CLTI Impulse Response (4A)

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- Requirements of an Impulse Response

Solutions of Differential Equations : h(t)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\, x(t)}{dt} + b_M x(t)$$

requirement at time t = 0

All the derivatives of h(t) up to ***N*** must match a corresponding derivatives of the impulse up to ***M*** at time t=0

requirement at time t ≠ 0

The linear combination of all the derivatives of h(t) must add to zero for any time t≠0

$y_h(t)u(t)$ is such a function

$y_h(t)$ is the homogeneous solution

Case 1 N > M

The derivatives of the $y_h(t)u(t)$ provide all the singularity functions necessary to match the impulse and derivatives of the impulse on the right side and no other terms need to be added

Case 2 N = M

Need to add an impulse term $K_0\delta(t)$.. and solve for K_0 by matching coefficients of impulses on both sides

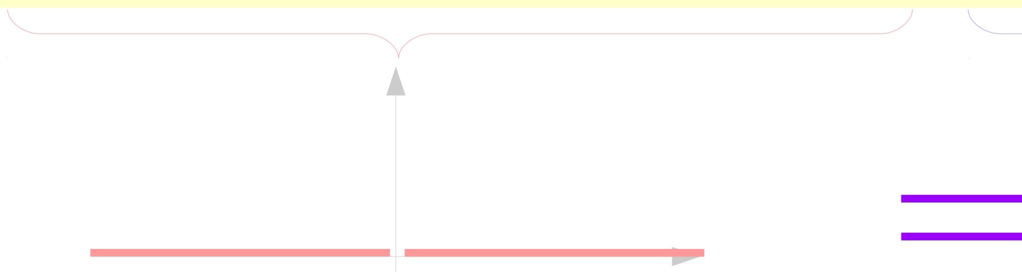
Case 3 N < M

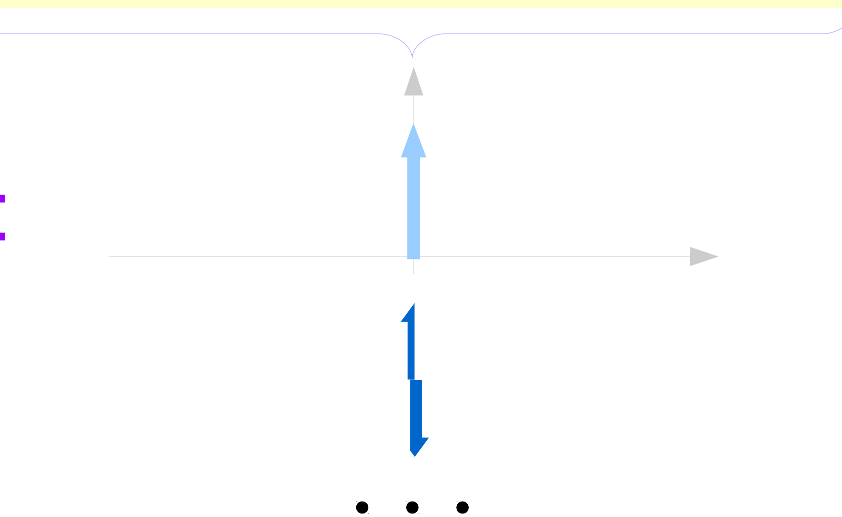
The n-th derivative of the function we add to $y_h(t)u(t)$ must have a term that matches the M-th derivative of the unit impulse. Must add

$$K_{m-n} u_{m-n}(t) + K_{m-n-1} u_{m-n-1}(t) + \cdots + K_0 u_0(t)$$
$$= K_{m-n} \delta^{(m-n)}(t) + K_{m-n-1} \delta^{(m-n-1)}(t) + \cdots + K_0 \delta(t)$$

Requirements at t≠0 (1)

$$\frac{d^N h(t)}{dt^N} + \boldsymbol{a_1}\frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + \boldsymbol{a_{N-1}}\frac{d\,h(t)}{dt} + \boldsymbol{a_N} h(t) = \boldsymbol{b_0}\frac{d^M \delta(t)}{dt^M} + \boldsymbol{b_1}\frac{d^{M-1}\delta(t)}{dt^{M-1}} + \cdots + \boldsymbol{b_{M-1}}\frac{d\,\delta(t)}{dt} + \boldsymbol{b_M}\delta(t)$$





requirements at *t ≠ 0*

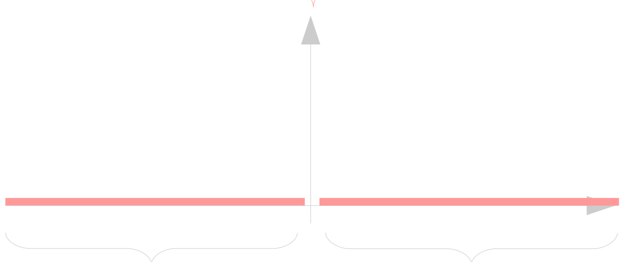
$$h^{(N)}(t) + a_1 h^{(N-1)}(t) \cdots + a_N h(t) = 0 \qquad (t \neq 0)$$

The linear combination
of all the derivatives of *h(t)*
must add to zero for any time t≠0

all the derivatives of δ(t)
exists only t=0.
It is zero for any time t≠0

Requirements at t≠0 (2)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$



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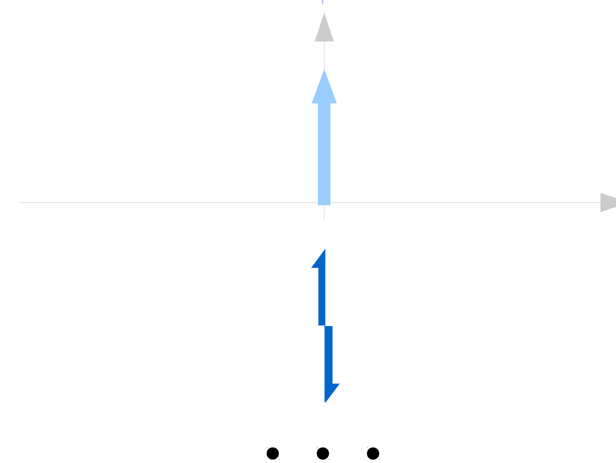


u(t)=0
: step function

the linear combination of all the derivatives of $y_p(t)$ results to zero
: homogeneous solution

$u(t) = 0$

$y_h^{(N)} + a_1 y_h^{(N-1)} + \cdots + a_N y_h = 0$

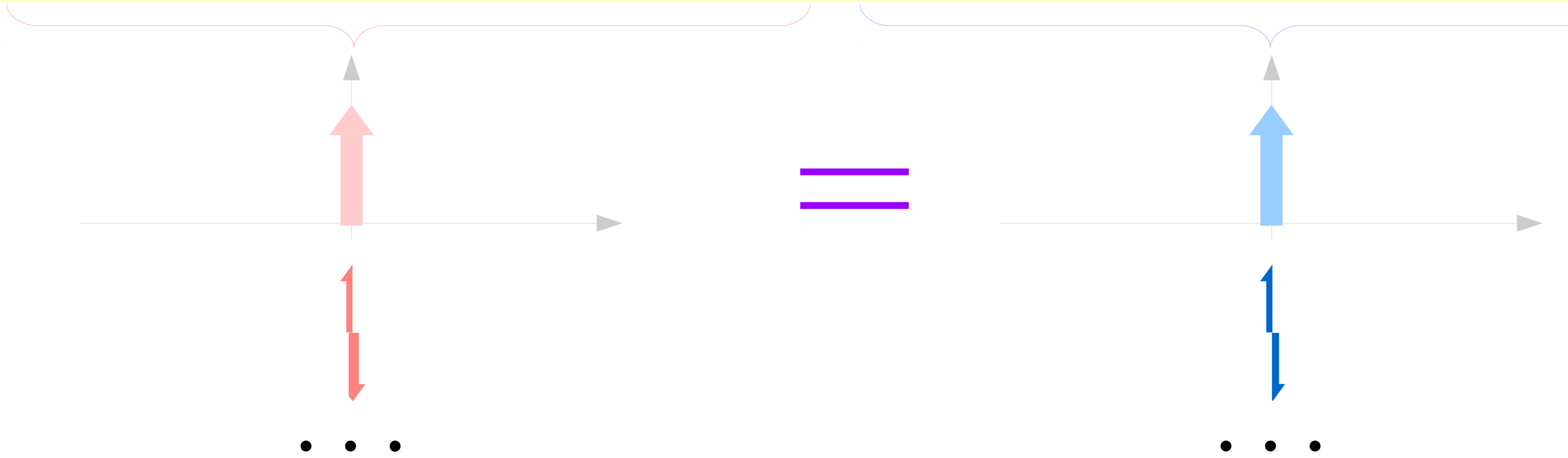
for $t < 0,\quad u(t) = 0$
for $t > 0,\quad y_h^{(N)} + a_1 y_h^{(N-1)} + \cdots + a_N y_h = 0$
derivatives of $\{y_h \cdot u\}$ *produce*
derivatives of δ *when* $t = 0$

$y_h(t)u(t)$ *when* $t \neq 0$

➡ *A possible candidate of* $h(t)$

Requirements at t=0 (1)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\, \delta(t)}{dt} + b_M \delta(t)$$



requirements at *t = 0*

All the <u>derivatives</u> of *h(t)* up to *N* must **match** the corresponding <u>derivatives</u> of an impulse *δ(t)* up to *M* at time t=0

Need to add a δ(t) and its derivatives in case that ($N \le M$)

Need to integrate $y_h(t) \cdot u(t)$ several times in case that ($N > M$)

Requirements at t=0 (2)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$

$$\frac{d^N}{dt^N}\{y_h(t)u(t)\} = \frac{d^{N-1}}{dt^{N-1}}\{c_0\delta(t)\} + \cdots$$

(N≤M) ⟷ $b_0 \frac{d^M \delta(t)}{dt^M}$ — the highest order derivatives of δ(t)

$y_h(t) \cdot u(t)$ gives the highest order (N-1)

Need to have the following terms

(N≤M)

$$m_{M-N} \frac{d^M}{dt^M}\delta(t) + \cdots + m_0 \frac{d^N}{dt^N}\delta(t)$$ ←

(N≤M) ⟷ $b_0 \frac{d^M \delta(t)}{dt^M}$ — the highest order derivatives of δ(t)

$$m_0 \frac{d^N}{dt^N}\delta(t) + \cdots + m_{M-N} \frac{d^M}{dt^M}\delta(t)$$ →

Requirements at t=0 (3)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\, \delta(t)}{dt} + b_M \delta(t)$$

$$m_0 \frac{d^N}{dt^N} \delta(t) + \cdots + m_{M-N} \frac{d^M}{dt^M} \delta(t)$$

all the derivatives of $y_h(t) \cdot u(t)$ may not include all the required the derivatives of $\delta(t)$ at the time $t=0$ in case that ***N* ≤ *M***.

