The Growth of Functions (2A)

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4/3/18

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Functions and Ranges

$$\left\{ \begin{array}{l} x^2+2x+1 \\ x^2 \\ 2x \\ 1 \end{array} \right.$$

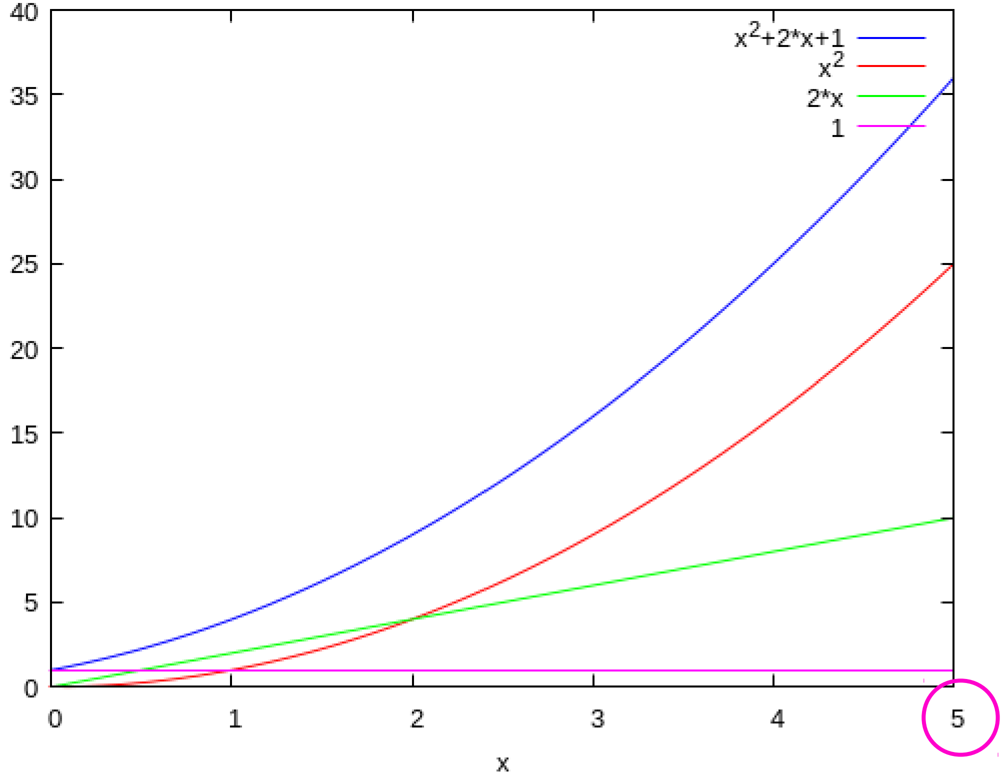
$$A_1 = [0,5]$$

$$A_2 = [0,100]$$

$$A_3 = [0,500]$$

Small Range

All are distinguishable



$x^2 + 2x + 1$

x^2

$2x$

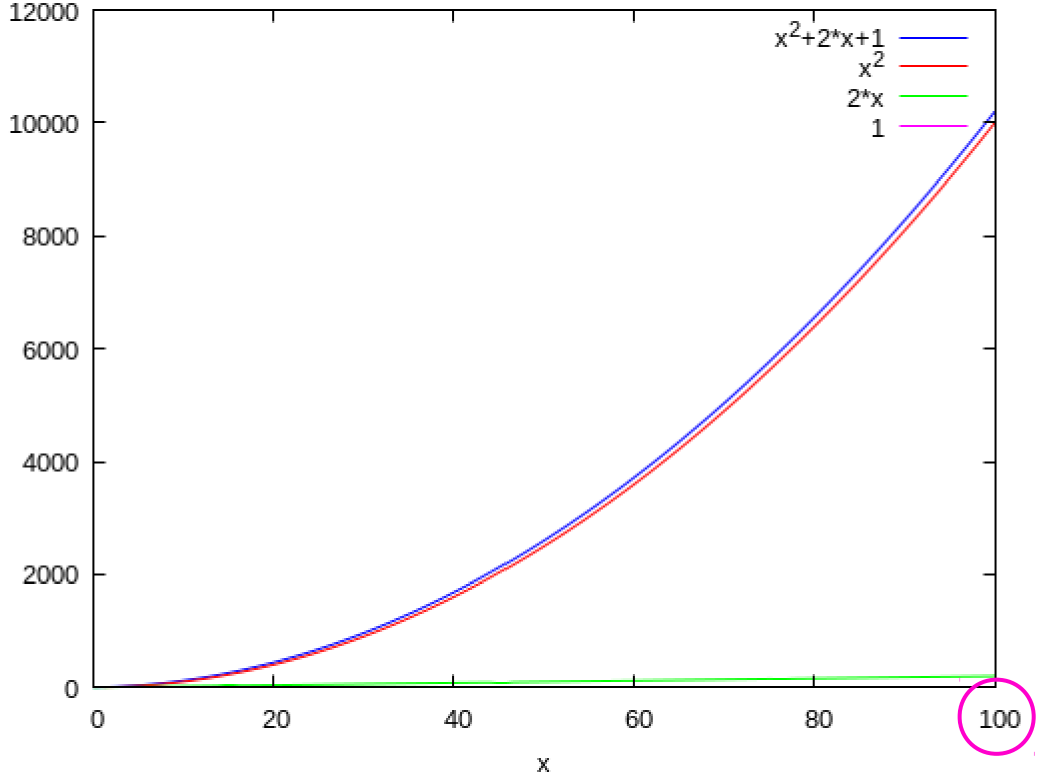
1

Zoom Out

$for \quad x > -0.5 \qquad x^2 < x^2 + 2x + 1$

Medium Range

similar



$x^2 + 2x + 1$ $10000 + 201$

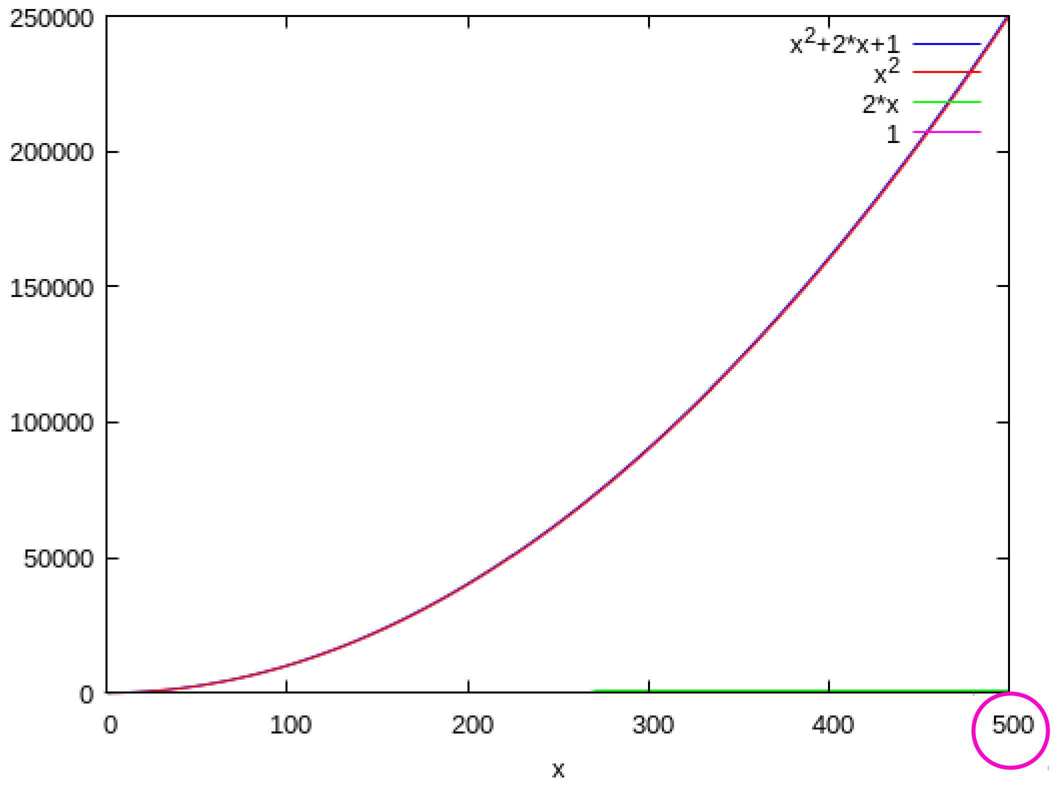
x^2 10000

$2x$

Zoom Out More

$for \quad x > -0.5$ $x^2 < x^2 + 2x + 1$

Indistinguishable



$x^2 + 2x + 1$ $\quad 250000 + 1001$

x^2 $\quad 250000$

$for \quad x > -0.5 \qquad x^2 < x^2 + 2x + 1$

Functions and Ranges

$$\left\{\begin{array}{l} 2 \cdot x^2 \\ x^2 + 2x + 1 \\ 2x \\ 1 \end{array}\right.$$

