)|X(t_1)}(x_1|x_2) \\ &= f_{X(t_2)}(x_2) \left( \frac{1}{2} \delta(x_1 - x_{11}) + \frac{1}{2} \delta(x_1 - x_{12}) \right) \end{aligned}$$

$$\begin{aligned} f_{X(t_1), X(t_2)}(x(t_1), x(t_2)) &= f_{X(t_2)}(x(t_2)) f_{X(t_2)|X(t_1)}(x(t_1)|x(t_2)) \\ &= f_{X(t_2)}(x(t_2)) \left( \frac{1}{2} \delta(x(t_1) - x_{11}) + \frac{1}{2} \delta(x(t_1) - x_{12}) \right) \end{aligned}$$

These depend only on  $t_2 - t_1$ ,  
 and thus **the second order pdf is stationary**

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

## Second order distribution (11)

given that  $x(t_2) = x_2 = \cos(\omega t_2 + \theta)$

find  $\theta$ , then  $x(t_1) = x_1 = \cos(\omega t_1 + \theta)$

has only two values with an equal probability 0.5

$$x(t_1) = \begin{cases} x_{11} = \cos(\omega t_1 + (\cos^{-1}(x(t_2)) - \omega t_2)) \\ x_{12} = \cos(\omega t_1 - (\cos^{-1}(x(t_2)) + \omega t_2)) \end{cases}$$

$$\begin{aligned} f_{X(t_1), X(t_2)}(x(t_1), x(t_2)) &= f_{X(t_2)}(x(t_2)) f_{X(t_2)|X(t_1)}(x(t_1)|x(t_2)) \\ &= f_{X(t_2)}(x(t_2)) \left( \frac{1}{2} \delta(x(t_1) - x_{11}) + \frac{1}{2} \delta(x(t_1) - x_{12}) \right) \end{aligned}$$

These depend only on  $t_2 - t_1$ ,  
 and thus **the second order pdf is stationary**

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

## Second order distribution (12)

$\delta(x(t_1) - x_{11})$  when  $x(t_1)$  is equal to  $x_{11} = \cos(\omega t_1 + \theta_1)$

$\delta(x(t_1) - x_{12})$  when  $x(t_1)$  is equal to  $x_{12} = \cos(\omega t_1 + \theta_2)$

$$f_{X(t_2)}(x(t_2)) = \frac{1}{2\pi\sqrt{1-x^2(t_2)}}$$

These depend only on  $t_2 - t_1$ ,  
 and thus **the second order pdf is stationary**

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

## Second-Order Stationary Process

$$f_X(x_1, x_2; t_1, t_2)$$

if  $X(t)$  is to be a **second-order stationary**

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta)$$

must be true for any time  $t_1, t_2$  and any real number  $\Delta$

the **second order density function**

does not change with a shift in time origin

## Second-Order Stationary Process

$f_X(x_1, x_2; t_1, t_2)$

- $f_X(x_1, x_2; t_1, t_2)$  is independent of  $t_1$  and  $t_2$   
the **second order density function**  
does not change with a shift in time origin

- the **autocorrelation function**

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)] = R_{XX}(\tau)$$





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Example:  $X(t) = \cos(\omega t + \Theta)$

- the random process  $X(t)$
- the **first-order** moments  $\mu_X$
- the **second-order** moments  $\sigma_X^2$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Example:  $X(t) = \cos(\omega t + \Theta)$

The **mean** of the process is obtained  
 by taking the **expectation** operator  
 with respect to the **random** parameter  $\Theta$  on both sides

$$X_t(\Theta) = \cos(\omega t + \Theta)$$

$$E_{\Theta} [X_t(\Theta)] = E_{\Theta} [\cos(\omega t + \Theta)]$$

note that the **expectation** integral is a linear operation:

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Example:  $X(t) = \cos(\omega t + \Theta)$

$$\begin{aligned}\mu_X &= E_{\Theta}[X_t(\Theta)] = E_{\Theta}[\cos(\omega t + \Theta)] \\ &= E_{\Theta}[\cos(\omega t)\cos(\Theta) - \sin(\omega t)\sin(\Theta)] \\ &= E_{\Theta}[\cos(\Theta)]\cos(\omega t) - E_{\Theta}[\sin(\Theta)]\sin(\omega t)\end{aligned}$$