Need to add a $\delta(t)$ and its derivatives in case that ($N \leq M$)

$$h(t) = y_h(t) u(t) + m_0 \delta(t) \qquad (N = M)$$

$$h(t) = y_h(t) u(t) + m_0 \delta(t) + m_1 \dot{\delta}(t) + \cdots + m_{M-N} \delta^{(M-N)}(t) \qquad (N < M)$$

Requirements at t=0 (4)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$

$$\frac{d^N}{dt^N}\{\quad\} = \frac{d^M}{dt^M}\{c_0 \delta(t)\} + \cdots$$

(N>M)

$$b_0 \frac{d^M \delta(t)}{dt^M}$$

the highest order derivatives of δ(t)

h(t) gives the highest order (M)
h(t) must have the following terms

$$h(t) = \underbrace{\int_{-\infty}^{t} \cdots \int_{-\infty}^{t}}_{N-M-1} y_h(t) u(t)\, dt \cdots dt$$

(N≤M)

$$\frac{d^N}{dt^N}\{h(t)\} = \frac{d^{M+1}}{dt^{M+1}}\left\{\frac{d^{N-M-1}}{dt^{N-M-1}}\{h(t)\}\right\} = \frac{d^{M+1}}{dt^{M+1}}\{y_h(t) u(t)\}$$

$$b_0 \frac{d^M \delta(t)}{dt^M}$$

the highest order derivatives of δ(t)

Requirements at t=0 (5)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$

All the derivatives of *h(t)* up to *N*
must generate the derivatives of an
impulse *δ(t)* only up to *M*
in case that ***N > M***

Need to integrate $y_h(t) \cdot u(t)$ several times
in case that (N>M)

$$h(t) = \overbrace{\int_{-\infty}^{t} \cdots \int_{-\infty}^{t}}^{N-M-1} y_h(t) u(t)\, dt \cdots dt \qquad (N > M)$$

- Impulse Response Representations in terms of $y_h(t) \cdot u(t)$

Derivatives of $y_h(t) \cdot u(t)$

$$h(t) = y_h(t) u(t)$$

$$u^{(i)}(t) = \delta^{(i-1)}(t)$$
$$f(t)\delta(t) = f(0)\delta(t)$$

$$h = y_h u$$
$$h^{(1)} = y_h^{(1)} u + y_h u^{(1)} \longrightarrow y_h(0)\delta(t)$$
$$h^{(2)} = y_h^{(2)} u + 2 y_h^{(1)} u^{(1)} + y_h u^{(2)} \longrightarrow 2 y_h^{(1)}(0)\delta(t) + y_h(0)\delta^{(1)}(t)$$
$$h^{(3)} = y_h^{(3)} u + 3 y_h^{(2)} u^{(1)} + 3 y_h^{(1)} u^{(2)} + y_h u^{(3)} \longrightarrow 3 y_h^{(2)}(0)\delta(t) + 3 y_h^{(1)}(0)\delta^{(1)}(t) + y_h(0)\delta^{(2)}(t)$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$h(t) = y_h(t) u(t)$$

All the <u>derivatives</u> of *h(t)* up to *N* incurs
the <u>derivatives</u> of an impulse *δ(t)* up to *N-1*

$$h^{(N)}(t) = \frac{d^N}{dt^N}\{y_h(t)u(t)\} \longrightarrow K_1\delta^{(N-1)}(t) + K_2\delta^{(N-2)}(t) + \cdots + K_{N-1}\delta^{(1)}(t) + K_N\delta(t)$$

Three different h(t)'s

$h(t) = y_h(t)u(t)$ ⇒ All the derivatives of $h(t)$ up to N incurs the derivatives of an impulse $\delta(t)$ up to $N-1$

$h^{(1)}(t) = y_h(t)u(t)$ ⇒ All the derivatives of $h(t)$ up to N incurs the derivatives of an impulse $\delta(t)$ up to $N-2$

$h^{(2)}(t) = y_h(t)u(t)$ ⇒ All the derivatives of $h(t)$ up to N incurs the derivatives of an impulse $\delta(t)$ up to $N-3$

Derivatives of three different h(t)'s

$h(t) = y_h(t)u(t)$

$h^{(N)}(t) \longrightarrow K_1\delta^{(N-1)}(t) + K_2\delta^{(N-2)}(t) + \cdots + K_{N-1}\delta^{(1)}(t) + K_N\delta(t)$

$h(t) = \int_{-\infty}^{t} y_h(t)u(t)\, dt$

$h^{(N)}(t) \longrightarrow K_1\delta^{(N-2)}(t) + K_2\delta^{(N-3)}(t) + \cdots + K_{N-1}\delta(t)$

$h(t) = \iint_{-\infty}^{t} y_h(t)u(t)\, dt\, dt$

$h^{(N)}(t) \longrightarrow K_1\delta^{(N-3)}(t) + \cdots + K_{N-2}\delta(t)$

All the derivatives of h(t) up to N

	$h^{(N)}(t)$	$+a_1 h^{(N-1)}(t)$	$+ \cdots$	$+a_{N-2} h^{(2)}(t)$	$+a_{N-1} h^{(1)}(t)$	$+a_N h^{(0)}(t)$
(a)	$\frac{d^N}{dt^N}\{y_h u\}$	$\frac{d^{N-1}}{dt^{N-1}}\{y_h u\}$		$\frac{d^2}{dt^2}\{y_h u\}$	$\frac{d}{dt}\{y_h u\}$	$y_h u$
(b)	$\frac{d^{N-1}}{dt^{N-1}}\{y_h u\}$	$\frac{d^{N-2}}{dt^{N-2}}\{y_h u\}$		$\frac{d}{dt}\{y_h u\}$	$y_h u$	$\int_{-\infty}^{t} y_h u \, dt$
(c)	$\frac{d^{N-2}}{dt^{N-2}}\{y_h u\}$	$\frac{d^{N-3}}{dt^{N-3}}\{y_h u\}$		$y_h u$	$\int_{-\infty}^{t} y_h u \, dt$	$\int_{-\infty}^{t}\int_{-\infty}^{t} y_h u \, dt\, dt$

$(N = M+1)$ $\quad (N-1 = M)$ $\quad \delta^{(N-1)} = \delta^{(M)}$ $\quad (M-N+1 = 0)$

$(N = M+2)$ $\quad (N-2 = M)$ $\quad \delta^{(N-2)} = \delta^{(M)}$ $\quad (M-N+1 = -1)$

$(N = M+3)$ $\quad (N-3 = M)$ $\quad \delta^{(N-3)} = \delta^{(M)}$ $\quad (M-N+1 = -2)$

Negative powers denote integration

(a)

$h(t) = y_h(t)u(t)$ ⇒ $h(t) = y_h(t)u(t)$ = $g(t)$

(b)

$h^{(1)}(t) = y_h(t)u(t)$ ⇒ $h(t) = \int_{-\infty}^{t} y_h(t)u(t)\, dt$ = $g^{(-1)}(t) \equiv \int_{-\infty}^{t} g(t)\, dt$

(c)

$h^{(2)}(t) = y_h(t)u(t)$ ⇒ $h(t) = \iint_{-\infty}^{t} y_h(t)u(t)\, dt\, dt$ = $g^{(-2)}(t) \equiv \int_{-\infty}^{t} \int_{-\infty}^{t} g(t)\, dt\, dt$

Derivative of three different h(t) : g(t), $g^{(1)}(t)$, $g^{(2)}(t)$

	$h^{(N)}(t)$	$+a_1 h^{(N-1)}(t)$	$+ \cdots$	$+a_{N-2} h^{(2)}(t)$	$+a_{N-1} h^{(1)}(t)$	$+a_N h^{(0)}(t)$
(a)c	$g^{(N)}(t)$	$g^{(N-1)}(t)$		$g^{(2)}(t)$	$g^{(1)}(t)$	$g(t)$
(b)	$g^{(N-1)}(t)$	$g^{(N-2)}(t)$		$g^{(1)}(t)$	$g(t)$	$g^{(-1)}(t)$
(c)	$g^{(N-2)}(t)$	$g^{(N-3)}(t)$		$g(t)$	$g^{(-1)}(t)$	$g^{(-2)}(t)$

$$g(t) = y_h(t) \cdot u(t)$$

$(N = M+1)$ $(M - N + 1 = 0)$

$(N = M+2)$ $(M - N + 1 = -1)$

$(N = M+3)$ $(M - N + 1 = -2)$

$$h(t) = g^{(M-N+1)}(t) = g^{(0)}(t)$$

$$h(t) = g^{(M-N+1)}(t) = g^{(-1)}(t)$$

$$h(t) = g^{(M-N+1)}(t) = g^{(-2)}(t)$$

Impulse response h(t) in terms of $y_h(t) \cdot u(t)$

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\, \delta(t)}{dt} + b_M \delta(t)$$

$$(N = M+1) \quad (N-1 = M) \quad \delta^{(N-1)} = \delta^{(M)} \quad (M-N+1 = 0) \quad h(t) = g^{(M-N+1)}(t) = g^{(0)}(t)$$

$$(N = M+2) \quad (N-2 = M) \quad \delta^{(N-2)} = \delta^{(M)} \quad (M-N+1 = -1) \quad h(t) = g^{(M-N+1)}(t) = g^{(-1)}(t)$$

$$(N = M+3) \quad (N-3 = M) \quad \delta^{(N-3)} = \delta^{(M)} \quad (M-N+1 = -2) \quad h(t) = g^{(M-N+1)}(t) = g^{(-2)}(t)$$

$$g(t) = y_h(t) u(t)$$

$$\begin{cases} h(t) = g^{(M-N+1)}(t) = \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} y_h(t) u(t)\, dt \cdots dt & (N > M) \\ h(t) = g(t) + m_0 \delta(t) & (N = M) \\ h(t) = g(t) + m_0 \delta(t) + m_1 \dot{\delta}(t) + \cdots + m_{M-N} \delta^{(M-N)}(t) & (N < M) \end{cases}$$

- Matching an impulse and its derivatives

Impulse Matching (N > M)

$M+1$

$$b_0 \frac{d^M \delta}{dt^M} + \cdots + b_{M-1}\frac{d\delta}{dt} + b_M \delta$$

RHS

$$\frac{d^N h}{dt^N} + a_1 \frac{d^{N-1} h}{dt^{N-1}} + \cdots + a_{N-M-1}\frac{d^{M+1} h}{dt^{M+1}} + a_{N-M}\frac{d^M h}{dt^M} + \cdots + a_{N-1}\frac{dh}{dt} + a_N h =$$

LHS

$M+1$

$$K_1 \delta^{(N-1)}(t) + K_2 \delta^{(N-2)}(t) + \cdots + K_{N-M}\delta^{(M)}(t) + K_{N-M+1}\delta^{(M-1)}(t) + \cdots + K_N \delta(t) \quad y_h(t)u(t)$$

must have zero coefficients

impulse related terms in the left hand side

Impulse Matching (N > M)

$M+1$

$b_0 \frac{d^M \delta}{dt^M} + \cdots + b_{M-1} \frac{d\delta}{dt} + b_M \delta$ RHS

OK

$N > M$

$\frac{d^N h}{dt^N} + a_1 \frac{d^{N-1} h}{dt^{N-1}} + \cdots + a_M \frac{d^{N-M} h}{dt^{N-M}} + a_{M+1} \frac{d^{N-M-1} h}{dt^{N-M-1}} + \cdots + a_{N-1} \frac{dh}{dt} + a_N h =$ LHS

