

# Monad P3 : Existential Types (1D)

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# Based on

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Haskell in 5 steps

[https://wiki.haskell.org/Haskell\\_in\\_5\\_steps](https://wiki.haskell.org/Haskell_in_5_steps)

# Three different usages for **forall**

Basically, there are 3 different common uses for the forall keyword (or at least so it seems), and each has its own Haskell extension:

## ScopedTypeVariables

specify types for code inside **where** clauses

## RankNTypes/Rank2Types,

The type is labeled "**Rank-N**" where N is the number of **forall**s which are nested and cannot be merged with a previous one.

## ExistentialQuantification

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

# Existential Quantification

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

# Hiding a type variable (1)

Normally when creating a new type using **type**, **newtype**, **data**, etc., every **type variable** that appears on the right-hand side must also appear on the left-hand side.

```
newtype ST s a = ST (State# s -> (# State# s, a #))
```

**Existential types** are a way of escaping

Existential types can be used for several different purposes. But what they do is to **hide a type variable** on the right-hand side.

[https://wiki.haskell.org/Existential\\_type](https://wiki.haskell.org/Existential_type)

# Hiding a type variable (2)

Normally, any type variable appearing on the right must also appear on the left:

```
data Worker x y = Worker {buffer :: b, input :: x, output :: y}
```

This is an **error**, since the **type b** of the **buffer** is not specified on the right (**b** is a **type variable** rather than a **type**) but also is not specified on the left (there's no **b** in the left part).

In Haskell98, you would have to write

```
data Worker b x y = Worker {buffer :: b, input :: x, output :: y}
```

[https://wiki.haskell.org/Existential\\_type](https://wiki.haskell.org/Existential_type)

# Hiding a type variable (3)

However, suppose that a **Worker** can use any type **b** so long as it belongs to some particular class.

Then every **function** that uses a **Worker** will have a type like

```
foo :: (Buffer b) => Worker b Int Int
```

In particular, failing to write an **explicit type signature** `(Buffer b)` will invoke the dreaded monomorphism restriction.

Using **existential types**, we can avoid this:

[https://wiki.haskell.org/Existential\\_type](https://wiki.haskell.org/Existential_type)



# Hiding a type variable (4)

Using existential type :

```
data Worker x y = forall b. Buffer b => Worker {buffer :: b, input :: x, output :: y}
foo :: Worker Int Int
```

The **type** of the **buffer** (**Buffer**) now does not appear in the **Worker** type at all.    **Worker x y**

Explicit type signature :

```
data Worker b x y = Worker {buffer :: b, input :: x, output :: y}
foo :: (Buffer b) => Worker b Int Int
```

[https://wiki.haskell.org/Existential\\_type](https://wiki.haskell.org/Existential_type)

# Hiding a type variable (5)

- it is now impossible for a function to demand a **Worker** having a specific type of **buffer**.
- the **type** of **foo** can now be derived automatically without needing an explicit type signature.  
(No monomorphism restriction.)
- since code now has no idea what **type** the buffer function returns, you are more limited in what you can do to it.

```
data Worker x y = forall b. Buffer b => Worker {buffer :: b, input :: x, output :: y}
foo :: Worker Int Int
```

[https://wiki.haskell.org/Existential\\_type](https://wiki.haskell.org/Existential_type)

# Hiding a type variable (6)

In general, when you use a **hidden type** in this way, you will usually want that **type** to belong to a **specific class**, or you will want to **pass some functions** along that can work on that type.

Otherwise you'll have some value belonging to a **random unknown type**, and you won't be able to do anything to it!

[https://wiki.haskell.org/Existential\\_type](https://wiki.haskell.org/Existential_type)

# Less specific types (1)

Note: You can use **existential types** to **convert** a **more specific type** into a **less specific one**.

**constrained type variables**

There is no way to perform the reverse conversion!