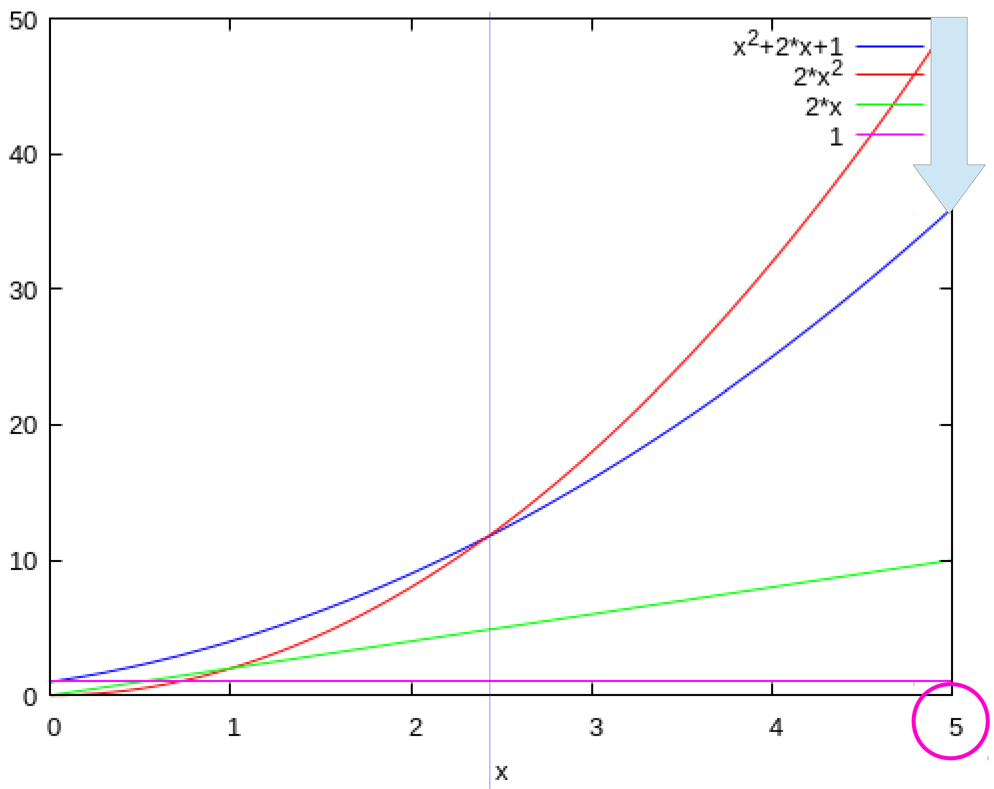
$$B_1 = [0, 5]$$

$$B_2 = [0, 100]$$

$$B_3 = [0, 500]$$

Small Range, $2x^2$

distinguishable



$2x^2$ 50

x^2+2x+1 $25+11$

$2x$

1

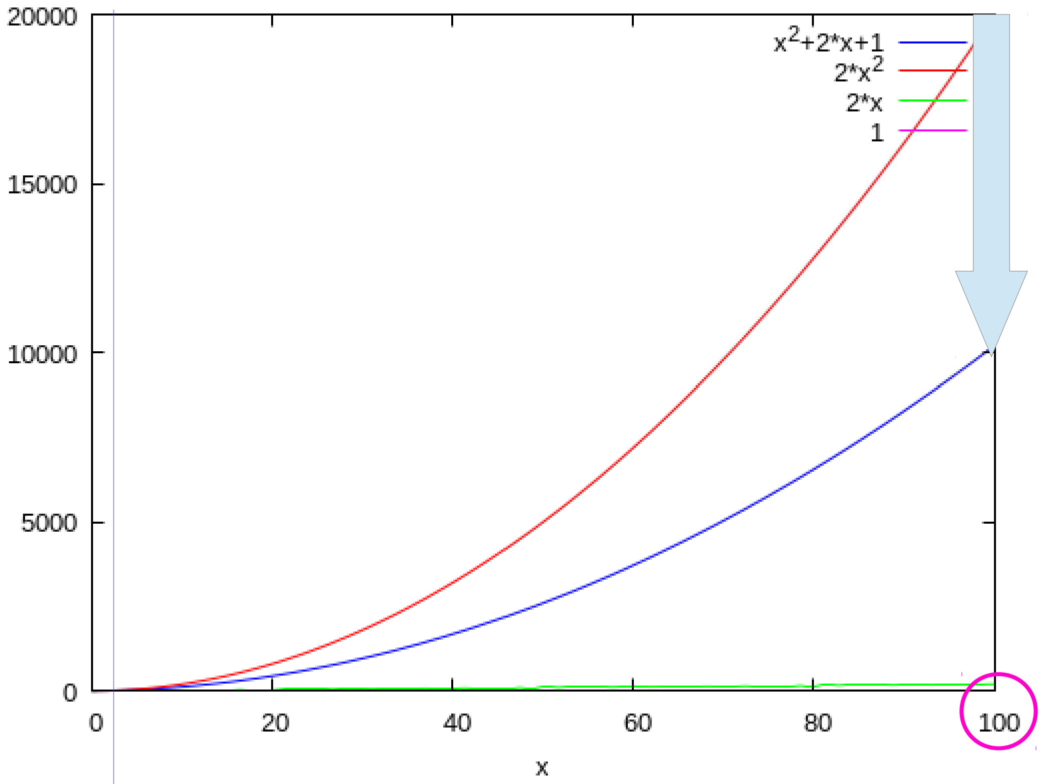
Zoom Out

$$\text{for} \quad x > 2.414 \qquad x^2+2x+1 < 2x^2$$

$$\text{for} \quad x > k \qquad f(n) < C\,g(n)$$

Medium Range, $2x^2$

distinguishable



$2x^2$ 20000

x^2+2x+1 10000+201

$2x$

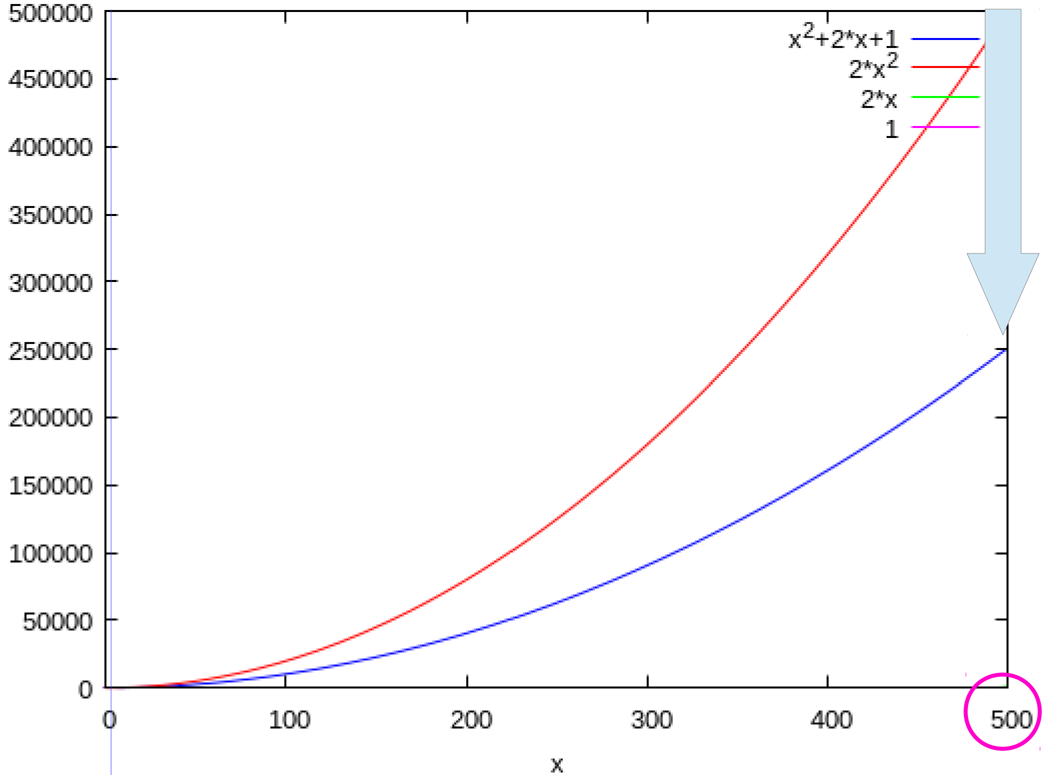
1

Zoom Out

for $x > 2.414$ $x^2+2x+1 < 2x^2$

for $x > k$ $f(n) < C g(n)$

Large Range, $2x^2$



distinguishable

$2x^2$ 500000

x^2+2x+1 250000 + 1001

$2x$

1

Zoom Out

for $x > 2.414$ $x^2+2x+1 < 2x^2$

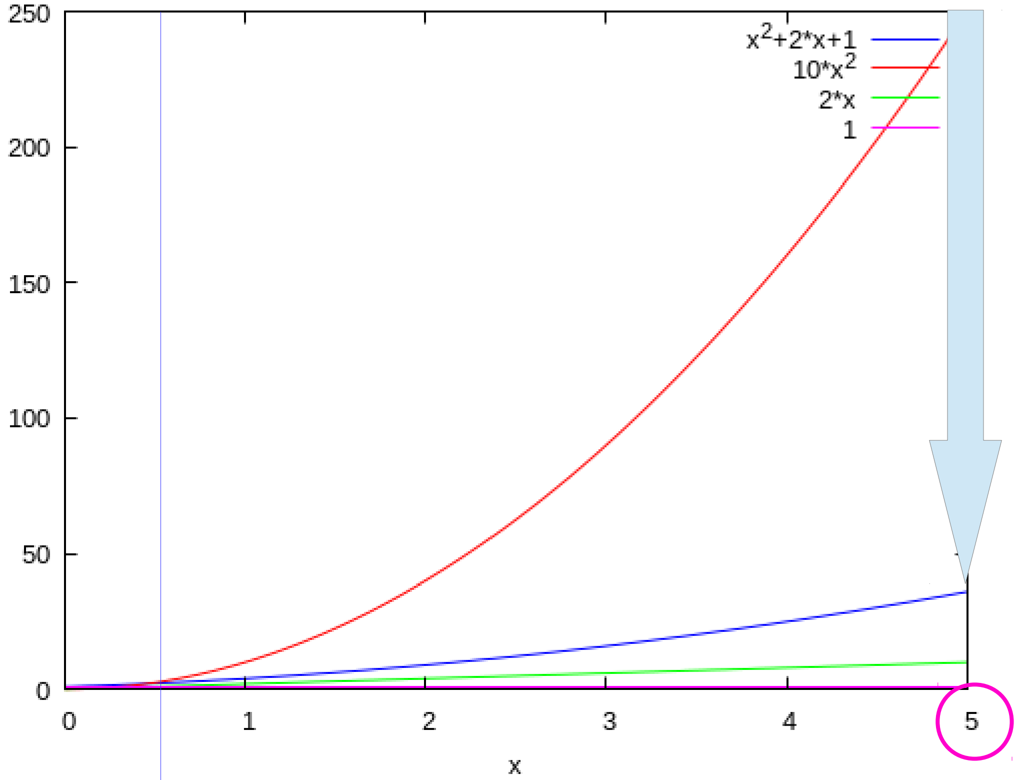
for $x > k$ $f(n) < C g(n)$

Functions and Ranges

$$\left\{ \begin{array}{l} 10 \cdot x^2 \\ x^2 + 2x + 1 \\ 2x \\ 1 \end{array} \right.$$