$M+1$

$K_1 \delta^{(M)}(t) + K_2 \delta^{(M-1)}(t) + \cdots + K_{M+1} \delta(t)$ $y_h(t) u(t)$

impulse related terms in the left hand side

do not have an impulse or its derivatives

$$g(t) = y_h(t) u(t)$$

$$h(t) = g^{(M-N+1)}(t) = \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} y_h(t) u(t)\, dt \cdots dt \qquad (N > M)$$

Impulse Matching (N = M)

$M+1$

$$b_0 \frac{d^M \delta}{dt^M} + b_1 \frac{d^{M-1}\delta}{dt^{M-1}} + \cdots + b_{M-1}\frac{d\delta}{dt} + b_M \delta$$

RHS

$N = M$

$$\frac{d^N h}{dt^N} + a_1 \frac{d^{N-1} h}{dt^{N-1}} + \cdots + a_{N-1}\frac{dh}{dt} + a_N h \quad =$$

LHS

$m_0 \delta(t)$ *give*

M

$$K_1 \delta^{(M)}(t) + K_2 \delta^{(M-1)}(t) + K_3 \delta^{(M-2)}(t) + \cdots + K_{M+1}\delta(t) \qquad y_h(t)u(t)$$

if $h(t) = y_h(t)u(t)$

if $h(t) = y_h(t)u(t) + \delta(t)$

$$g(t) = y_h(t)u(t)$$

$$h(t) = g(t) + m_0\delta(t) \qquad (N = M)$$

Impulse Matching (N < M)

$M+1$

$$b_0 \frac{d^M \delta}{dt^M} + b_1 \frac{d^{M-1}\delta}{dt^{M-1}} + b_2 \frac{d^{M-2}\delta}{dt^{M-2}} + \cdots + b_{M-1}\frac{d\delta}{dt} + b_M \delta$$

RHS

$N = M-1$

$$\frac{d^N h}{dt^N} + a_1 \frac{d^{N-1}h}{dt^{N-1}} + \cdots + a_{N-1}\frac{dh}{dt} + a_N h \quad =$$

LHS

$m_0\delta(t) + m_1\delta^{(1)}(t)$ *give*

$M-1$

$$K_1\delta^{(M)}(t) + K_2\delta^{(M-1)}(t) + K_3\delta^{(M-2)}(t) + \cdots + K_{M+1}\delta(t) \quad y_h(t)u(t)$$

if $h(t) = y_h(t)u(t)$

if $h(t) = y_h(t)u(t) + \delta(t)+\delta'(t)$

$$g(t) = y_h(t)u(t)$$

$$h(t) = g(t) + m_0\delta(t) + m_1\dot{\delta}(t) + \cdots + m_{M-N}\delta^{(M-N)}(t) \qquad (N < M)$$

Integrals of $y_h(t) \cdot u(t)$: has the same form as $y_h(t) \cdot u(t)$

$$\frac{d^2 h(t)}{dt^2} + \boldsymbol{a_1} \frac{d\, h(t)}{dt} + \boldsymbol{a_2} h(t)$$

linear equation with constant coefficients

$$y_h(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t} \qquad m_1, m_2 = (-a_1 \pm \sqrt{a_1^2 - 4a_2})/2$$

$$y_h(t)u(t) = (C_1 e^{m_1 t} + C_2 e^{m_2 t}) u(t) \leftarrow \int_{-\infty}^{t} y_h(t)u(t)\, dt \leftarrow \int_{-\infty}^{t} \int_{-\infty}^{t} y_h(t)u(t)\, dt\, dt$$

$$y_h(t) = c_1 e^{m_1 x} + c_2 t e^{m_1 t} \qquad m_1, m_2 = -a_1/2$$

$$y_h(t)u(t) = (C_1 e^{m_1 x} + C_2 t e^{m_1 t}) u(t) \leftarrow \int_{-\infty}^{t} y_h(t)u(t)\, dt \leftarrow \int_{-\infty}^{t} \int_{-\infty}^{t} y_h(t)u(t)\, dt\, dt$$

$$y_h(t) = c_1 e^{s_1 x} + c_2 e^{s_2 x} \qquad s_1, s_2 = (-a_1 \pm i \sqrt{4a_2 - a_1^2})/2$$

$$y_h(t)u(t) = (C_1 e^{s_1 x} + C_2 e^{s_2 x}) u(t) \leftarrow \int_{-\infty}^{t} y_h(t)u(t)\, dt \leftarrow \int_{-\infty}^{t} \int_{-\infty}^{t} y_h(t)u(t)\, dt\, dt$$

Integrals of $y_h(t) \cdot u(t)$

$$\int_{-\infty}^{t} \cdots \int_{-\infty}^{t} y_h(t) u(t)\, dt \cdots dt \;\rightarrow\; y_h(t) u(t)$$

linear equation with constant coefficients

$$g(t) = y_h(t) u(t)$$

$$\begin{cases} h(t) = g^{(M-N+1)}(t) = \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} y_h(t) u(t)\, dt \cdots dt = y_h(t) u(t) & (N > M) \\ h(t) = g(t) + m_0 \delta(t) & (N = M) \\ h(t) = g(t) + m_0 \delta(t) + m_1 \dot{\delta}(t) + \cdots + m_{M-N} \delta^{(M-N)}(t) & (N < M) \end{cases}$$

$$\begin{cases} h(t) = y_h(t) u(t) & (N > M) \\ h(t) = y_h(t) u(t) + m_0 \delta(t) & (N = M) \\ h(t) = y_h(t) u(t) + m_0 \delta(t) + m_1 \dot{\delta}(t) + \cdots + m_{M-N} \delta^{(M-N)}(t) & (N < M) \end{cases}$$

Case Examples

$(N > M)$

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d h(t)}{dt} + a_N h(t) =$$

$$b_0 \frac{d^{N-1} \delta(t)}{dt^{N-1}} + b_1 \frac{d^{N-2} \delta(t)}{dt^{N-2}} + \cdots + b_{N-2} \frac{d \delta(t)}{dt} + b_{N-1} \delta(t)$$

$M = N-1$

$(N = M)$

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d h(t)}{dt} + a_N h(t) =$$

$$b_0 \frac{d^N \delta(t)}{dt^N} + b_1 \frac{d^{N-1} \delta(t)}{dt^{N-1}} + \cdots + b_{N-1} \frac{d \delta(t)}{dt} + b_N \delta(t)$$

$M = N$

$(N < M)$

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d h(t)}{dt} + a_N h(t) = \delta(t)$$

$$b_0 \frac{d^{N+1} \delta(t)}{dt^{N+1}} + b_1 \frac{d^N \delta(t)}{dt^N} + \cdots + b_N \frac{d \delta(t)}{dt} + b_{N+1} \delta(t)$$

$M = N+1$

in most systems $N \geq M$

$$h(t) =$$

$$y_h(t) u(t)$$

$$h(t) =$$

$$y_h(t) u(t) + m_0 \delta(t)$$

seldom used $N < M$

$$h(t) =$$

$$y_h(t) u(t) + m_0 \delta(t) + m_1 \dot{\delta}(t)$$

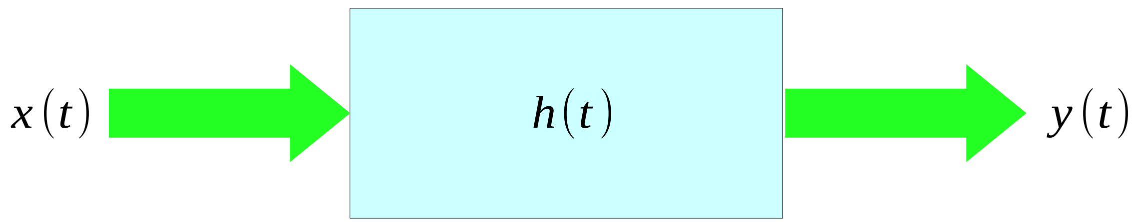
- An Impulse Response of a Causal LTI System

ODE's and Causal LTI Systems

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\, \delta(t)}{dt} + b_M \delta(t)$$

N > M : (N-M) integrator

N < M : (M-N) differentiator – magnify high frequency components of noise (seldom used)



$N \geq M$ *in most systems*

$h(t) = y_h(t)u(t)$ $\quad (N > M)$

$h(t) = y_h(t)u(t) + m_0 \delta(t)$ $\quad (N = M)$

h(t) can have at most a δ(t) for most systems

$y(t) \leftarrow \boxed{h(t)} \leftarrow x(t)$

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = b_0 x^{(N)} + b_1 x^{(N-1)} + \cdots + b_{N-1} x^{(1)} + b_N x$$

$h(t) \leftarrow \boxed{h(t)} \leftarrow \delta(t)$

$$h^{(N)} + a_1 h^{(N-1)} + \cdots + a_{N-1} h^{(1)} + a_N h = b_0 \delta^{(N)} + b_1 \delta^{(N-1)} + \cdots + b_{N-1} \delta^{(1)} + b_N \delta$$

if h contain δ: $C\delta^{(N)} \cdots \quad \cdots = b_0 \delta^{(N)}$ — the highest order derivatives of δ(t)

cannot be matched

if h contain $\delta^{(1)}$: $C\delta^{(N+1)} \cdots \quad \cdots \neq b_0 \delta^{(N)}$ — the highest order derivatives of δ(t)

cannot be matched

if h contain $\delta^{(2)}$: $C\delta^{(N+2)} \cdots \quad \cdots \neq b_0 \delta^{(N)}$ — the highest order derivatives of δ(t)

$t=0$

h(t) can have at most an impulse $b_0 \delta(t)$
no derivatives of $\delta(t)$ possible at all

in most systems $N \geq M$

h(t) can have at most a δ(t) (N ≥ M)

$N = M \qquad Q(D)\,y(t) = P(D)\,x(t)$

$N \qquad\qquad M \qquad (N \ge M)$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\,h(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\,\delta(t)$$

If $\delta'(t)$ is included in $h(t)$
then the highest order term
in the LHS

the highest order term
in the RHS

$\dfrac{d^N h}{dt^N} \to \delta^{(N+1)}(t)$ $\quad \delta^{(N+1)}(t) \;\ne\; \delta^{(N)}(t)$ ➡ contradiction

h(t) cannot contain $\delta^{(i)}(t)$ at all ➡ *h(t)* can contain ***at most*** δ(t)
only when $N = M$

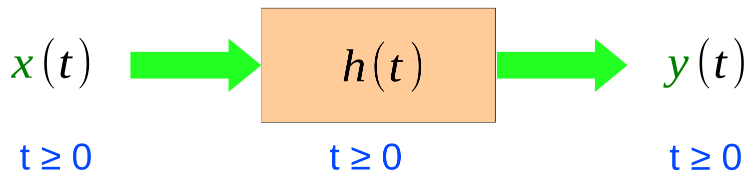
- An Impulse Response and System Responses

Causality

Causal Signals:
the input signals
starts at time t=0

Causal System:
the response h(t) cannot
begin before the input

Causal Signals:
the output signals
starts at time t=0



$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau)\, d\tau$$

$x(\tau) = 0 \quad (\tau < 0)$ — causal signal: x(t)=0 (t < 0)

$h(t-\tau) = 0 \quad (t-\tau < 0)$ — causal signal: h(t)=0 (t < 0)