[https://wiki.haskell.org/Existential\\_type](https://wiki.haskell.org/Existential_type)

## Less specific types (2)

This illustrates **creating a heterogeneous list**,  
all of whose members implement "**Show**",  
and progressing through that list to show these items:

```
data Obj = forall a. (Show a) => Obj a
```

```
xs :: [Obj]
```

```
xs = [Obj 1, Obj "foo", Obj 'c']
```

```
doShow :: [Obj] -> String
```

```
doShow [] = ""
```

```
doShow ((Obj x):xs) = show x ++ doShow xs
```

With output: `doShow xs ==> "1\"foo\"'c"`

[https://wiki.haskell.org/Existential\\_type](https://wiki.haskell.org/Existential_type)

# Existentials in terms of forall (1)

It is also possible to express existentials with RankNTypes as **type expressions** directly (without a **data** declaration)

```
forall r. (forall a. Show a => a -> r) -> r
```

(the leading forall r. is optional unless the expression is part of another expression).

the equivalent type **Obj** :

```
data Obj = forall a. (Show a) => Obj a
```

[https://wiki.haskell.org/Existential\\_type](https://wiki.haskell.org/Existential_type)

# Existentials in terms of forall (2)

The conversions are:

**fromObj :: Obj -> forall r. (forall a. Show a => a -> r) -> r**

**fromObj (Obj x) k = k x**

**toObj :: (forall r. (forall a. Show a => a -> r) -> r) -> Obj**

**toObj f = f Obj**

[https://wiki.haskell.org/Existential\\_type](https://wiki.haskell.org/Existential_type)

# Existentials

**Existential types**, or '**existentials**' for short, provide a way of 'squashing' a group of types into one, single type.

**Existentials** are part of GHC's type system **extensions**.

They aren't part of Haskell98, and as such you'll have

to either compile any code that contains them

with an extra command-line parameter of

**`-XExistentialQuantification`**,

or put at the top of your sources that use existentials.

**`{-# LANGUAGE ExistentialQuantification #-}`**

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)



# forall type variables

Example: A polymorphic function

```
map :: (a -> b) -> [a] -> [b]
```

Example: Explicitly quantifying the type variables

```
map :: forall a b. (a -> b) -> [a] -> [b]
```

instantiating the general type of map to a more specific type

```
a = Int and b = String
```

```
(Int -> String) -> [Int] -> [String]
```

Example: Two equivalent type statements

```
id :: a -> a
```

```
id :: forall a . a -> a
```

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# Heterogeneous Lists

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

# Type hider

Suppose we have a group of values.

they may not be all the same **type**,

but they are all **members** of some **class**

thus, they have a certain **property**

It might be useful to throw all these **values** into a **list**.

normally this is impossible because **lists elements**

must be of **the same type**

(**homogeneous** with respect to **types**).

**existential types** allow us to loosen this requirement

by defining a **type hider** or **type box**:

```
data ShowBox = forall s. Show s => SB s
```

```
heteroList :: [ShowBox]
```

```
heteroList = [SB (), SB 5, SB True]
```

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# Heterogeneous list example (1)

```
data ShowBox = forall s. Show s => SB s
```

```
-- type hider
```

```
heteroList :: [ShowBox]
```

```
heteroList = [SB (), SB 5, SB True]
```

[SB (), SB 5, SB True] calls the **constructor** on three values of different types, to place them all into a single list virtually **the same type** for each one.

Use the **forall** in the constructor

```
SB :: forall s. Show s => s -> ShowBox.
```

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# Heterogeneous list example (2)

```
data ShowBox = forall s. Show s => SB s
heteroList :: [ShowBox]
heteroList = [SB (), SB 5, SB True]
```

When passing **heteroList** type parameters to a function  
we cannot take out the **values** inside the **SB**  
because their type might **Bool**, **Int**, **Char**, ...

**But each of the elements can be**  
converted to a **string** via **show**.

In fact, that's the only thing we know about them.