$$C_1 = [0, 5]$$
$$C_2 = [0, 100]$$
$$C_3 = [0, 500]$$

Small Range, $10x^2$



distinguishable

$10x^2$ 250

x^2+2x+1 $25+11$

$2x$

1

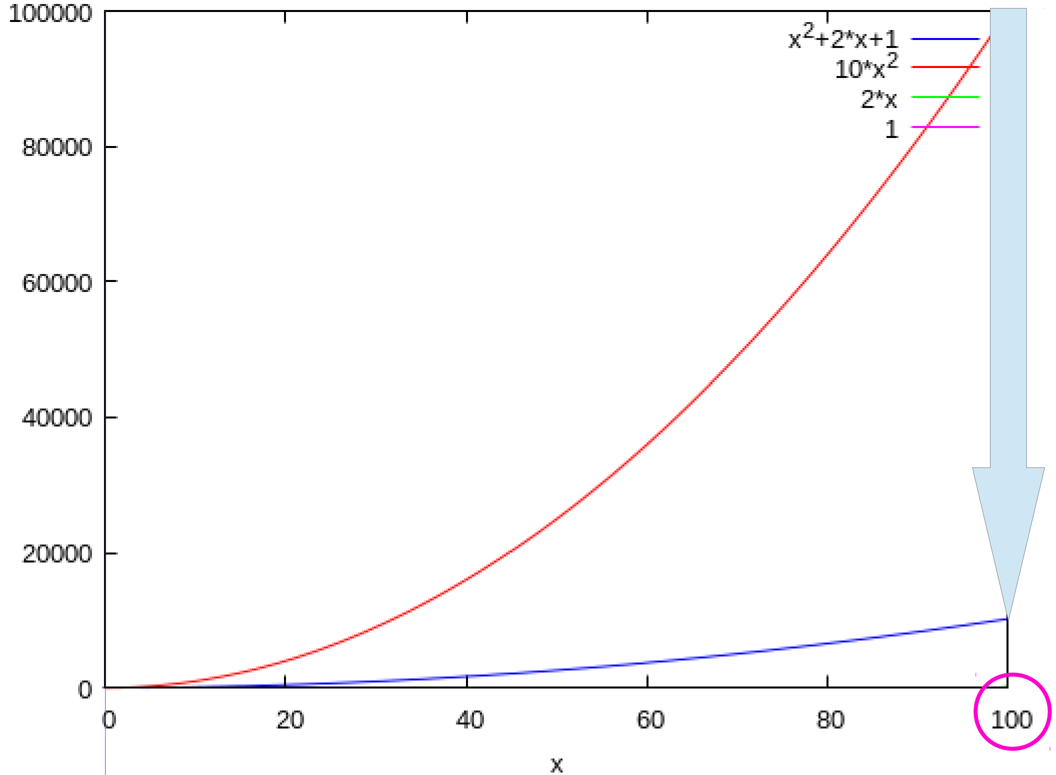
Zoom Out

$for \quad x > 0.462$ $\qquad x^2+2x+1 < 10x^2$

$for \quad x > k$ $\qquad f(n) < C g(n)$

Medium Range, $10x^2$

distinguishable



$10x^2$ 100000

x^2+2x+1 $10000+201$

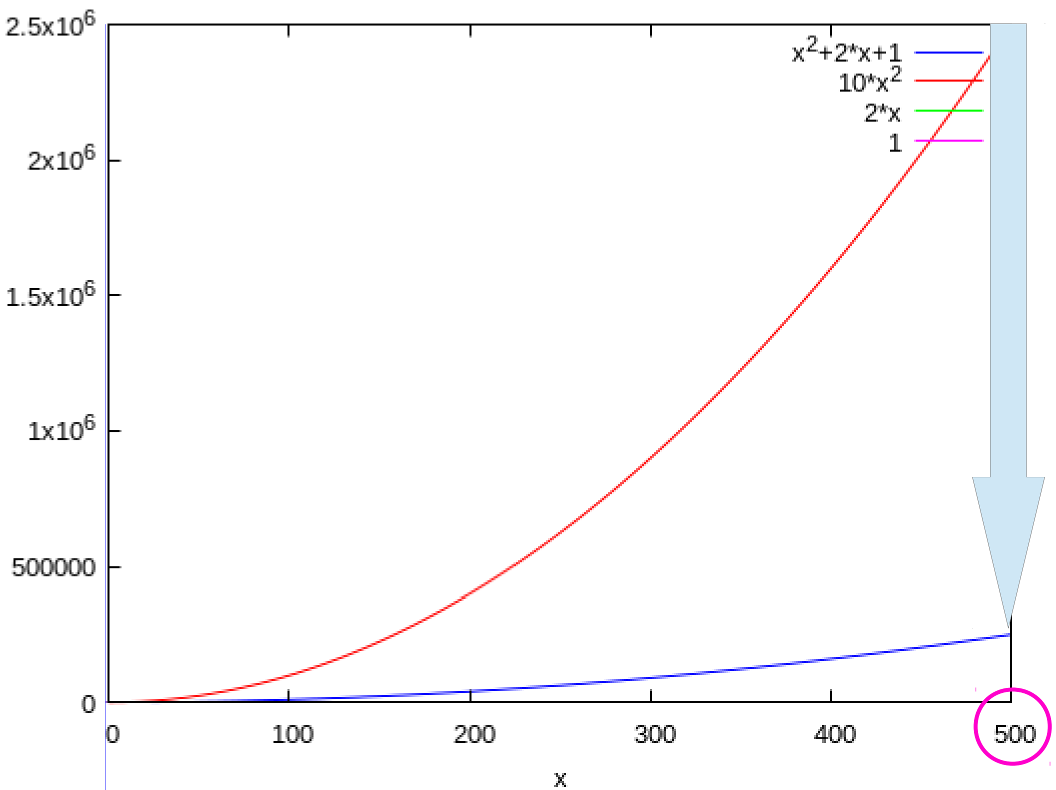
Zoom Out

for $x > 0.462$ $x^2+2x+1 < 10x^2$

for $x > k$ $f(n) < C\,g(n)$

Large Range, $10x^2$

distinguishable



$\mathbf{10}\,x^2$ 2500000

x^2+2x+1 $250000+1001$

for $x > 0.462$ $x^2+2x+1 < \mathbf{10}\,x^2$

for $x > k$ $f(n) < \mathbf{C}\,g(n)$

Functions and Ranges

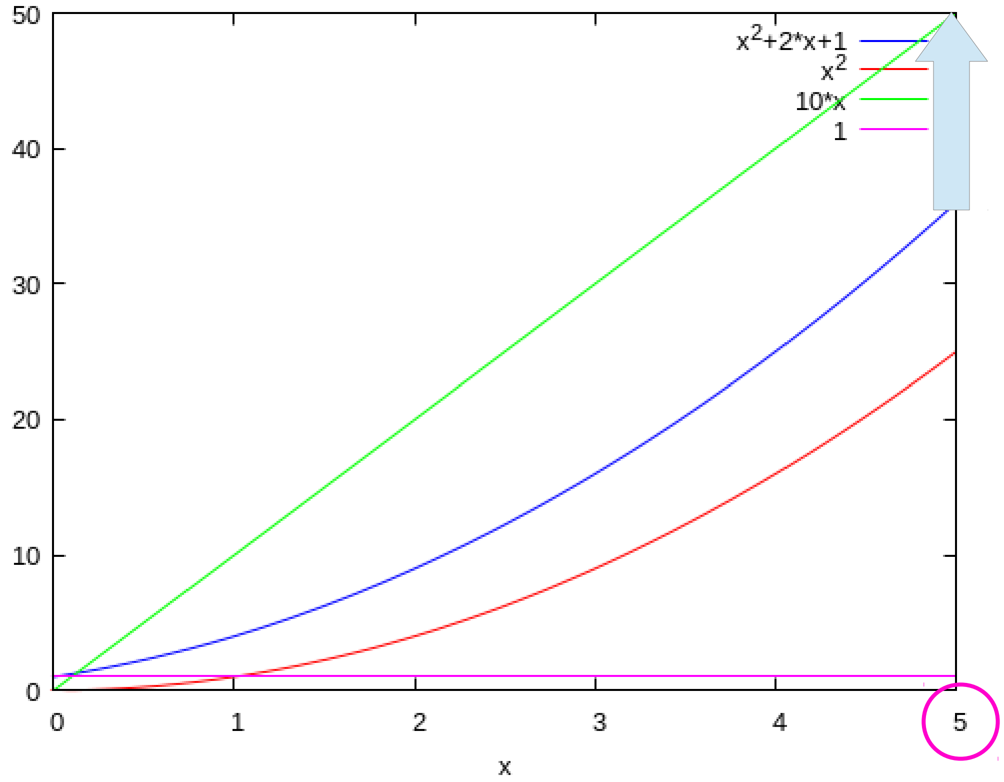
$$\left\{ \begin{array}{l} 10 \cdot x \\ x^2 + 2x + 1 \\ x^2 \\ 1 \end{array} \right.$$