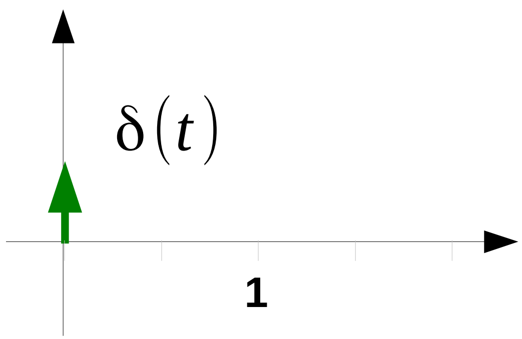
causal signal: y(t)=0 (t < 0)

$$y(t) = x(t) * h(t) = \int_{0^-}^{t} x(\tau) h(t-\tau)\, d\tau$$

$$= \int_{0^-}^{t} x(t-\tau) h(\tau)\, d\tau$$

0^- to include an impulse function δ(t) in h(t)

h(t) as a ZSR (t ≥ 0)

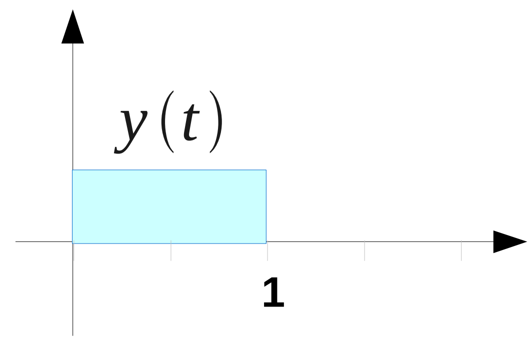


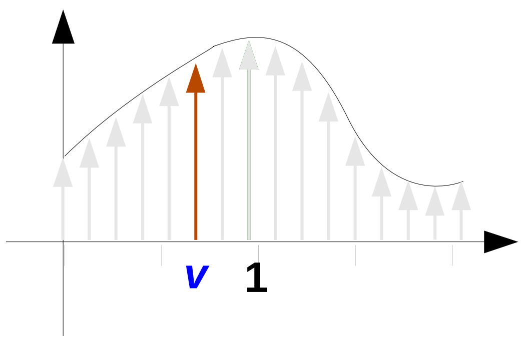
t ≥ 0 t ≥ 0 t ≥ 0

$$(b_0, b_1, \cdots, b_N)$$

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

$$(1, a_1, \cdots, a_N)$$



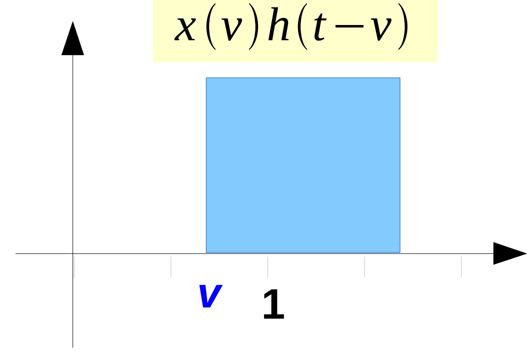


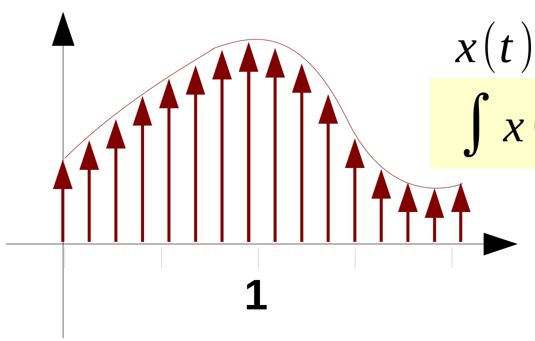
$$\delta(t-v) \Rightarrow h(t-v)$$

$$x(v)\,\delta(t-v) \Rightarrow x(v)\,h(t-v)$$

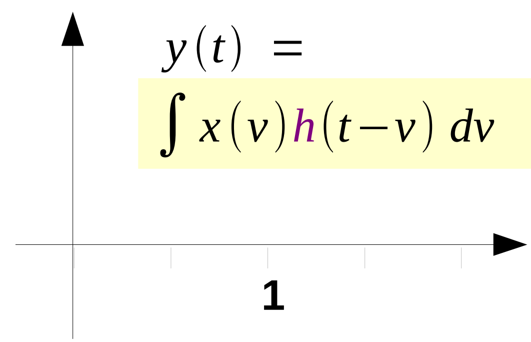
input value at time v
→ x(v)
delayed impulse response
→ x(v) h(t - v)

$$x(v)h(t-v)$$





$$x(t) = \int x(v)\delta(t-v)\,dv \quad \Rightarrow \quad y(t) = \int x(v)h(t-v)\,dv$$



Zero State (initially at rest at $t=0^-$)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1}\delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$

$$h(t) = y_h(t)u(t)$$

The solution of the IVP with the following I.C.

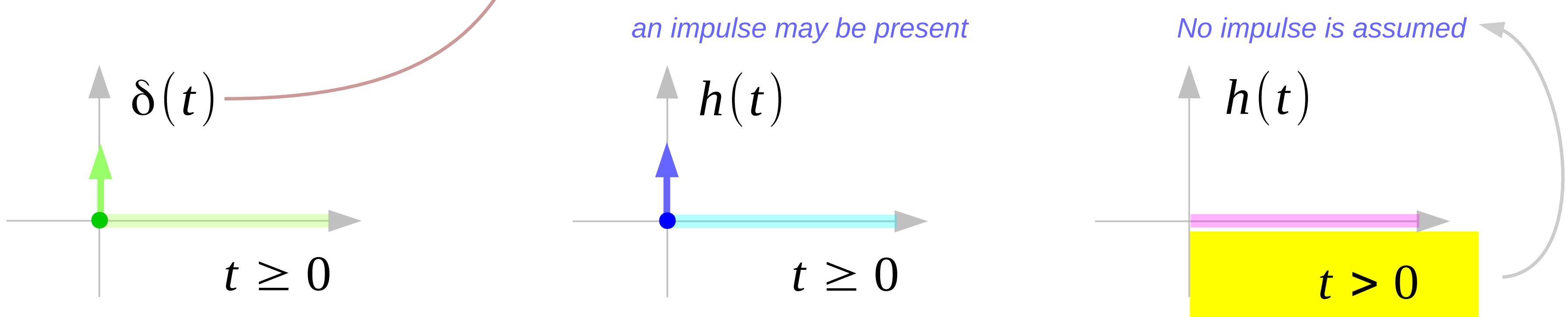
Interval of Validity : (t > 0)

Zero Input (no input for t > 0)

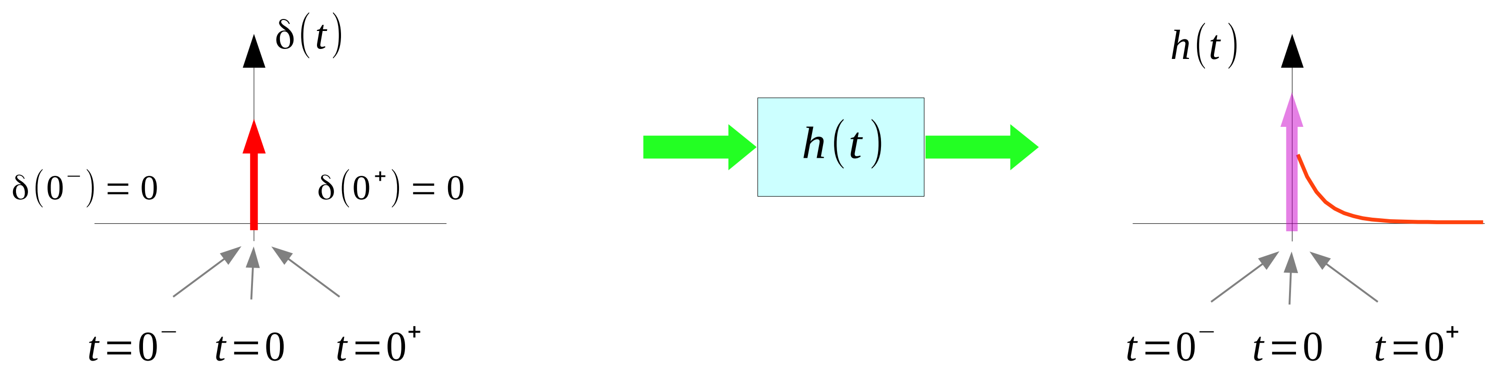
$\delta(t) = 0 \quad for \quad t > 0$

creates nonzero i. c

$$\begin{aligned} h^{(N-1)}(0^+) &= k_{N-1} \\ h^{(N-2)}(0^+) &= k_{N-2} \\ \vdots \quad & \quad \vdots \\ h^{(1)}(0^+) &= k_1 \\ h(0^+) &= k_0 \end{aligned}$$



Impulse Response h(t)



* general **y(t)** cannot be a **ZIR** for a general input **x(t)** (generally x(t) ≠ 0 for t > 0)

* impulse input vanishes (x(t) = δ(t) = 0 for t > 0)

$(N \geq M)$

h(t) : ZIR with the newly created I.C. (t > 0)

$$
\begin{aligned}
h^{(N-1)}(0^+) &= k_{N-1} \\
h^{(N-2)}(0^+) &= k_{N-2} \\
\vdots \quad & \quad \vdots \\
h^{(1)}(0^+) &= k_1 \\
h(0^+) &= k_0
\end{aligned}
$$

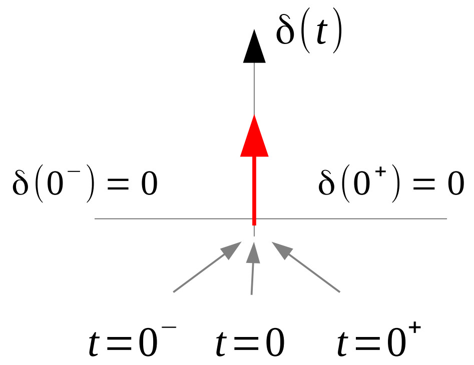
non-zero i. c.