Since the random parameter  $\Theta$  is uniformly distributed

$$\begin{aligned}\mu_X &= E_{\Theta}[\cos(\Theta)]\cos(\omega t) - E_{\Theta}[\sin(\Theta)]\sin(\omega t) \\ &= \cos(\omega t) \left(\frac{1}{2\pi}\right) \int_0^{2\pi} \cos(\theta) d\theta - \sin(\omega t) \left(\frac{1}{2\pi}\right) \int_0^{2\pi} \sin(\theta) d\theta \\ &= 0\end{aligned}$$

## Example: $X(t) = \cos(\omega t + \Theta)$

The variance of the random process  $X(t)$

$$\sigma_X^2 = E_{\Theta}[(x_t(\Theta) - \mu_X)^2] = E_{\Theta} [[x_t(\Theta)]^2] - \mu_X^2$$

Substituting the mean of the process

$$\begin{aligned} \sigma_X^2 &= \left(\frac{1}{2\pi}\right) \int_0^{2\pi} \cos^2(\omega t + \theta) d\theta \\ &= \left(\frac{1}{2\pi}\right) \int_0^{2\pi} [1 + \cos(2\omega t + 2\theta)] d\theta \\ &= \frac{1}{2} \end{aligned}$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Example:  $X(t) = \cos(\omega t + \Theta)$

the average power of the random sinusoidal signal  $X(t)$

$$P_{ave}^X = \sigma_X^2 = \frac{1}{2}$$

the same as the average power of a sinusoid the phase is not random

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Example:  $X(t) = \cos(\omega t + \Theta)$ 

the correlation between the R.Vs  $x(t_1)$  and  $x(t_2)$  denoted as  $R_{XX}(t_1, t_2)$

$$\begin{aligned}R_{XX}(t_1, t_2) &= E_{\Theta}[x(t_1)x(t_2)] = \int_0^{2\pi} \cos[\omega t_1 + \theta] \cos[\omega t_2 + \theta] d\theta \\ &= \left(\frac{1}{4\pi}\right) \int_0^{2\pi} \cos[\omega(t_1 + t_2) + 2\theta] d\theta \\ &\quad + \left(\frac{1}{4\pi}\right) \int_0^{2\pi} \cos[\omega(t_1 - t_2)] d\theta \\ &= \left(\frac{1}{2}\right) \cos[\omega(t_1 - t_2)]\end{aligned}$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>



Example:  $X(t) = \cos(\omega t + \Theta)$

The covariance of R.Vs  $X(t_1)$  and  $X(t_2)$  denoted  $C_{XX}(t_1, t_2)$

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) = \left(\frac{1}{2}\right) \cos[\omega(t_1 - t_2)]$$

The correlation coefficient of the R.Vs  $X(t_1)$  and  $X(t_2)$  denoted  $\rho_{XX}(t_1, t_2)$

$$\rho_{XX}(t_1, t_2) = \cos[\omega(t_1 - t_2)]$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Example:  $X(t) = \cos(\omega t + \Theta)$

Looking at the **mean** and the **variance**  
of the random process  $X(t)$

we can see that they are shift-invariant and  
consequently the process is **first-order stationary**.

The ACF and other second-order statistics of the process are  
dependent only on the variable  $\tau = t_1 - t_2$ .

The random process  $X(t)$  is therefore a **WSS** process also.

The ACF can then expressed in terms of the variable  $\tau = t_1 - t_2$  as:

$$R_{XX}(\tau) = \left(\frac{1}{2}\right) \cos(\omega\tau)$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

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Example A.1:  $X(t) = \cos(\omega t)$ 

A **white noise** is not necessarily strictly stationary.

Let  $\omega$  be a **random variable uniformly distributed** in the interval  $(0, 2\pi)$

define the time series  $\{X(t)\}$

$$X(t) = \cos(\omega t) \quad (t = 1, 2, \dots)$$

[https://en.wikipedia.org/wiki/Stationary\\_process](https://en.wikipedia.org/wiki/Stationary_process)

Example A.1:  $X(t) = \cos(\omega t)$ 

Then

$$E[X(t)] = \frac{1}{2\pi} \int_0^{2\pi} \cos(t\omega) d\omega = 0$$

$$\text{Var}(X(t)) = \frac{1}{2\pi} \int_0^{2\pi} \cos^2(t\omega) d\omega = 1/2$$

$$\text{Cov}(x(t), x(s)) = \frac{1}{2\pi} \int_0^{2\pi} \cos(t\omega) \cos(s\omega) d\omega = 0 \quad \forall t \neq s$$

So  $\{X(t)\}$  is a **white noise**,  
however it is not strictly stationary.