$$D_1 = [0, 5]$$

$$D_2 = [0, 100]$$

$$D_3 = [0, 500]$$

Small Range, *10x*



$\mathbf{10}\,x$

x^2+2x+1

distinguishable

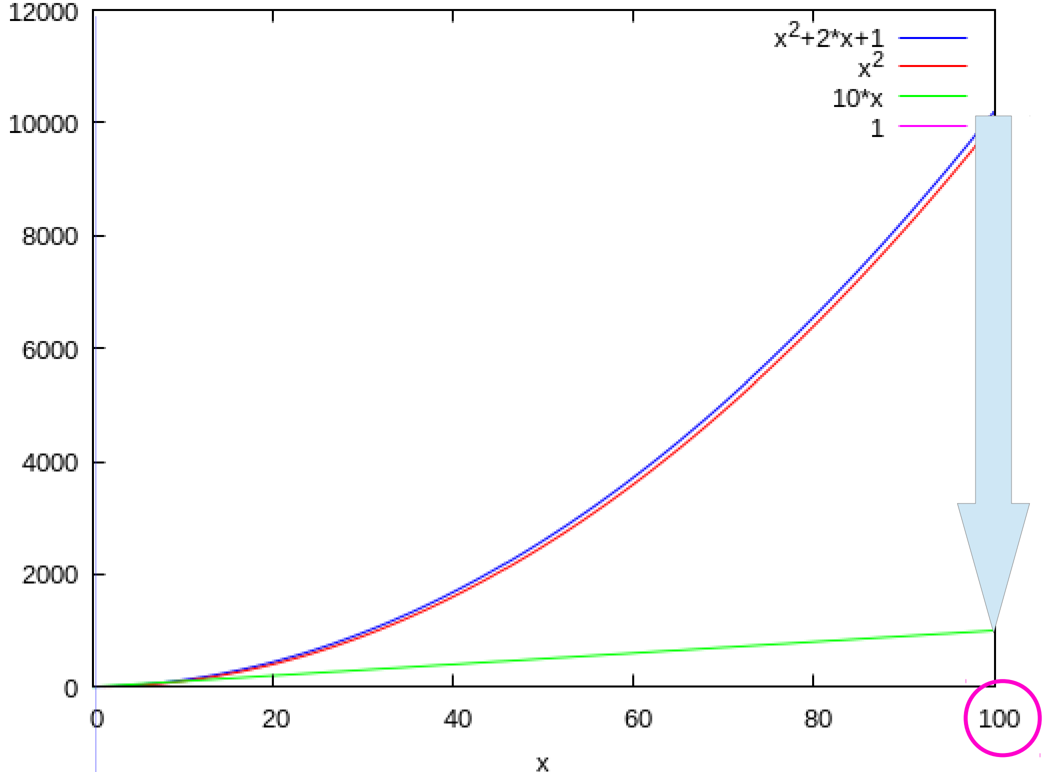
x^2

1

Zoom Out

for $0.127 < x < 7.873$ $\quad x^2+2x+1 < \mathbf{10}\,x$

Medium Range, *10x*



x^2+2x+1

x^2

indistinguishable

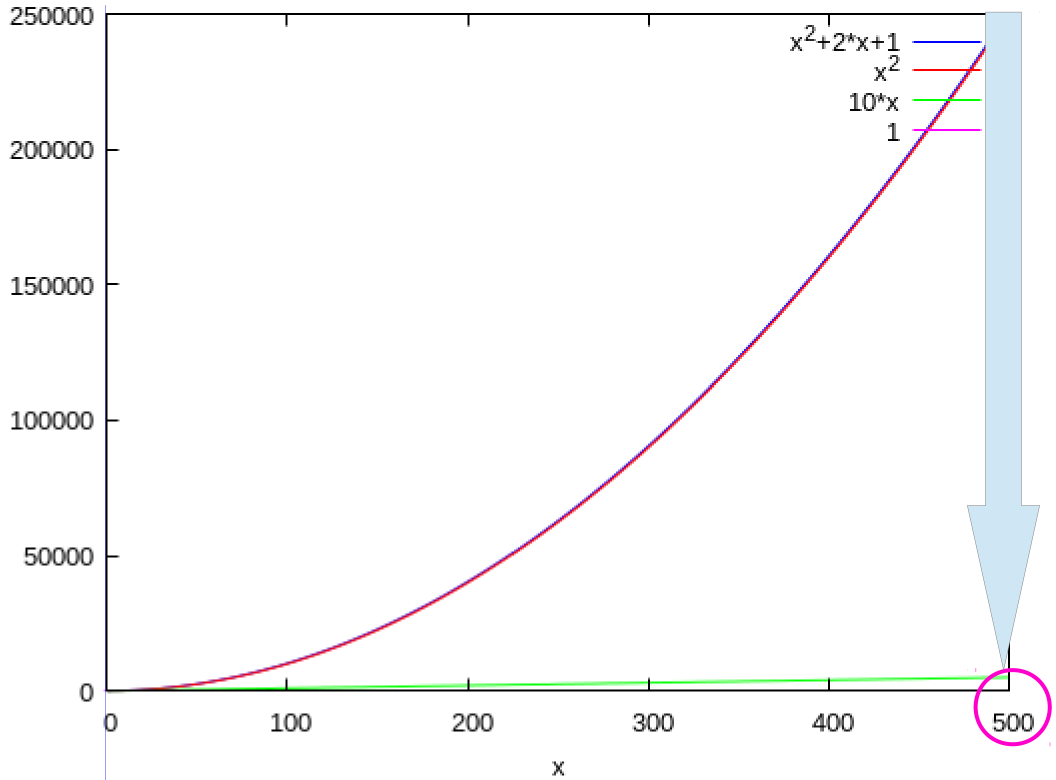
$10x$

Zoom Out

for $x > 7.873$ $\quad$ $10x < x^2+2x+1$

for $x > k$ $\quad$ $C\,g(n) < f(n)$

Large Range, *10x*



x^2+2x+1

x^2

indistinguishable

$10x$

for $x > 7.873$ $\quad$ $10x < x^2+2x+1$

for $x > k$ $\quad$ $C\,g(n) < f(n)$

Big-**O** Definition

Let f and g be functions (Z→R or R→R)
from the set of integers or
the set of real numbers
to the set of real numbers.

We say f(x) is **O(**g(x)**)** "f(x) is **big-oh** of g(x)"

If there are constants C and k such that

$|f(x)| \le C|g(x)|$ whenever $x > k$.

g(x) : **upper bound** of f(x)

Big-Ω Definition

Let f and g be functions (Z → R or R → R)
from the set of integers or
the set of real numbers
to the set of real numbers.

We say f(x) is **Ω(**g(x)**)** "f(x) is **big-omega** of g(x)"

If there are constants C and k such that

$C|g(x)| \leq |f(x)|$ whenever $x > k$.

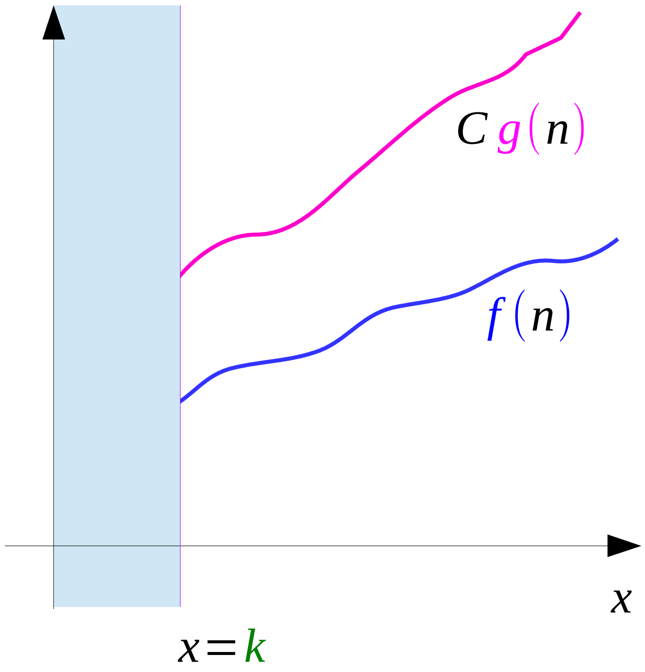
g(x) : **lower bound** of f(x)

Big-O Definition

$$\text{for} \quad k < x$$

$$f(x) \leq C|g(x)|$$

$$f(x) \text{ is } O(g(x))$$



g(x) : **upper bound** of f(x)