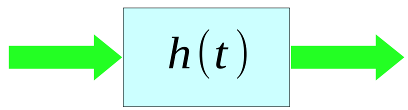
$\exists\, i, \quad k_i \neq 0$

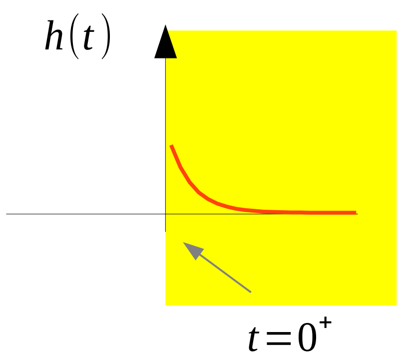
$t \geq 0^+$ $(t \neq 0)$	$h(t) = \mathit{char\ mode\ terms}$
$t = 0$	$h(t) = b_0 \delta(t)$ at most an impulse
$t \geq 0$	$h(t) = b_0 \delta(t)$ + $\mathit{char\ mode\ terms}$

h(t) : **ZSR** to an impulse input

Assumed Finite Jumps







- generates energy storage
- creates nonzero i. c. at $t=0^+$

$$\begin{aligned} h^{(N-1)}(0^-) &= 0 \\ h^{(N-2)}(0^-) &= 0 \\ &\vdots \\ h^{(1)}(0^-) &= 0 \\ h(0^-) &= 0 \end{aligned}$$

initially at rest

$$\begin{aligned} h^{(N)}(0) &= f_{N-1}(k_i, \delta^{(i)}) \\ h^{(N-1)}(0) &= f_{N-2}(k_i, \delta^{(i)}) \\ &\vdots \\ h^{(2)}(0) &= f_1(k_i, \delta^{(i)}) \\ h^{(1)}(0) &= f_0(k_i, \delta^{(i)}) \end{aligned}$$

Impulse matching

$$\begin{aligned} h^{(N-1)}(0^+) &= k_{N-1} \\ h^{(N-2)}(0^+) &= k_{N-2} \\ &\vdots \\ h^{(1)}(0^+) &= k_1 \\ h(0^+) &= k_0 \end{aligned}$$

non-zero initial conditions

$\exists\, i, \quad k_i \neq 0$

Assumed finite jumps

Total Response

zero input response

+

zero state response

$[-\infty, \quad 0^-]$

$y(t) = y_{zi}(t) \Leftarrow t \le 0^-$

because the input has not started yet

continuous at $t = 0$

$y(0^-) = y_{zi}(0^-) = y_{zi}(0^+)$

$\dot{y}(0^-) = \dot{y}_{zi}(0^-) = \dot{y}_{zi}(0^+)$

$[0^+, \quad +\infty]$

$y(t) = y_{zi}(t) + y_{zs}(t)$

$y(0^+) \neq y(0^-)$

possible discontinuity at $t = 0$

$y(0^+) = y_{zi}(0^+) + y_{zs}(0^+)$

$\dot{y}(0^+) = \dot{y}_{zi}(0^+) + \dot{y}_{zs}(0^+)$

natural response

+

forced response

$y_h(0^-) \neq y_{zi}(0^-)$

$\dot{y}_h(0^-) \neq \dot{y}_{zi}(0^-)$

$y_p(0^-) \neq y_{zi}(0^-)$

$\dot{y}_p(0^-) \neq \dot{y}_{zi}(0^-)$

$[0^+, \quad +\infty]$

$y(t) = y_h(t) + y_p(t)$

$y(0^+) = y_{zi}(0^+) + y_{zs}(0^+)$

$\dot{y}(0^+) = \dot{y}_{zi}(0^+) + \dot{y}_{zs}(0^+)$

$y(0^+) = y_h(0^+) + y_p(0^+)$

$\dot{y}(0^+) = \dot{y}_h(0^+) + \dot{y}_p(0^+)$

Interval of validity $t > 0$

- Determining the Coefficients of an Impulse Response

Impulse Matching Method Summary

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$

$$\begin{aligned} h(0^-) &= 0 \\ h^{(1)}(0^-) &= 0 \\ &\vdots \\ h^{(N-2)}(0^-) &= 0 \\ h^{(N-1)}(0^-) &= 0 \end{aligned} \qquad \begin{aligned} h(0^+) &= k_0 \\ h^{(1)}(0^+) &= k_1 \\ &\vdots \\ h^{(N-2)}(0^+) &= k_{N-2} \\ h^{(N-1)}(0^+) &= k_{N-1} \end{aligned} \qquad \begin{aligned} h^{(1)}(t) &= f_0(k_i, \delta^{(i)}) \\ h^{(2)}(t) &= f_1(k_i, \delta^{(i)}) \\ &\vdots \\ h^{(N-1)}(t) &= f_{N-2}(k_i, \delta^{(i)}) \\ h^{(N)}(t) &= f_{N-1}(k_i, \delta^{(i)}) \end{aligned}$$

initially at rest

assumed finite jumps

$$h^{(N)}(0) + a_1 h^{(N-1)}(0) + \cdots + a_{N-1} h^{(1)}(0) + a_N h(0) = b_0 \delta^{(N)}(t) + b_1 \delta^{(N-1)}(t) + \cdots + b_{N-1} \delta^{(1)}(t) + b_N \delta(t)$$

Impulse matching

$$h(t) = \left(c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} + \cdots + c_{N-2} e^{\lambda_{N-2} t} + c_{N-1} e^{\lambda_{N_1} t}\right) \cdot u(t)$$

Differentiate each initial condition (N-1 = M)

$$\begin{aligned} h(0^-) &= 0 \\ h^{(1)}(0^-) &= 0 \\ &\vdots \\ h^{(N-2)}(0^-) &= 0 \\ h^{(N-1)}(0^-) &= 0 \end{aligned}$$

initially at rest

$$\begin{aligned} h(0^+) &= k_0 \\ h^{(1)}(0^+) &= k_1 \\ &\vdots \\ h^{(N-2)}(0^+) &= k_{N-2} \\ h^{(N-1)}(0^+) &= k_{N-1} \end{aligned}$$

assumed finite jumps

$$\begin{aligned} h^{(1)}(t) &= k_0\delta(t) \\ h^{(2)}(t) &= k_1\delta(t) + k_0\delta^{(1)}(t) \\ &\vdots \\ h^{(N-1)}(t) &= k_{N-2}\delta(t) + k_{N-3}\delta^{(1)}(t) + \cdots + k_0\delta^{(N-2)}(t) \\ h^{(N)}(t) &= k_{N-1}\delta(t) + k_{N-2}\delta^{(1)}(t) + \cdots + k_1\delta^{(N-2)}(t) + k_0\delta^{(N-1)}(t) \end{aligned}$$

(N > M : N-1 = M)

substitute $h^{(m)}(t)$

$$h^{(N)}(t) + a_1 h^{(N-1)}(t) + \cdots + a_{N-1} h^{(1)}(t) + a_N h(t) = b_0\delta^{(M)}(t) + b_1\delta^{(M-1)}(t) + \cdots + b_{M-1}\delta^{(1)}(t) + b_M\delta(t)$$

N variables / N equations

Find the finite jumps

$$h^{(1)}(t) = k_0\delta(t)$$
$$h^{(2)}(t) = k_1\delta(t) + k_0\delta^{(1)}(t)$$
$$\vdots \qquad \vdots$$
$$h^{(N-1)}(t) = k_{N-2}\delta(t) + k_{N-3}\delta^{(1)}(t) + \cdots + k_0\delta^{(N-2)}(t)$$
$$h^{(N)}(t) = k_{N-1}\delta(t) + k_{N-2}\delta^{(1)}(t) + \cdots + k_1\delta^{(N-2)}(t) + k_0\delta^{(N-1)}(t)$$

(N > M : N-1 = M)

substitute $h^{(m)}(t)$

Impulse matching

$$h^{(N)}(t) + a_1 h^{(N-1)}(t) + \cdots + a_{N-1} h^{(1)}(t) + a_N h(t) \Leftrightarrow b_0\delta^{(M)}(t) + b_1\delta^{(M-1)}(t) + \cdots + b_{M-1}\delta^{(1)}(t) + b_M\delta(t)$$

Impulse matching		*Find coefficients*	*Find the finite jumps*
$d_0\delta^{(N-1)}(t)$	$\Leftrightarrow b_0\delta^{(M)}(t)$	d_0	k_0
$d_1\delta^{(N-2)}(t)$	$\Leftrightarrow b_1\delta^{(M-1)}(t)$	d_1	k_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$d_{N-2}\delta^{(1)}(t)$	$\Leftrightarrow b_1\delta^{(1)}(t)$	d_{N-2}	k_{N-2}
$d_{N-1}\delta^{(0)}(t)$	$\Leftrightarrow b_M\delta^{(0)}(t)$	d_{N-1}	k_{N-1}

N variables / N equations

Find the homogeneous solution

$$
\begin{aligned}
h(0^-) &= 0 \\
h^{(1)}(0^-) &= 0 \\
&\vdots \\
h^{(N-2)}(0^-) &= 0 \\
h^{(N-1)}(0^-) &= 0
\end{aligned}
$$

initially at rest

$$
\begin{aligned}
h(0^+) &= k_0 \\
h^{(1)}(0^+) &= k_1 \\
&\vdots \\
h^{(N-2)}(0^+) &= k_{N-2} \\
h^{(N-1)}(0^+) &= k_{N-1}
\end{aligned}
$$

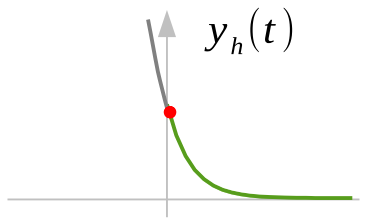
found finite jumps

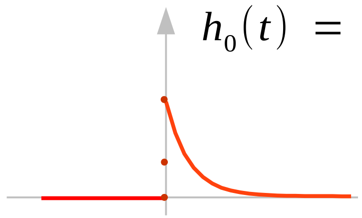
substitute

$$y_h(t) = \left(c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} + \cdots + c_{N-2} e^{\lambda_{N-2} t} + c_{N-1} e^{\lambda_{N_1} t}\right)$$

(N > M : N-1 = M)

$k_0, k_1, \ldots, k_{N-2}, k_{N-1}$ → $c_0, c_1, \ldots, c_{N-2}, c_{N-1}$





found homogeneous response

found impulse response

$$h(t) = y_h(t)u(t)$$

Case: N-2 = M

initially at rest	assumed finite jumps	
$h(0^-) = 0$	$h(0^+) = 0$	$h^{(1)}(t) = 0$ ➡ continuous (no jump)
$h^{(1)}(0^-) = 0$	$h^{(1)}(0^+) = k_1$	$h^{(2)}(t) = k_1\delta(t) + 0$
$\vdots$	$\vdots$	$\vdots$
$h^{(N-2)}(0^-) = 0$	$h^{(N-2)}(0^+) = k_{N-2}$	$h^{(N-1)}(t) = k_{N-2}\delta(t) + k_{N-3}\delta^{(1)}(t) + \cdots + 0$
$h^{(N-1)}(0^-) = 0$	$h^{(N-1)}(0^+) = k_{N-1}$	$h^{(N)}(t) = k_{N-1}\delta(t) + k_{N-2}\delta^{(1)}(t) + \cdots + k_1\delta^{(N-2)}(t) + 0$

(N > M : N-2 = M)

substitute $h^{(m)}(t)$

$$h^{(N)}(t) + a_1 h^{(N-1)}(t) + \cdots + a_{N-1} h^{(1)}(t) + a_N h(t) = b_0\delta^{(M)}(t) + b_1\delta^{(M-1)}(t) + \cdots + b_{M-1}\delta^{(1)}(t) + b_M\delta(t)$$