[https://en.wikipedia.org/wiki/Stationary\\_process](https://en.wikipedia.org/wiki/Stationary_process)

Example A.2:  $X(t) = \cos(t + U)$ 

a **stationary process** example  
for which any single realisation has  
an apparently noise-free structure,

Let  $U$  have a uniform distribution on  $(0, 2\pi]$  and  
define the time series  $\{X(t)\}$  by

$$X(t) = \cos(t + U) \quad \text{for } t \in \mathbb{R}$$

then  $\{X(t)\}$  is **strictly stationary (SSS)**.

[https://en.wikipedia.org/wiki/Stationary\\_process](https://en.wikipedia.org/wiki/Stationary_process)

Example A.2:  $X(t) = \cos(t + U)$ 

Show that  $X(t)$  is a **WSS** process.

We need to check two conditions:

$$\mu_X(t) = \mu_X \quad \text{for } t \in \mathbb{R}$$

$$R_X(t_1, t_2) = R_X(t_1 - t_2) \quad \text{for } t_1, t_2 \in \mathbb{R}$$

[https://www.probabilitycourse.com/chapter10/10\\_1\\_4\\_stationary\\_processes.php](https://www.probabilitycourse.com/chapter10/10_1_4_stationary_processes.php)

Example A.2:  $X(t) = \cos(t + U)$ 

$$\begin{aligned}\mu_X(t) &= E[X(t)] \\ &= E[\cos(t + U)] \\ &= \frac{1}{2\pi} \int_0^{2\pi} \cos(t + u) du \\ &= 0, \quad \text{for all } t \in \mathbb{R}.\end{aligned}$$

[https://www.probabilitycourse.com/chapter10/10\\_1\\_4\\_stationary\\_processes.php](https://www.probabilitycourse.com/chapter10/10_1_4_stationary_processes.php)



Example A.2:  $X(t) = \cos(t + U)$ 

$$\begin{aligned}R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\&= E[\cos(t_1 + U)\cos(t_2 + U)] \\&= E\left[\frac{1}{2}\cos(t_1 + t_2 + 2U) + \frac{1}{2}\cos(t_1 - t_2)\right] \\&= E\left[\frac{1}{2}\cos(t_1 + t_2 + 2U)\right] + E\left[\frac{1}{2}\cos(t_1 - t_2)\right] \\&= \frac{1}{2\pi} \int_0^{2\pi} \cos(t_1 + t_2 + u) du + \frac{1}{2}\cos(t_1 - t_2) \\&= 0 + \frac{1}{2}\cos(t_1 - t_2) = \frac{1}{2}\cos(t_1 - t_2), \quad \text{for all } t_1, t_2 \in \mathbb{R}.\end{aligned}$$

Example A.3:  $X(t) = \alpha \cos(\omega t + \Theta)$ 

The random phase signal  $X(t) = \alpha \cos(\omega t + \Theta)$   
where  $\Theta \in U[0, 2\pi]$  is **SSS**

it is known that the **first order pdf** is

$$f_{X(t)}(x) = \frac{1}{\pi\alpha\sqrt{1 - (x/\alpha)^2}}, \quad -\alpha < x < +\alpha$$

which is independent of  $t$ , and is therefore **stationary**

<http://isl.stanford.edu/~abbas/ee278/lect07.pdf>

Example A.3:  $X(t) = \alpha \cos(\omega t + \Theta)$ 

To find the **second order pdf**,  
note that if we are given the value of  $X(t)$  at one point, say  $t_1$ ,  
there are (at most) two possible **sample functions**

- $X(t_1) = x_1$ 
  - at  $t_1$ , two sinusoid waves intersect with each other
- $X(t_2) = x_{21}$  or  $x_{22}$ 
  - at  $t_2$ , two sinusoid waves do not intersect with each other

<http://isl.stanford.edu/~abbas/ee278/lect07.pdf>

Example A.3:  $X(t) = \alpha \cos(\omega t + \Theta)$ 

The second order pdf can thus be written as

$$\begin{aligned} f_{X(t_1), X(t_2)}(x_1, x_2) &= f_{X(t_1)}(x_1) f_{X(t_2)|X(t_1)}(x_2|x_1) \\ &= f_{X(t_1)}(x_1) \left( \frac{1}{2} \delta(x_2 - x_{21}) + \frac{1}{2} \delta(x_2 - x_{22}) \right) \end{aligned}$$

which depends only on  $t_2 - t_1$ ,  
and thus the second order pdf is **stationary**