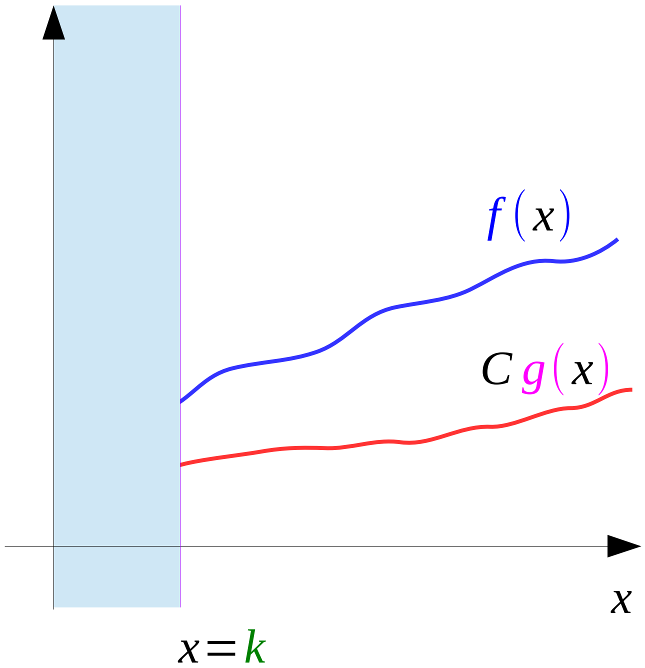
g(x) has a simpler form than f(x)
is usually a single term

Big-Ω Definition

$$\text{for} \quad k < x$$

$$f(x) \geq C|g(x)|$$

$$f(x) \text{ is } \Omega(g(x))$$



g(x) : **lower bound** of f(x)

g(x) has a simpler form than f(x)
is usually a single term

Big-Θ definition

$$\text{for} \quad k < x$$

$$f(x) \leq C|g(x)| \iff f(x) \text{ is } O(g(x))$$

$$C|g(x)| \leq f(x) \iff f(x) \text{ is } \Omega(g(x))$$

$$C_1|g(x)| \leq f(x) \leq C_2|g(x)| \iff f(x) \text{ is } \Theta(g(x))$$

Big-Θ = Big-Ω and Big-O

$for \quad k < x$

$$C_1|g(x)| \le f(x) \le C_2|g(x)| \quad \Longleftrightarrow \quad f(x) \; is \; \Theta(g(x))$$

$$\Omega(g(x)) \wedge O(g(x)) \quad \Longleftrightarrow \quad \Theta(g(x))$$

Big-O, Big-Ω, Big-Θ Examples

$for \quad x > -0.5$ $\quad 1x^2 < x^2+2x+1$ $\quad x^2+2x+1 \ is \ \Omega(x^2)$

lower bound

upper bound

$for \quad x > 2.414$ $\quad x^2+2x+1 < 2x^2$ $\quad x^2+2x+1 \ is \ O(x^2)$

$x^2+2x+1 \ is \ \Theta(x^2)$

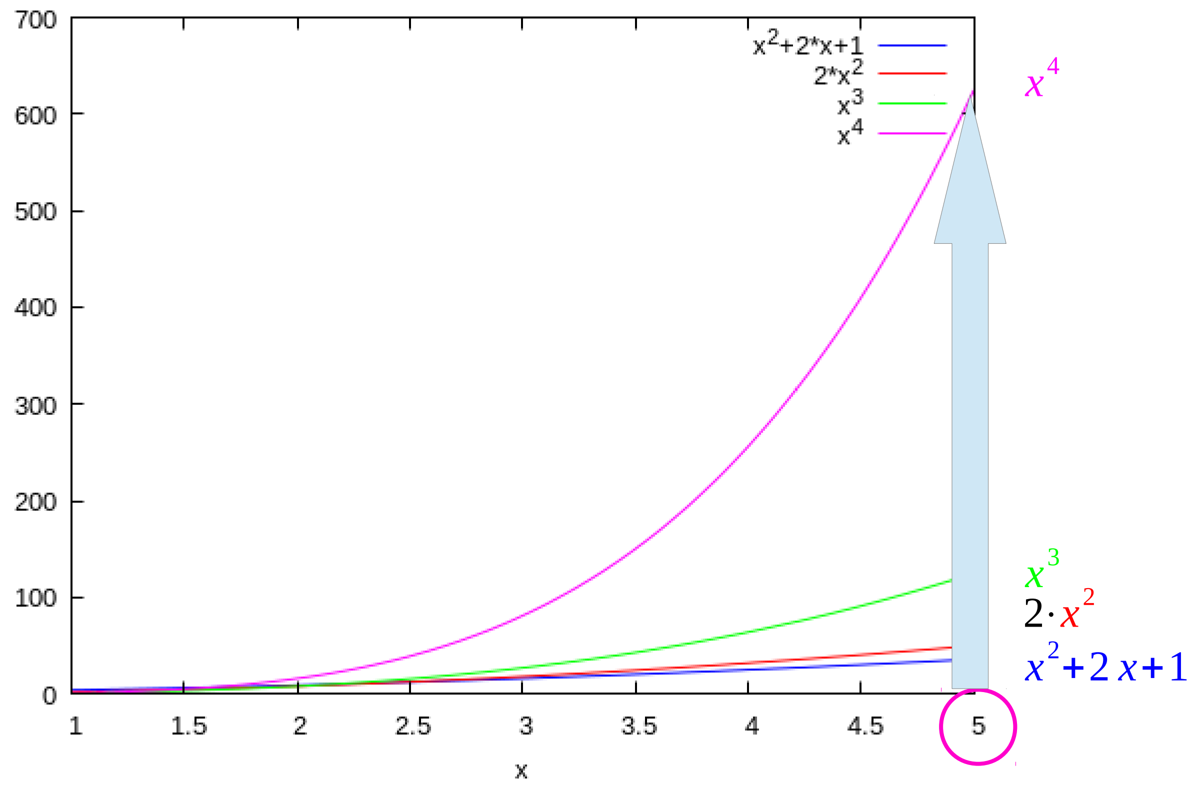
upper bound

$for \quad x > 0.462$ $\quad x^2+2x+1 < 10x^2$ $\quad x^2+2x+1 \ is \ O(x^2)$

$for \quad x > 7.873$ $\quad 10x < x^2+2x+1$ $\quad x^2+2x+1 \ is \ \Omega(x)$

lower bound

Many Larger Upper Bounds



x^2+2x+1 *is* $O(x^2)$

x^2+2x+1 *is* $O(x^3)$

x^2+2x+1 *is* $O(x^4)$

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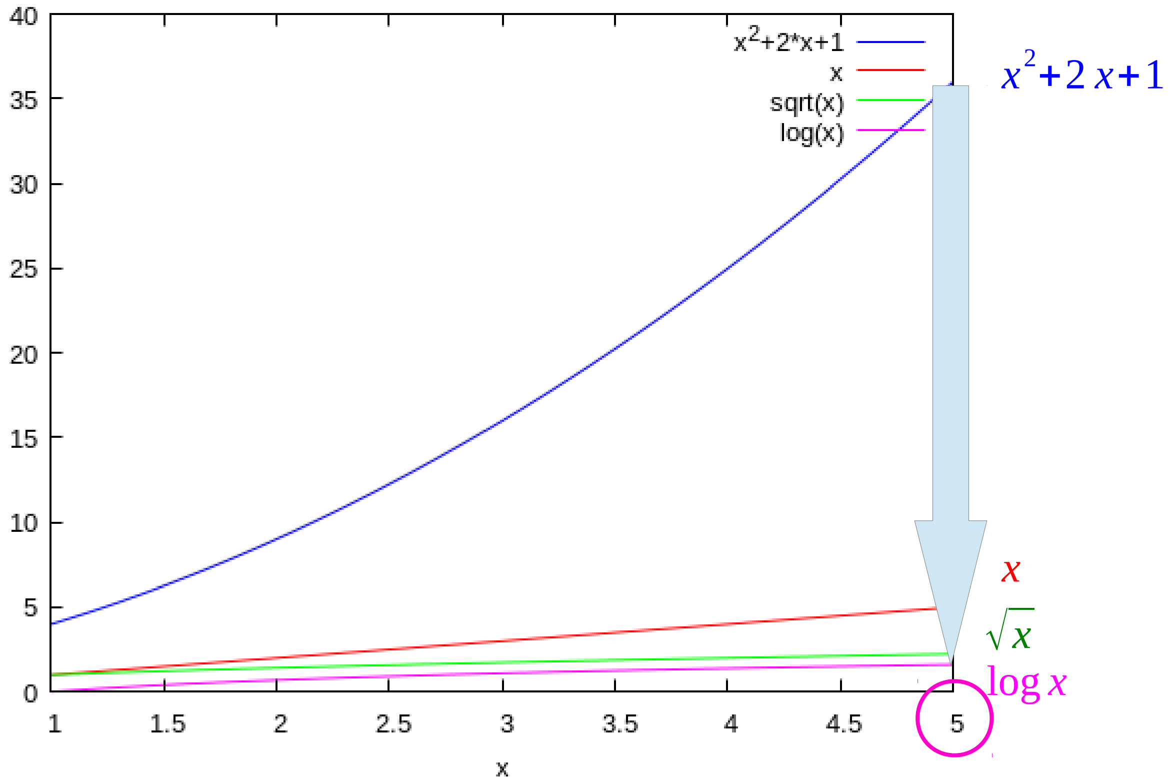
•
•

the least upper bound?