Impulse matching		Find	Find	Find
$d_0\delta^{(N-1)}(t)$	$\Leftrightarrow b_0\delta^{(M)}(t)$	d_0	$k_0 = 0$	c_0
$d_1\delta^{(N-2)}(t)$	$\Leftrightarrow b_1\delta^{(M-1)}(t)$	d_1	k_1	c_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$d_{N-2}\delta^{(1)}(t)$	$\Leftrightarrow b_1\delta^{(1)}(t)$	d_{N-2}	k_{N-2}	c_{N-2}
$d_{N-1}\delta^{(0)}(t)$	$\Leftrightarrow b_M\delta^{(0)}(t)$	d_{N-1}	k_{N-1}	c_{N-1}

$$h(t) = y_h(t)u(t)$$

A δ(t) needs to be added to characteristic modes (N = M)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1}\delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$

$$\begin{aligned}
h^{(1)}(0) &= k_0 \delta(t) + b_0 \delta^{(1)}(t) \\
h^{(2)}(0) &= k_1 \delta(t) + k_0 \delta^{(1)}(t) + b_0 \delta^{(2)}(t) \\
&\vdots \\
h^{(N-1)}(0) &= k_{N-2}\delta(t) + \cdots + k_1 \delta^{(N-1)}(t) + k_0 \delta^{(N-2)}(t) + b_0 \delta^{(N-1)}(t) \\
h^{(N)}(0) &= k_{N-1}\delta(t) + k_{N-2}\delta^{(1)}(t) + \cdots + k_1 \delta^{(N-2)}(t) + k_0 \delta^{(N-1)}(t) + b_0 \delta^{(N)}(t)
\end{aligned}$$

$$h(t) = b_0 \delta(t) + y_h(t) u(t)$$

$\delta^{(N)}(t)$ $\delta^{(N-1)}(t)$ $\delta^{(1)}(t)$

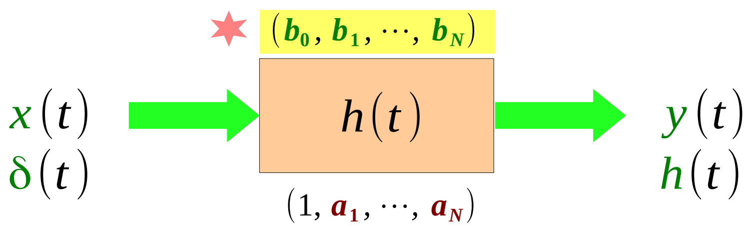
$$h^{(N)}(0) + a_1 h^{(N-1)}(0) + \cdots + a_{N-1} h^{(1)}(0) + a_N h(0) = b_0 \delta^{(N)}(t) + b_1 \delta^{(N-1)}(t) + \cdots + b_{N-1} \delta^{(1)}(t) + b_N \delta(t)$$

Impulse matching

- Superposition of an Impulse Response and its Derivatives

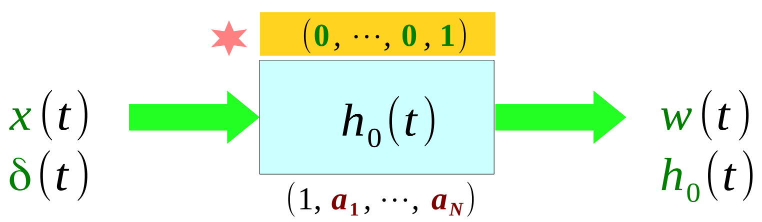
A General System S and a Base System S0

General System S



$h(t)$: the impulse response of **S**

Base System S0



$h_0(t)$: the impulse response of **S0**

ZIR of a Base System S0

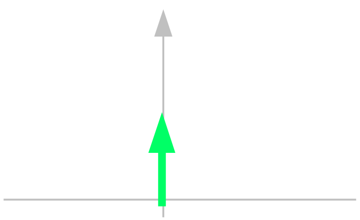
N ≥ M : $h_0(t)$ include no impulse $\delta(t)$ (t > 0)

➡ $h_0(t)$ include characteristic modes only

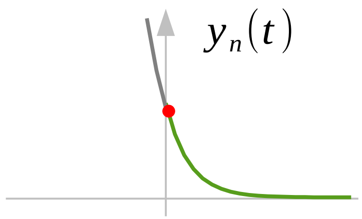
➡ System **S** and **S0** have the same characteristic modes

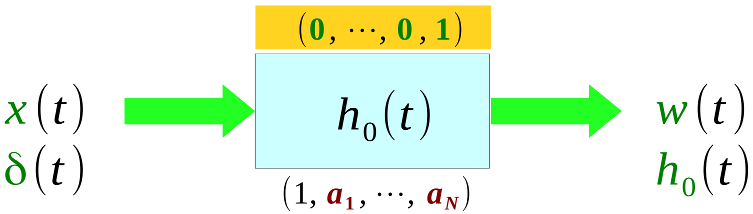
$y_n(t)$: viewed as the zero input response of **S0**

cf) natural response all the lumped char modes
homogeneous response

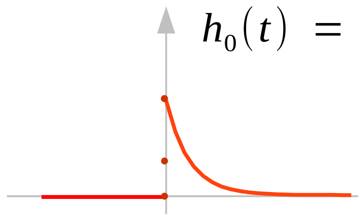


an impulse function
generates energy storage
creates nonzero initial
condition at $t=0^+$





$h_0(t) = y_n(t)u(t)$



Superposition of inputs (1)

$$h^{(N)}(t) + a_1 h^{(N-1)}(t) + \cdots + a_{N-1} h^{(1)}(t) + a_N h(t) = b_0 \delta^{(N)}(t) + b_1 \delta^{(N-1)}(t) + \cdots + b_{N-1} \delta^{(1)}(t) + b_N \delta(t)$$

$$h^{(N)} + a_1 h^{(N-1)} + \cdots + a_{N-1} h^{(1)} + a_N h(t) = \delta^{(N)}$$

$$h^{(N)} + a_1 h^{(N-1)} + \cdots + a_{N-1} h^{(1)} + a_N h(t) = \delta^{(N-1)}$$

$$\vdots$$

$$h^{(N)} + a_1 h^{(N-1)} + \cdots + a_{N-1} h^{(1)} + a_N h(t) = \delta^{(1)}$$

$$h^{(N)} + a_1 h^{(N-1)} + \cdots + a_{N-1} h^{(1)} + a_N h(t) = \delta$$

N	$h(t)$	⇒ $h_N(t)$	$= h_0^{(N)}(t)$
$N-1$	$h(t)$	⇒ $h_{N-1}(t)$	$= h_0^{(N-1)}(t)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	$h(t)$	⇒ $h_1(t)$	$= h_0^{(1)}(t)$
0	$h(t)$	⇒ $h_0(t)$	$= h_0^{(0)}(t)$

$$h(t) = b_0 h_N + b_1 h_{N-1} + \cdots + b_{N-1} h_1 + b_N h_0$$

$$= b_0 h_0^{(N)} + b_1 h_0^{(N-1)} + \cdots + b_{N-1} h_0^{(1)} + b_N h_0^{(0)}$$

Superposition of inputs (2)

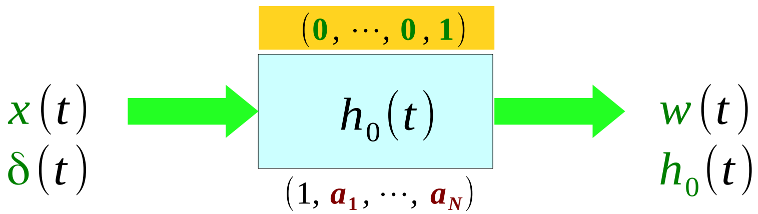
$$h^{(N)}(t) + a_1 h^{(N-1)}(t) + \cdots + a_{N-1} h^{(1)}(t) + a_N h(t) = b_0 \delta^{(N)}(t) + b_1 \delta^{(N-1)}(t) + \cdots + b_{N-1} \delta^{(1)}(t) + b_N \delta(t)$$

$$h(t) = b_0 h_0^{(N)} + b_1 h_0^{(N-1)} + \cdots + b_{N-1} h_0^{(1)} + b_N h_0^{(0)}$$

$$h^{(N)}(t) + a_1 h^{(N-1)}(t) + \cdots + a_{N-1} h^{(1)}(t) + a_N h(t) = \delta(t)$$

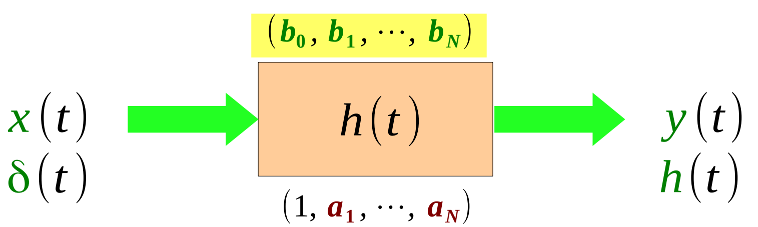
$\Rightarrow$ $h_0(t)$

h(t) of a General System S via $y_n(t)$



$$h_0 = \frac{1}{a_N}\left(\delta - h_0^{(N)} - a_1 h_0^{(N-1)} - \cdots - a_{N-1} h_0^{(1)}\right)$$

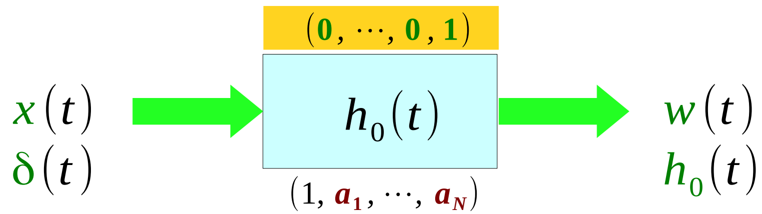
causal $\quad h_0(t) = y_n(t)\,u(t)$



$$h(t) = b_0 h_0^{(N)}(t) + b_1 h_0^{(N-1)}(t) + \cdots + b_N h_0^{(0)}(t)$$

causal $\quad h(t) = b_0\{y_n u\}^{(N)} + b_1\{y_n u\}^{(N-1)} + \cdots + b_N\{y_n u\}^{(0)}$

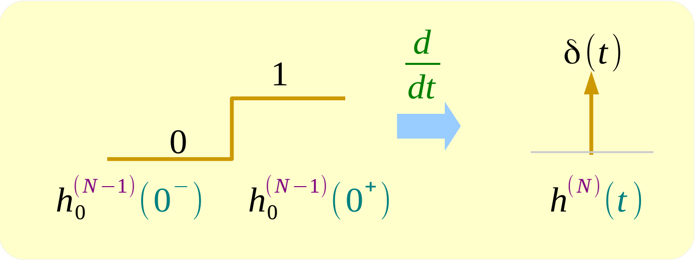
Impulse Matching of $h_0(t)$



$$h_0 = \frac{1}{a_N}\left(\delta - h_0^{(N)} - a_1 h_0^{(N-1)} - \cdots - a_{N-1} h_0^{(1)}\right)$$