<http://isl.stanford.edu/~abbas/ee278/lect07.pdf>

## Example A.3: $X(t) = \alpha \cos(\omega t + \Theta)$

- if we know that  $X(t_1) = x_1$  and  $X(t_2) = x_2$ ,  
the sample path is totally determined  
except when  $x_1 = x_2 = 0$ ,
- when  $x_1 = x_2 = 0$ ,  
two paths may be possible
- thus all **n-th order pdfs are stationary**

<http://isl.stanford.edu/~abbas/ee278/lect07.pdf>

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Example B.1:  $X(t) = Y$ 

Let  $Y$  be any scalar **random variable**,  
and define a time-series  $\{X(t)\}$ , by

$$X(t) = Y \quad \text{for all } t.$$

Then  $\{X(t)\}$  is a **stationary** time series

- **realisations** consist of a series of **constant** values,
- a different **constant** value for each **realisation**.

[https://en.wikipedia.org/wiki/Stationary\\_process](https://en.wikipedia.org/wiki/Stationary_process)

Example B.1:  $X(t) = Y$ 

$$X(t) = Y \quad \text{for all } t.$$

$X(t)$  is a **first-order stationary**

$$f_X(x_1; t_1) = f_X(x_1; t_1 + \Delta) = \text{const}$$

$X(t)$  is a **second-order stationary**

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta) = \text{const}$$

$X(t)$  is to be a  **$N^{\text{th}}$ -order stationary**

$$f_X(x_1, \dots, x_N; t_1, \dots, t_N) = f_X(x_1, \dots, x_N; t_1 + \Delta, \dots, t_N + \Delta) = \text{const}$$



Example B.2:  $Z(t) = X(t) + Y(t)$ 

Let  $X(t)$  and  $Y(t)$  be two jointly **WSS** random processes.

Consider the random process  $Z(t)$

$$Z(t) = X(t) + Y(t)$$

Show that  $Z(t)$  is **WSS**.

[https://www.probabilitycourse.com/chapter10/10\\_1\\_4\\_stationary\\_processes.php](https://www.probabilitycourse.com/chapter10/10_1_4_stationary_processes.php)

Example B.2:  $Z(t) = X(t) + Y(t)$ 

Since  $X(t)$  and  $Y(t)$  are jointly WSS, we conclude

$$\mu_{X(t)} = \mu_X$$

$$\mu_{Y(t)} = \mu_Y$$

$$R_X(t_1, t_2) = R_X(t_1 - t_2)$$

$$R_Y(t_1, t_2) = R_Y(t_1 - t_2)$$

$$R_{XY}(t_1, t_2) = R_{XY}(t_1 - t_2)$$

[https://www.probabilitycourse.com/chapter10/10\\_1\\_4\\_stationary\\_processes.php](https://www.probabilitycourse.com/chapter10/10_1_4_stationary_processes.php)

Example B.2:  $Z(t) = X(t) + Y(t)$ 

Since  $X(t)$  and  $Y(t)$  are jointly WSS, we conclude

$$\begin{aligned}\mu_Z(t) &= E[X(t) + Y(t)] \\ &= E[X(t)] + E[Y(t)] \\ &= \mu_X + \mu_Y.\end{aligned}$$

[https://www.probabilitycourse.com/chapter10/10\\_1\\_4\\_stationary\\_processes.php](https://www.probabilitycourse.com/chapter10/10_1_4_stationary_processes.php)

Example B.2:  $Z(t) = X(t) + Y(t)$ 

Since  $X(t)$  and  $Y(t)$  are jointly WSS, we conclude

$$\begin{aligned}R_Z(t_1, t_2) &= E[(X(t_1) + Y(t_1))(X(t_2) + Y(t_2))] \\ &= E[X(t_1)X(t_2)] + E[X(t_1)Y(t_2)] \\ &\quad + E[Y(t_1)X(t_2)]E[Y(t_1)Y(t_2)] \\ &= R_X(t_1 - t_2) + R_{XY}(t_1 - t_2) \\ &\quad + R_{YX}(t_1 - t_2) + R_Y(t_1 - t_2).\end{aligned}$$

[https://www.probabilitycourse.com/chapter10/10\\_1\\_4\\_stationary\\_processes.php](https://www.probabilitycourse.com/chapter10/10_1_4_stationary_processes.php)