Many Smaller Lower Bound



x^2+2x+1 *is* $\Omega(x^2)$

x^2+2x+1 *is* $\Omega(x)$

x^2+2x+1 *is* $\Omega(\sqrt{x})$

x^2+2x+1 *is* $\Omega(\log x)$

•
•
•

the <u>greatest</u> lower bound?

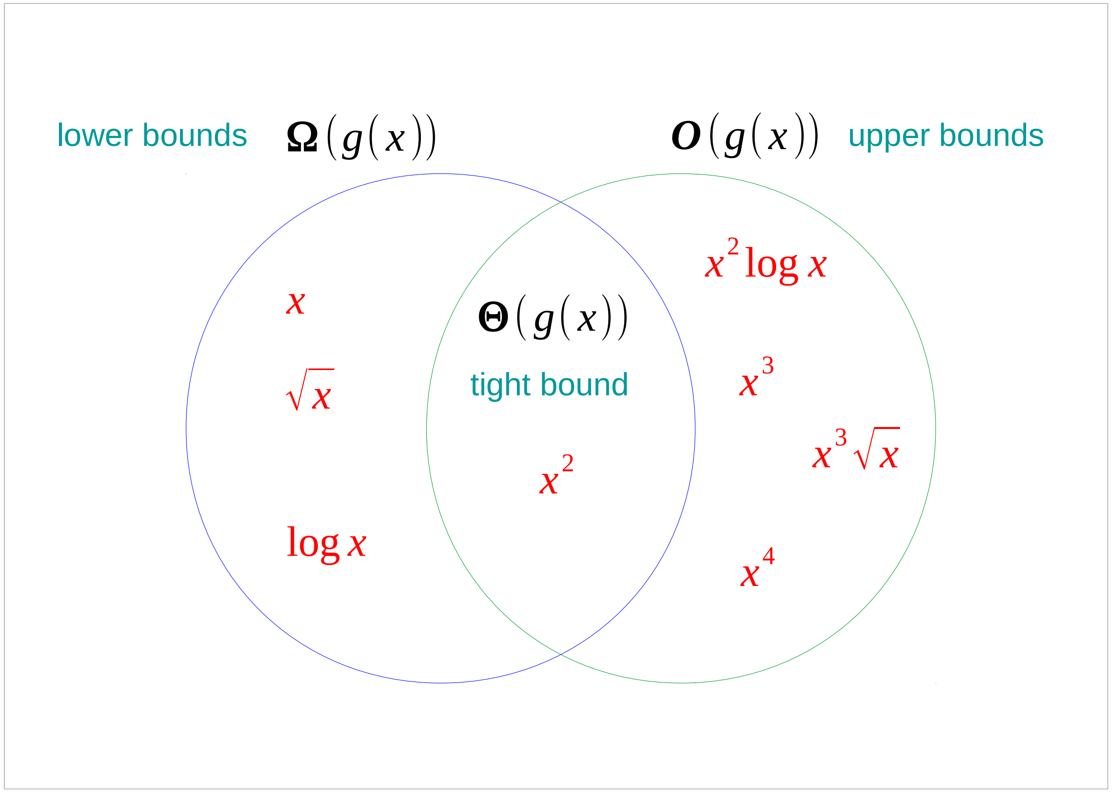
Upper and Lower Bounds

x^2+2x+1 *is* $O(x^2)$	⟺	$x^2+2x+1 \le C\,x^2$	upper bound	the least
x^2+2x+1 *is* $O(x^3)$	⟺	$x^2+2x+1 \le C\,x^3$	upper bound	
x^2+2x+1 *is* $O(x^4)$	⟺	$x^2+2x+1 \le C\,x^4$	upper bound	
⋮		⋮		

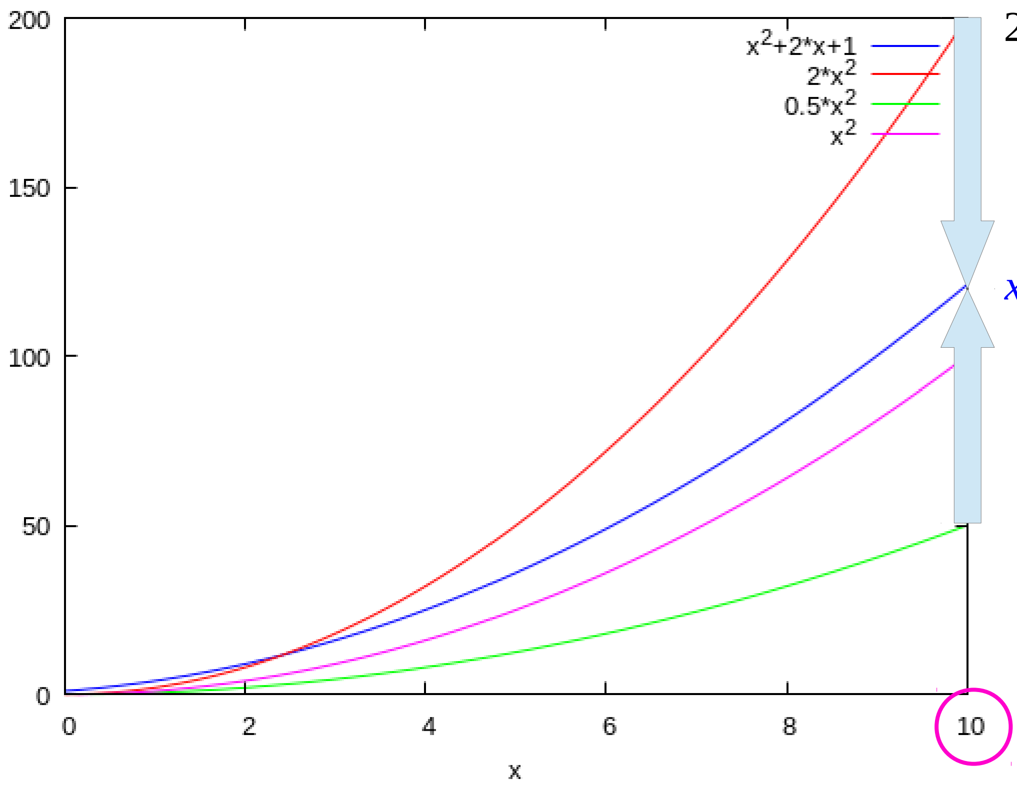
x^2+2x+1 *is* $O(x^2)$	⟺	$x^2+2x+1 \ge C\,x^2$	upper bound	the greatest
x^2+2x+1 *is* $\Omega(x)$	⟺	$x^2+2x+1 \ge C\,x$	lower bound	
x^2+2x+1 *is* $\Omega(\sqrt{x})$	⟺	$x^2+2x+1 \ge C\sqrt{x}$	lower bound	
x^2+2x+1 *is* $\Omega(\log x)$	⟺	$x^2+2x+1 \ge C\log x$	lower bound	
⋮		⋮		

Simultaneously Lower and Upper Bound

$$f(x) = x^2 + 2x + 1$$



Big-Θ Examples (1)



$2 \cdot x^2$

$x^2 + 2 * x + 1$

$1 \cdot x^2$

$0.5 \cdot x^2$

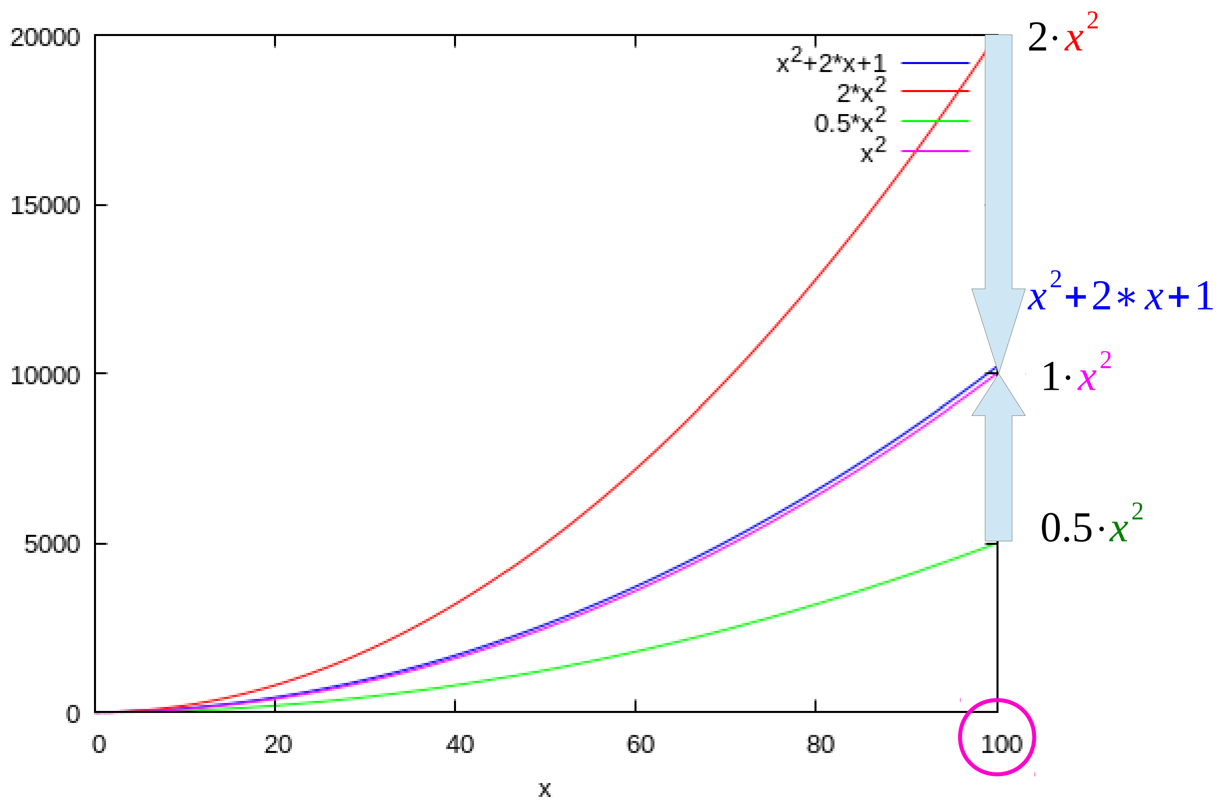
$x^2 + 2x + 1$ *is* $\boldsymbol{O}(x^2)$