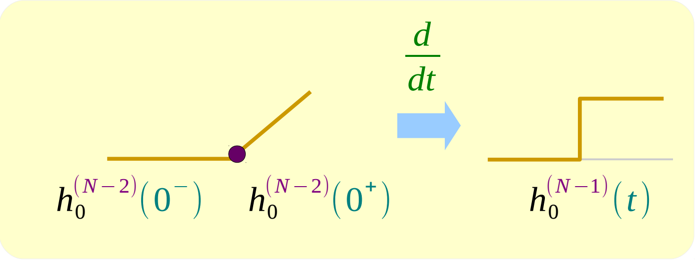
causal $\quad h_0(t) = y_n(t)u(t)$

the only finite jump



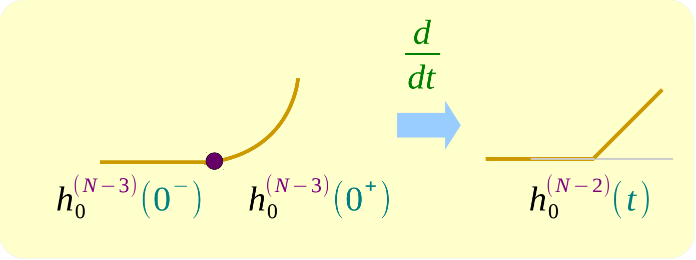
$$h_0^{(N-1)}(0^+) = 1$$

continuous



$$h_0^{(N-2)}(0^+) = 0$$

continuous



$$h_0^{(N-3)}(0^+) = 0$$

New Initial Condition created by δ(t)

$$h^{(N)} + a_1 h^{(N-1)} + \cdots + a_{N-1} h^{(1)} + a_N h(t) = \delta$$

Single Impulse at the right hand side creates a unique initial condition

integrate integrate integrate integrate

$$\frac{d^N h_0(t)}{dt^N} + a_1 \frac{d^{N-1} h_0(t)}{dt^{N-1}} + a_2 \frac{d^{N-2} h_0(t)}{dt^{N-2}} + \cdots + a_{N-1} \frac{d h_0(t)}{dt} + a_N h_0(t) = \delta(t)$$

$$\delta(t) \qquad u(t) \qquad t\,u(t) \qquad t^{N-2}\,u(t) \qquad t^{N-1}\,u(t)$$

differentiate differentiate differentiate differentiate

A unit ***impulse*** *at t = 0*

A unit jump ***discontinuity*** *at t = 0*

no jump discontinuity at t = 0

$$h_0^{(N-1)}(0^+) = \mathbf{1}$$

$$h_0^{(N-2)}(0^+) = h_0^{(N-1)}(0^+) = \cdots \; h_0^{(1)}(0^+) = h_0(0^+) = \mathbf{0}$$

$h_0(t)$ and $y_n(t)$

$$h^{(N)} + a_1 h^{(N-1)} + \cdots + a_{N-1} h^{(1)} + a_N h(t) = \delta$$

M = 0

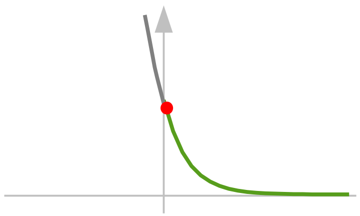
$$h_0^{(N-1)}(0^+) = 1 \qquad h_0^{(N-2)}(0^+) = h_0^{(N-1)}(0^+) = \cdots h_0^{(1)}(0^+) = h_0(0^+) = 0$$

N > M : $h_0(t)$ include no impulse $\delta(t)$ (t > 0)

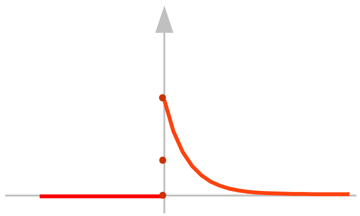
➡ $h_0(t)$ include characteristic modes only

➡ System **S** and **S0** have the same characteristic modes

$y_n(t)$



$h_0(t) = y_n(t) u(t)$



$$y_n^{(N)} + a_1 y_n^{(N-1)} + \cdots + a_{N-1} y_n^{(1)} + a_N y_n(t) = 0$$

homogeneous solution

$$y_n^{(N-1)}(0^+) = 1 \qquad y_n^{(N-2)}(0^+) = y_n^{(N-1)}(0^+) = \cdots y_n^{(1)}(0^+) = y_n(0^+) = 0$$

Solve

$$y_n(t) = \left(c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} + \cdots + c_{N-2} e^{\lambda_{N-2} t} + c_{N-1} e^{\lambda_{N_1} t}\right)$$

➡

$$h_0(t) = y_n(t) u(t)$$

Simplified Impulse Matching

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot h(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot \delta(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot h_0(t) = \delta(t)$$

$$Q(D) \cdot h(t) = P(D) \cdot \delta(t) \qquad h(t) = P(D) \cdot h_0(t)$$

$$Q(D) \cdot h_0(t) = \delta(t)$$

$t \geq 0^+$	h(t)	= characteristic mode terms
$t \geq 0$	h(t)	$= A_0 \delta(t)$ + characteristic mode terms

Simplified Impulse Matching Method

$$h(t) = b_0 \delta(t) + [P(D) h_0(t)] \cdot u(t)$$

$h_0(t)$ *linear combination of characteristic modes with the following initial conditions*

$$h_0(0^+) = h_0^{(1)}(0^+) = h_0^{(2)}(0^+) \cdots = h_0^{(M-2)}(0^+) = 0 \qquad h_0^{(M-1)}(0^+) = 1$$

Simplified Impulse Matching Method (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

Derived System S

shares the same characteristic modes

Base System S0

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = x(t)$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = \mathbf{0}$$

$y_n(t)$ *linear combination of characteristic modes with the following initial conditions*

$$y_n^{(N-1)}(0^+) = 1 \qquad y_n^{(N-2)}(0^+) = \cdots = y_n^{(1)}(0^+) = y_n(0^+) = 0$$

$y_h(t)$ *Yet, another linear combination of characteristic modes*

$$h(t) = b_0 \delta(t) + [P(D) y_n(t)] u(t)$$

becomes

$$h(t) = y_h(t) u(t)$$

Simplified Impulse Matching Method (2)

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) w(t) = x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_0 D^M + \cdots + b_{N-1} D + b_N) x(t)$$

$$y(t) = (b_0 D^M + \cdots + b_{N-1} D + b_N) w(t)$$

$$Q(D) w(t) = x(t)$$

$$Q(D) P(D) w(t) = P(D) x(t)$$

$$y(t) = P(D) w(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y_n(t) = \delta(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) h(t) = (b_0 D^M + \cdots + b_{N-1} D + b_N) \delta(t)$$

$$h(t) = (b_0 D^M + \cdots + b_{N-1} D + b_N) y_n(t)$$

$$Q(D) y_n(t) = \delta(t)$$

$$Q(D) P(D) y_n(t) = P(D) \delta(t)$$

$$h(t) = P(D) y_n(t)$$

No interval restriction

Causality is considered *causal* $y_n(t) u(t)$

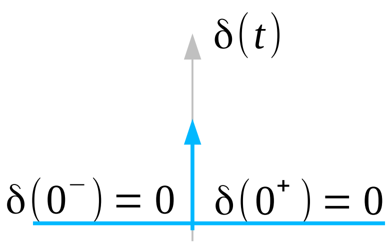
$$h(t) = P(D)[y_n(t) u(t)] \Rightarrow [P(D) y_n(t)] u(t)$$

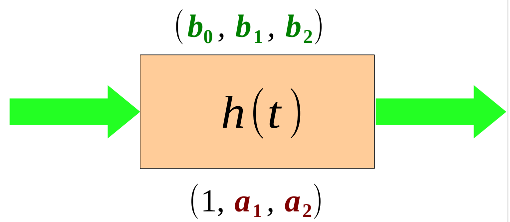
$$h(t) = b_o \delta(t) + P(D) y_n(t), \quad t \geq 0$$

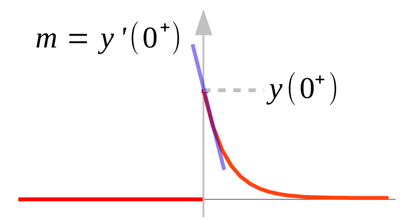
$$h(t) = b_o \delta(t) + [P(D) y_n(t)] u(t)$$

Impulse Input

$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{d\,y(t)}{dt} + a_2\, y(t) = b_0 \frac{d^2 x(t)}{dt^2} + b_1 \frac{d\,x(t)}{dt} + b_2\, x(t)$$

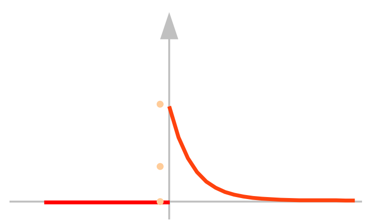


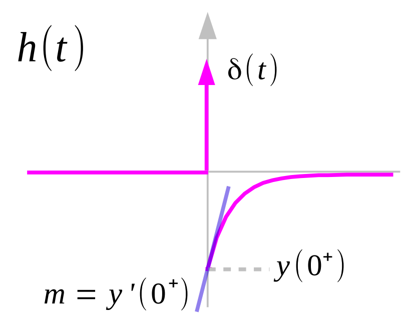




Case **N>M** (b_0=0)
No delta function

Any two functions that have finite values everywhere and **differ** in value only at a finite number of points are **equivalent** in the system response or transform





Case **N=M** (b_0≠0)
No delta function

All initial conditions are zero at $t=0^-$

generates energy storage creates nonzero initial condition at $t=0^+$

$y(0^-) = 0$
$y'(0^-) = 0$

$y(0^+) = K_1$
$y'(0^+) = K_2$

IVP (Initial Value Problem)

Derivatives of a delta function at the output

$$\frac{d^2 h(t)}{dt^2} + a_1 \frac{dh(t)}{dt} + a_2 h(t) = b_0 \frac{d^2 \delta(t)}{dt^2} + b_1 \frac{d\delta(t)}{dt} + b_2 \delta(t)$$

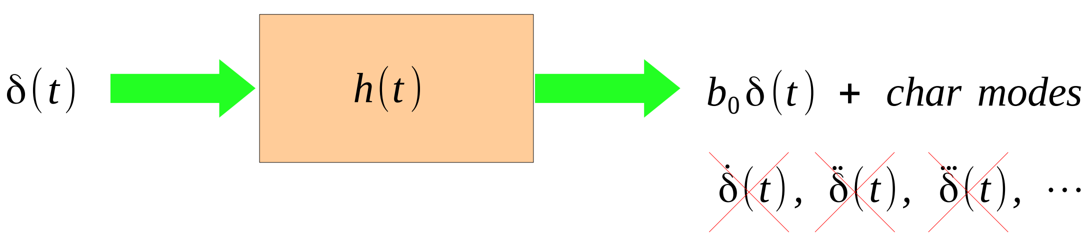
if $h(t)$ contains $\delta^{(1)}(t)$ $(b_0 \neq 0)$

$$\delta^{(3)}(t) + \cdots \neq b_0 \delta^{(2)}(t) + b_1 \delta^{(1)}(t) + b_2 \delta(t)$$

the impulse response $h(t)$ can contain at most $\delta(t)$ $(b_0 \neq 0)$

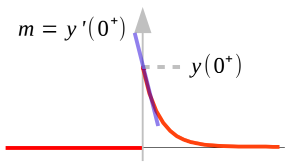
the impulse response $h(t)$ cannot contain any derivatives of $\delta(t)$

N ≥ M

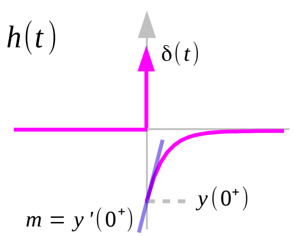


Impulse Response h(t)

$$\frac{d^2 h(t)}{dt^2} + a_1 \frac{dh(t)}{dt} + a_2 h(t) = b_0 \frac{d^2 \delta(t)}{dt^2} + b_1 \frac{d\delta(t)}{dt} + b_2 \delta(t)$$