Example B.3:  $X(t) = \pm \sin t, \pm \cos t$ 

Let

$$X(t) = \begin{cases} +\sin t & p_0 = \frac{1}{4} \\ -\sin t & p_1 = \frac{1}{4} \\ +\cos t & p_2 = \frac{1}{4} \\ -\cos t & p_3 = \frac{1}{4} \end{cases}$$

$$E[X(t)] = 0$$

$$R_X(t_1, t_2) = \frac{1}{2} \cos(t_2 - t_1)$$

thus  $X(t)$  is WSS

Example B.3:  $X(t) = \pm \sin t, \pm \cos t$ 

Let

$$X(t) = \begin{cases} +\sin t & p_0 = \frac{1}{4} \\ -\sin t & p_1 = \frac{1}{4} \\ +\cos t & p_2 = \frac{1}{4} \\ -\cos t & p_3 = \frac{1}{4} \end{cases}$$

But  $X(0)$  and  $X(\frac{\pi}{4})$  do not have the same pmf (different ranges), so the first order pmf is not stationary, and the process is not **SSS**

<http://isl.stanford.edu/~abbas/ee278/lect07.pdf>

# Outline

- 1 Random Phase Oscillator
  - Problem definition
  - First order distribution
    - Uniform random variable  $\Theta$
    - Uniform random variable  $T$
  - Second order distribution
  - Mean and variance
- 2 Stationary Process Examples
  - Examples - A
  - Examples - B
- 3 Cyclo-stationary Process Examples
  - Examples

# Cyclostationary Process (1)

- A cyclostationary process is one whose distribution is periodic in time.
- Cyclostationarity is a characteristic of a **probability distribution**.
- A **random variable** whose **distribution** changes periodically with time, or is periodically time-varying, is referred to as a **cyclostationary process**.

<https://www.wavewalkerdsp.com/2022/04/20/cyclostationary-and-stationary-processes-with-examples/>



## Cyclostationary Process (2)

- Let's return to the example of **flipping a coin** but with a caveat: the coin will only be flipped on Mondays and not flipped all other days of the week.
- However, the **distribution** is periodically time-varying when focusing on each week day individually.
- For example, the distribution of the coin flip on Monday is defined as

$$p_{C,weekly}(x|\text{day} = \text{Monday}) = \begin{cases} 0.5, & x = \text{heads} \\ 0.5, & x = \text{tails.} \end{cases}$$

<https://www.wavewalkerdsp.com/2022/04/20/cyclostationary-and-stationary-processes-with-examples/>

## Cyclostationary Process (3)

- the distribution of the coin flip on Monday is defined as

$$p_{C,weekly}(x|\text{day} = \text{Monday}) = \begin{cases} 0.5, & x = \text{heads} \\ 0.5, & x = \text{tails}. \end{cases}$$

- the **distribution is stationary**, however it occurs every 7 days.
- A **cyclostationary process** is one whose **distribution is stationary** but with a given period which in this case is 7 days.

<https://www.wavewalkerdsp.com/2022/04/20/cyclostationary-and-stationary-processes-with-examples/>

## Cyclostationary Process (4)

- Continuing the example, the coin flip distribution for Tuesday is defined by

$$p_{C,weekly}(x|\text{day} = \text{Tuesday}) = \begin{cases} 1, & x = \text{not flipped.} \end{cases}$$

- Again the **distribution** is **stationary** but with a period of every 7 days.
- The distribution is the same for all other days of the week.

<https://www.wavewalkerdsp.com/2022/04/20/cyclostationary-and-stationary-processes-with-examples/>

## Transforming Cyclostationary into Stationary (1)

- A cyclostationary process can be transformed into a stationary process by averaging out the probabilities across time.
- This results in distortion because it does not retain the full information about the nature of the cyclostationary process behind the coin flips.

<https://www.wavewalkerdsp.com/2022/04/20/cyclostationary-and-stationary-processes-with-examples/>

## Transforming Cyclostationary into Stationary (2)

- Each day of the week is equally likely, with probability  $1/7 \approx 0.14$ .
- The probability of the coin landing on heads or tails is still 50% but the coin is only flipped once on Mondays, therefore there is a probability of  $0.5 \cdot 1/7 \approx 0.07$  the coin landing on heads and probability  $0.5 \cdot 1/7 \approx 0.07$  landing on tails.
- Since the coin is not flipped 6 out of the 7 days there is a probability of  $6/7 \approx 0.86$  that occurring.

<https://www.wavewalkerdsp.com/2022/04/20/cyclostationary-and-stationary-processes-with-examples/>