$x^2 + 2x + 1$ *is* $\boldsymbol{\Theta}(x^2)$

$x^2 + 2x + 1$ *is* $\boldsymbol{\Omega}(x^2)$

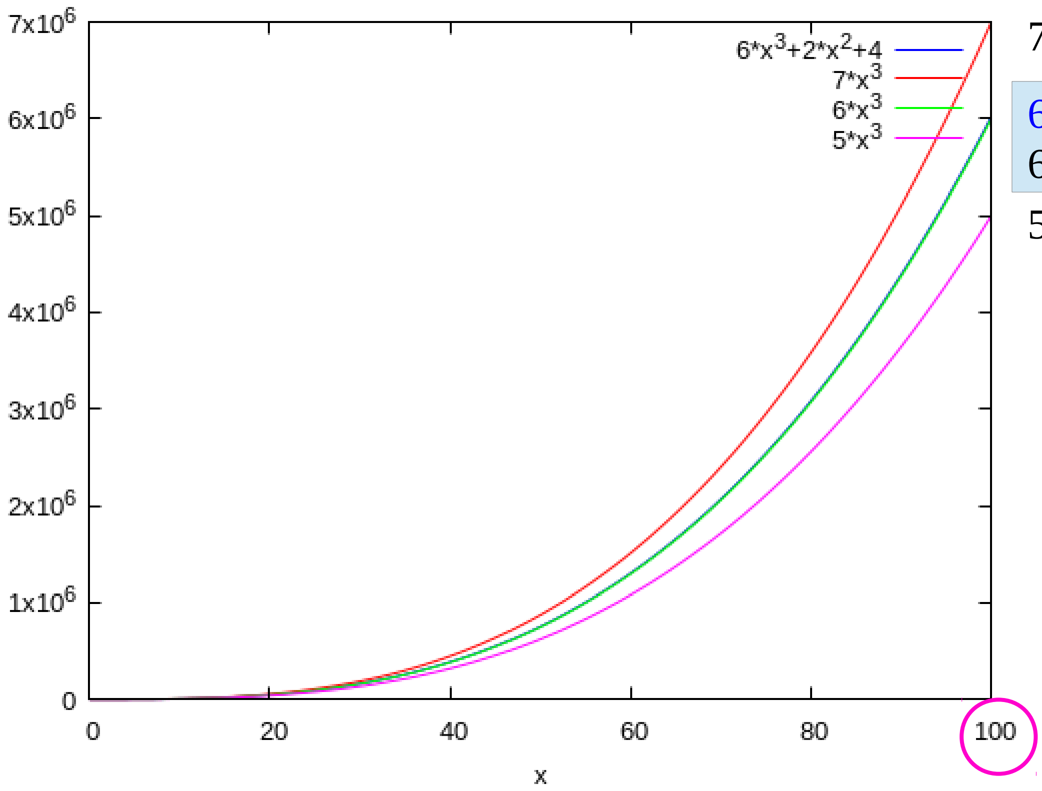
Zoom Out

Big-Θ Examples (2)



$x^2 + 2x + 1$ *is* $\Theta(x^2)$

Big-Θ Examples (3)



$7 \cdot x^3$

$6x^3 + 2x^2 + 4$
$6 \cdot x^3$

indistinguishable

$5 \cdot x^3$

$\mathbf{6}x^3 + 2x^2 + 4$ *is* $\Theta(x^3)$

$\mathbf{6}x^3 + 2x^2 + 4 \approx \mathbf{6} \cdot x^3$

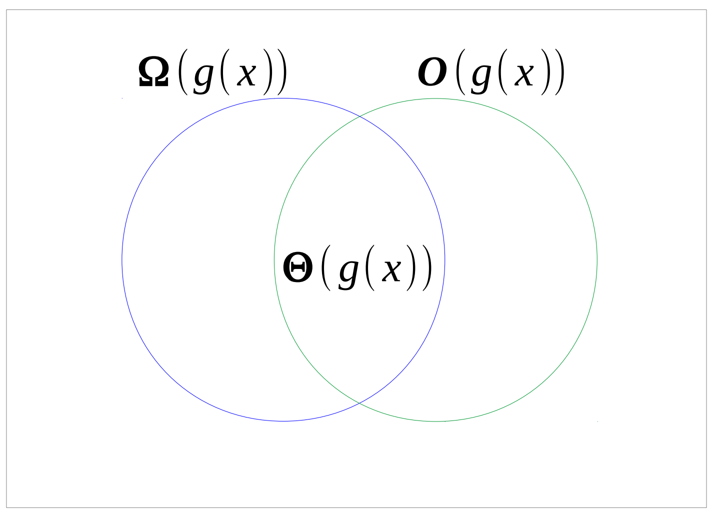
Tight bound Implications

$f(x)$ is $\Theta(g(x)) \Longrightarrow f(x)$ is $O(g(x))$

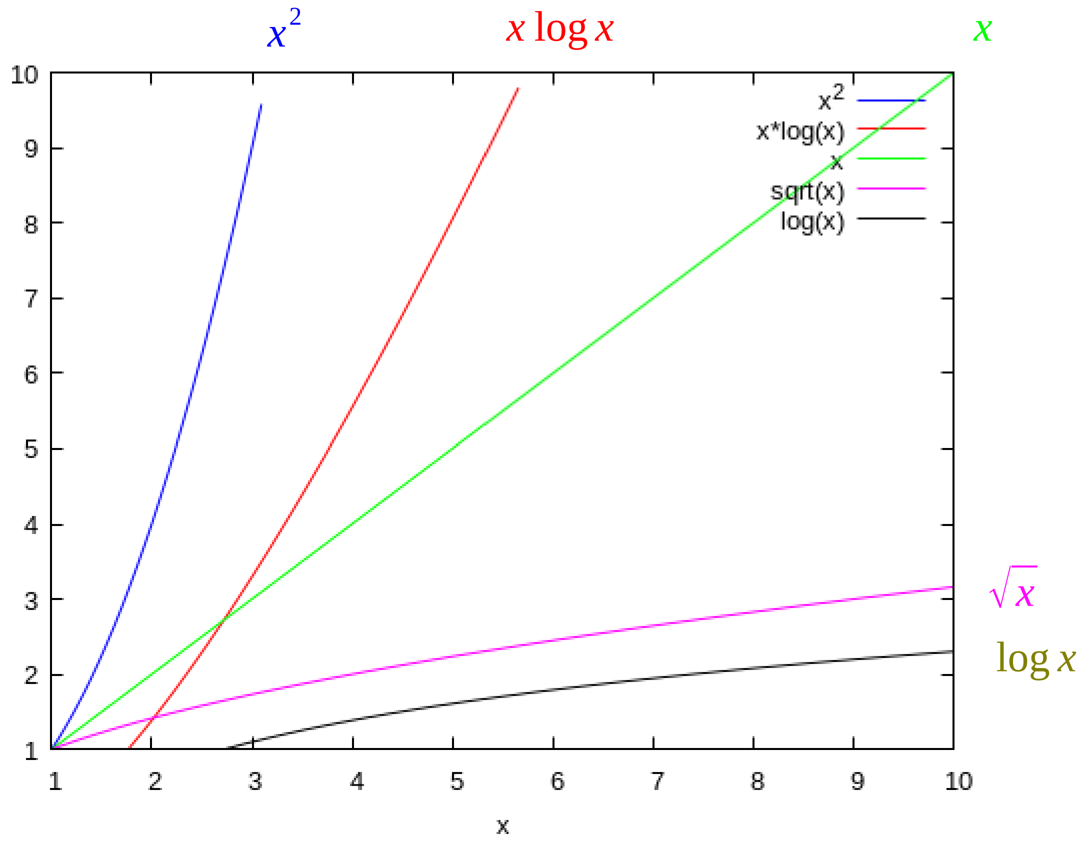
$f(x)$ is $\Theta(g(x)) \Longrightarrow f(x)$ is $\Omega(g(x))$