Case **N>M** ($b_0 = 0$)
No delta function



Case **N=M** ($b_0 \neq 0$)
No delta function

$t \geq 0^+$ $(t \neq 0)$: h(t) = characteristic mode terms only

$t = 0$: h(t) can have at most an impulse $b_0 \delta(t)$ or finite jump

$$h(t) = b_0 \delta(t) + \textit{char mode terms} \quad t \geq 0$$

(b_0, b_1, b_2)

$h(t)$

$(1, a_1, a_2)$

Impulse Matching

-determine $y(0^+), \; y'(0^+)$

-then determine $c_0, \; c_1$

$$h(t) = b_0 \delta(t) + \left(\sum_i c_i e^{-\lambda_i t} \right) u(t)$$

$(0, 0, 1)$

$h_0(t)$

$(1, a_1, a_2)$

Simplified Impulse Matching

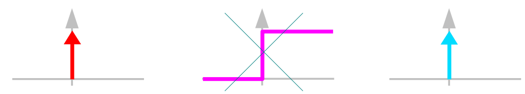
-use $y(0^+) = 0, \; y'(0^+) = 1$

-then determine $e_0, \; e_1$

$$h(t) = b_0 \delta(t) + \left[(b_0 D^2 + b_1 D + b_2) \left(\sum_i e_i e^{-\lambda_i t} \right) \right] u(t)$$

Base System Impulse Matching

$$y_n^{(N)}(t) + a_1 y_n^{(N-1)}(t) + \quad \cdots \quad + a_{N-1} y_n^{(1)}(t) + a_N y_n(t) = \delta(t)$$



$y_n(t)$ **must** continuous

$$y_n(0^+) = y_n(0^-) = 0$$

If $y_n(t)$ is discontinuous, then there should exist derivatives of delta : $\delta^{(i)}(t)$

$y_n^{(1)}(t)$ **must** continuous

$$y_n^{(1)}(0^+) = y_n^{(1)}(0^-) = 0$$

If $y_n^{(1)}(t)$ is discontinuous, then there should exist derivatives of delta : $\delta^{(i)}(t)$

$y_n^{(N-1)}(t)$ **must** be discontinuous

$$y_n^{(N-1)}(0^+) = 1, \quad y_n^{(N-1)}(0^-) = 0$$

$y_n(t)$ contains no impulse, but characteristic modes only

$N > M \; (=0)$

Impulse Response & Particular Solution

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,x(t)}{dt} + b_M\, x(t)$$

y(t) = h(t) : impulse response

when x(t) = δ(t) : forcing function

causal system

interval of existence / uniqueness

forcing function

	$t < 0$	$t = 0$	$t > 0$
x(t) =	0	δ(t)	0

particular solution	$y_p(t)$	0	$A\delta(t)$	0
homogeneous solution	$y_h(t)$	0	$y_h(t)$	$y_h(t)$

A=0 (N>M)

A≠0 (N=M)

impulse response

$h(t) = y_h(t)u(t)$

$h(t) = y_h(t)u(t) + A\delta(t)$ only when N=M

Impulse Response of Differential Equations

(t > 0) Homogeneous solution

$$a_0 \frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) = 0$$

(t = 0) Particular solution

$$a_0 \frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t)$$
$$= b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\delta(t)}{dt} + b_M \delta(t)$$

$y_h(t)$ **homogeneous solution**

– *characteristic modes only*

$y_p(t)$ **particular solution**

– linear combination of the *forcing function* x(t) and *all its unique derivatives* $x^{(i)}(t)$

$$\begin{cases} h(t) = y_h(t)u(t) \\ h(t) = y_h(t)u(t) + A\delta(t) \quad \text{only when } N=M \end{cases}$$

$$y_p(t) \Leftarrow \begin{cases} y(t) = b_0\delta(t) \\ y^{(i)}(t) = d_i\delta^{(i)}(t) \end{cases} \Leftarrow \begin{cases} x(t) = \delta(t) \\ x^{(i)}(t) = \delta^{(i)}(t) \end{cases}$$

linear combination of all the derivatives of h(t) must add to zero for any time $t \neq 0$

⬅ linear combination of an **impulse** and **its unique derivatives** (the doublet, the triplet, etc) : all these exist at time $t = 0$

Particular Solution at t=0

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,x(t)}{dt} + b_M\, x(t)$$

when x(t) = δ(t) : forcing function

$$y_p(t) = b_0 \delta^{(M)}(t) + b_1 \delta^{(M-1)}(t) + \cdots + b_{M-1} \delta^{(1)}(t) + b_M \delta(t)$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = x(t)$$

when x(t) = δ(t) : forcing function

$$y_p(t) = \delta(t)$$

Particular Solutions for x(t)=δ(t)

Method of Undetermined coefficients

Particular solution

$$a_0 \frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$

h(t) is differentiated up to *n* times ⇐ must match ⇒ δ(t) is differentiated up to *m* times

Finding particular solution

We can determine whether h(t) can contain an impulse or its unique derivatives

$$\int_{0^-}^{0^+} h(t)\,dt = 0$$

when h(t) does not have an impulse or its derivatives

$$\int_{0^-}^{0^+} h(t)\,dt \neq 0$$

Otherwise

h(t) can have at most a δ(t)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) = b_0 \frac{d^N \delta(t)}{dt^N} + b_1 \frac{d^{N-1} \delta(t)}{dt^{N-1}} + \cdots + b_{N-1} \frac{d\delta(t)}{dt} + b_N \delta(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) h(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N) \delta(t)$$

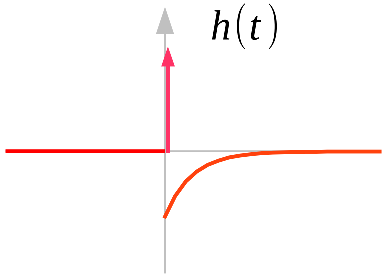
$M = N$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) h(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N) \delta(t)$$

If $\delta^{(1)}(t)$ is included in $h(t)$, then the highest order term

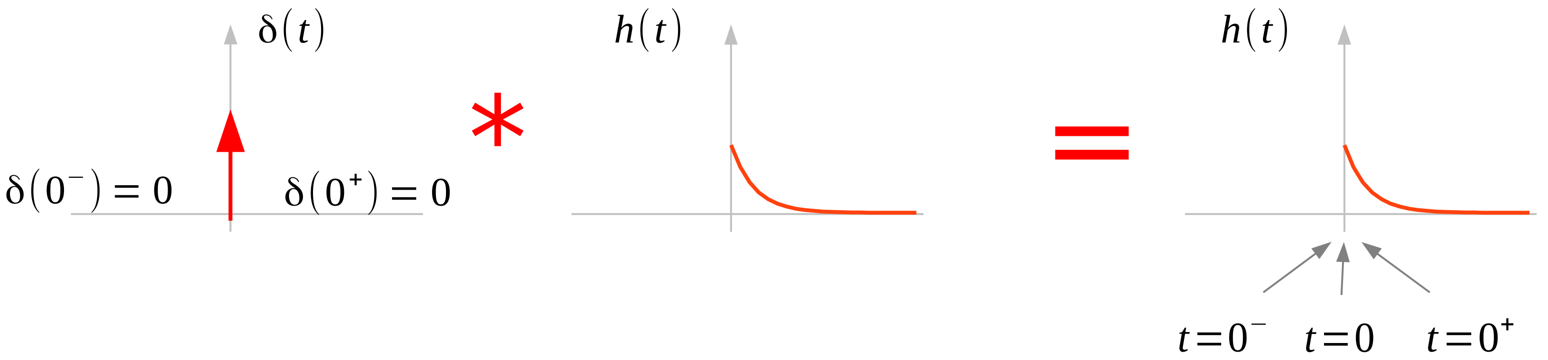
$D^N \delta^{(1)}(t) \rightarrow \delta^{(N+1)}(t)$ $\qquad$ $b_0 D^N \delta(t) \rightarrow \delta^{(N)}(t)$ $\qquad$ Contradiction!!!

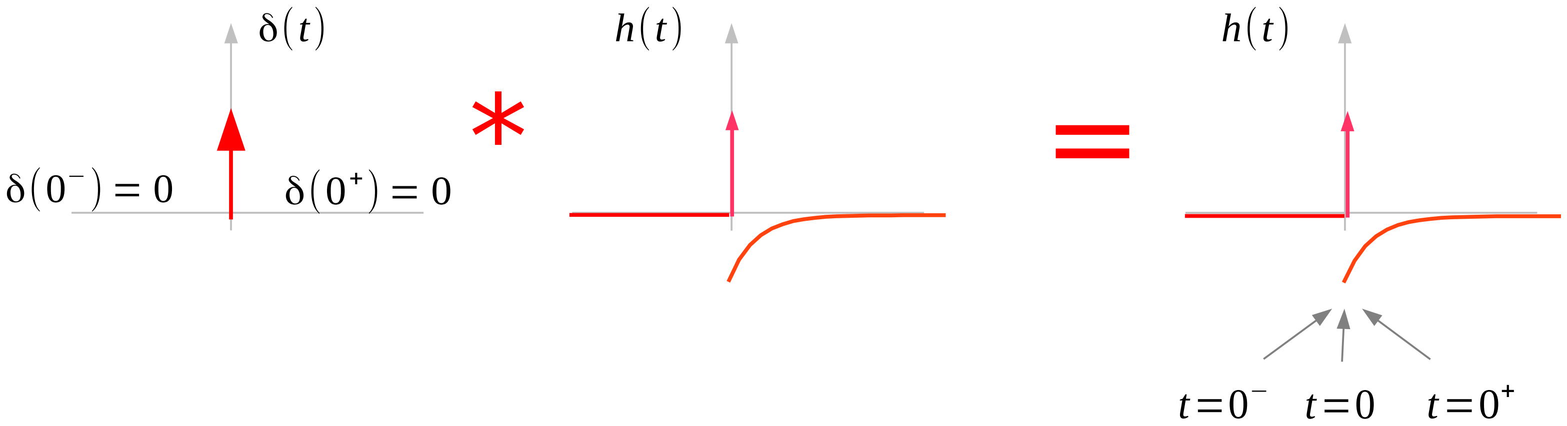
h(t) cannot contain $\delta^{(i)}(t)$ at all ➡ *h(t)* can contain at most δ(t) $\quad M \le N$



$$h(t) = b_0 \delta(t) + char\ mode\ terms \quad t \ge 0$$

Impulse Response h(t)





Initial Conditions & Total Response

$\delta(t)$ → $h(t)$ → $b_0 \delta(t) + \textit{char mode terms} \quad t \geq 0$

$\dot{\delta}(t)$ → $h(t)$ → $b_0 \dot{\delta}(t) + \frac{d}{dt} \textit{char mode terms} \quad t \geq 0$

$\ddot{\delta}(t)$ → $h(t)$ → $b_0 \ddot{\delta}(t) + \frac{d}{dt} \textit{char mode terms} \quad t \geq 0$

linear combination of an **impulse** and **its unique derivatives** (the doublet, the triplet, etc) : all these exist at time t = 0

$u(t)$ → $h(t)$ → $c + \int \textit{char mode terms} \, dt \quad t \geq 0$

$r(t)$ → $h(t)$ → $c + \int \int \textit{char mode terms} \, dt \, dt \quad t \geq 0$

no **impulse** and **its unique derivatives** at time t = 0

Generally, the interval of interest for y(t) is t > 0

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] B.P. Lathi, Linear Systems and Signals (2nd Ed)