$f(x)$ is $\Theta(g(x)) \not\Longleftarrow f(x)$ is $O(g(x))$

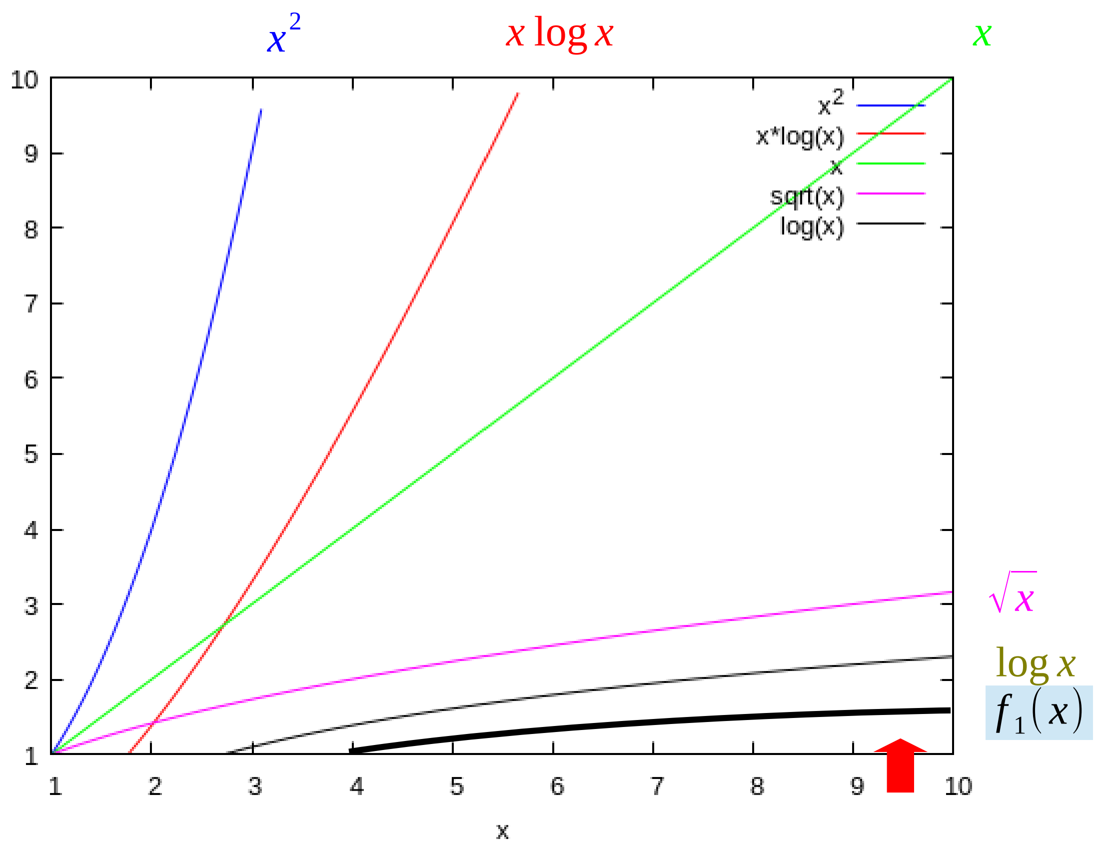
$f(x)$ is $\Theta(g(x)) \not\Longleftarrow f(x)$ is $\Omega(g(x))$



Common Growth Functions

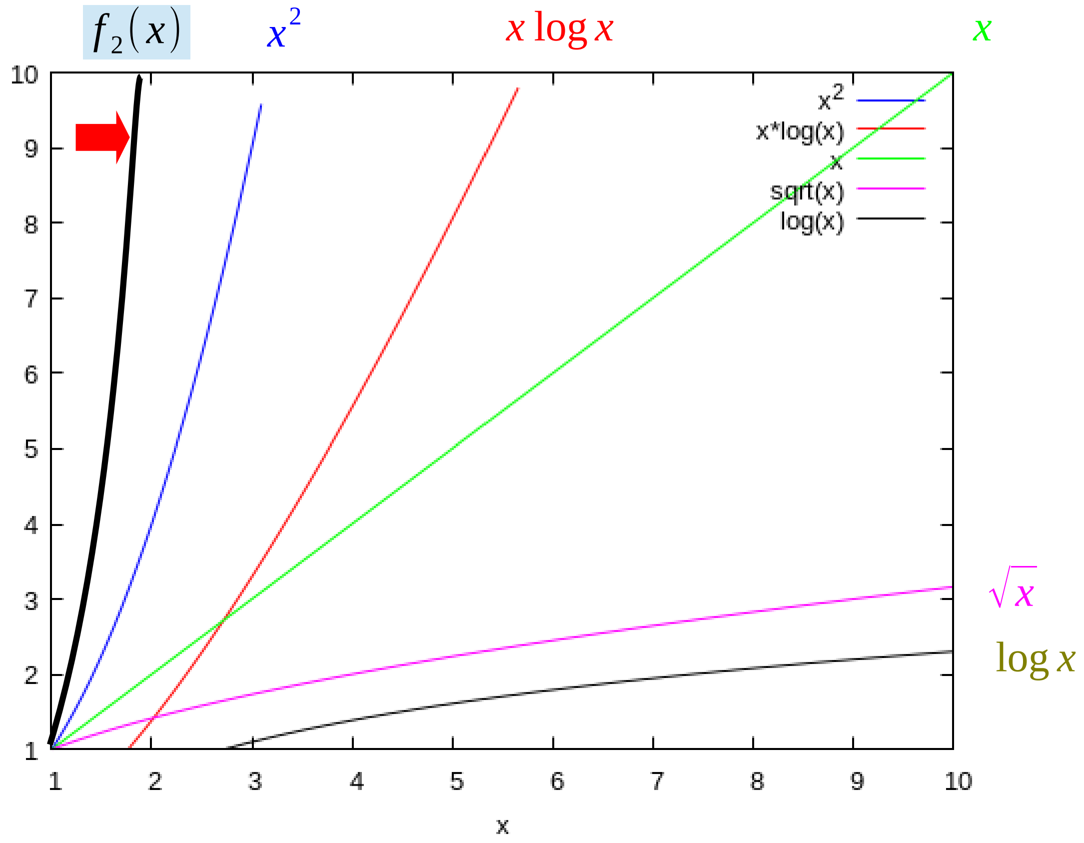


Upper bounds



$f_1(x)$ *is* $\boldsymbol{O}(\log x)$ ➡ $\boldsymbol{O}(\sqrt{x})$ ➡ $\boldsymbol{O}(x)$ ➡ $\boldsymbol{O}(x \log x)$ ➡ $\boldsymbol{O}(x^2)$

Lower bounds



$f_2(x)$ *is* $\mathbf{\Omega}(x^2)$ ➡ $\mathbf{\Omega}(x \log x)$ ➡ $\mathbf{\Omega}(x)$ ➡ $\mathbf{\Omega}(\sqrt{x})$ ➡ $\mathbf{\Omega}(\log x)$

Example 1

$f(n) = n^6 + 3n$ $\quad$ $f(n) = O(n^6)$ $\quad$ $f(n) = \Omega(n)$

$f(n) = 2^n + 12$ $\quad$ $f(n) = O(2^n)$ $\quad$ $f(n) = \Omega(1)$

$f(n) = 2^n + 3^n$ $\quad$ $f(n) = O(3^n)$ $\quad$ $f(n) = \Omega(2^n)$

$f(n) = n^n + n$ $\quad$ $f(n) = O(n^n)$ $\quad$ $f(n) = \Omega(n)$

https://discrete.gr/complexity/

Example 2

$f(n) = n^6 + 3n$	$f(n) = O(n^6)$	$f(n) = \Omega(n^6)$	$f(n) = \Theta(n^6)$
$f(n) = 2^n + 12$	$f(n) = O(2^n)$	$f(n) = \Omega(2^n)$	$f(n) = \Theta(2^n)$
$f(n) = 2^n + 3^n$	$f(n) = O(3^n)$	$f(n) = \Omega(3^n)$	$f(n) = \Theta(3^n)$
$f(n) = n^n + n$	$f(n) = O(n^n)$	$f(n) = \Omega(n^n)$	$f(n) = \Theta(n^n)$

https://discrete.gr/complexity/

Example 3

$$\Theta(n) \rightarrow O(n)$$

$$\Theta(n) \rightarrow O(n^2)$$

$$\Theta(n^2) \rightarrow O(n^3)$$

$$\Theta(n) \nrightarrow O(1)$$

$$O(1) \rightarrow \Theta(1)$$

$$O(n) \nrightarrow \Theta(1)$$

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Example 4

$\Theta(n)$	$O(n)$	upper bound	tight	
$\Theta(n^2)$	$O(n^3)$	upper bound		
$\Theta(1)$	$O(n)$	upper bound		
$\Theta(n)$	$O(1)$	upper bound	wrong	
$\Theta(n)$	$O(2n)$	upper bound	tight	$O(2n) = O(n)$

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Example 5

$\Theta(1)$	$O(1)$	tight upper bound	$O(n)$	upper bound
$\Theta(\sqrt{n})$	$O(\sqrt{n})$	tight upper bound	$O(n)$	upper bound
$\Theta(n)$	$O(n)$	tight upper bound	$O(n^2)$	upper bound
$\Theta(n^2)$	$O(n^2)$	tight upper bound	$O(n^3)$	upper bound
$\Theta(n^3)$	$O(n^3)$	tight upper bound	$O(n^4)$	upper bound

$\Theta(1)$	$\Omega(1)$	tight lower bound		
$\Theta(\sqrt{n})$	$\Omega(\sqrt{n})$	tight lower bound	$\Omega(1)$	lower bound
$\Theta(n)$	$\Omega(n)$	tight lower bound	$\Omega(\sqrt{n})$	lower bound
$\Theta(n^2)$	$\Omega(n^2)$	tight lower bound	$\Omega(n)$	lower bound
$\Theta(n^3)$	$\Omega(n^3)$	tight lower bound	$\Omega(n^2)$	lower bound

https://discrete.gr/complexity/

References

[1] http://en.wikipedia.org/
[2]