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# Heterogeneous list example (3)

```
instance Show ShowBox where
```

```
  show (SB s) = show s
```

```
f :: [ShowBox] -> IO ()
```

```
f xs = mapM_ print xs
```

```
main = f heteroList
```

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# Heterogeneous list example (4)

Example: Using our heterogeneous list

```
instance Show ShowBox where
  show (SB s) = show s      -- (*) see the comment in the text below
f :: [ShowBox] -> IO ()
f xs = mapM_ print xs
main = f heteroList
```

Example: Types of the functions involved

```
print :: Show s => s -> IO ()      -- print x = putStrLn (show x)
mapM_ :: (a -> m b) -> [a] -> m ()
mapM_ print :: Show s => [s] -> IO ()
```

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# mapM, mapM\_, and map (1)

The core idea is that **mapM** maps an "action" (ie function of type **a -> m b**) over a list and gives you **all** the results as **m [b]**

**mapM\_** does the same thing, but never collects the results, returning a **m ()**.

If you care about the **results** of your **a -> m b** function, use **mapM**.  
If you only care about the **effect**, but not the resulting value, use **mapM\_**, because it can be more **efficient**

<https://stackoverflow.com/questions/27609062/what-is-the-difference-between-mapm-and-mapm-in-haskell/27609146>



# mapM, mapM\_, and map (2)

Always use **mapM\_** with functions of the type **a -> m ()**,  
like **print** or **putStrLn**.  
these functions return **()** to signify that only the **effect** matters.

If you used **mapM**, you'd get a **list of ()** (ie **[(), (), ()]**),  
which would be completely useless  
but waste some memory.

If you use **mapM\_**, you would just get a **()**,  
but it would still print everything.

<https://stackoverflow.com/questions/27609062/what-is-the-difference-between-mapm-and-mapm-in-haskell/27609146>

# mapM, mapM\_, and map (3)

Normal **map** is something different:

it takes a normal function (**a -> b**)

instead of one using a monad (**a -> m b**).

This means that it cannot have any sort of **effect**

besides returning the **changed list**.

You would use it if you want to **transform a list**

using a normal function.

**map\_** doesn't exist because, since you don't have any effects,

you always care about the **results** of using **map**.

<https://stackoverflow.com/questions/27609062/what-is-the-difference-between-mapm-and-mapm-in-haskell/27609146>

forall

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

# A set of possible values

One way to think about **forall** is to consider **types** as a set of possible values.

**Bool** is the set **{True, False, ⊥}**  
(remember that **bottom**, **⊥**, is a member of every type!),

**Integer** is the set of integers (and bottom),

**String** is the set of all possible strings (and bottom), and so on.

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# Intersection of the specified types

**forall** serves as a way to assert a **commonality** or **intersection** of the specified types (i.e. sets of values).

**forall a. a** is the **intersection** of all types.

this **subset** turns out to be the set  $\{\perp\}$ ,

since it is an **implicit value in every type**.

that is, [the **type** whose **only available value is bottom**]

However, since **every Haskell type includes bottom,  $\{\perp\}$** ,  
this quantification in fact stipulates all Haskell types.

But the only permissible operations on it are

those available to [a **type** whose **only available value is bottom**]

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# A list of bottoms type (1)

1. The list `[forall a. a]`
2. The list `[forall a. Show a => a]`
3. The list `[forall a. Num a => a]`
4. The list `forall a. [a]`

**a list of bottoms.**     `[⊥]` , `[⊥,⊥]` , ...

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# A list of bottoms type (2)

The list, `[forall a. a]`, is the **type of a list**  
whose **elements** all have the type `forall a. a`, i.e.  
**a list of bottoms.** `[⊥]` , `[⊥,⊥]` , ...

The list, `[forall a. Show a => a]`, is the **type of a list**  
whose **elements** all have the type `forall a. Show a => a`.

the **Show** class constraint requires the possible types  
also to be **a member of the class, Show.**

However, `⊥` is still the only value common to all these types, `{⊥}`,  
so this too is **a list of bottoms.** `[forall a. a]`

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# A list of bottoms type (3)

The list, `[forall a. Num a => a]`, requires each element to be a member of the class, Num.

Consequently, the possible values include **numeric literals**, which have the specific type, `forall a. Num a => a`, as well as **bottom**.

`forall a. [a]` is the type of **the list** whose elements all have the same type **a**.

since we cannot presume any particular type at all, this too is **a list of bottoms**.

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)



# Intersections over types

most **intersections over types** just lead to **bottoms**  $\perp\perp\perp\perp$   
**types** generally don't have **any values in common**  
presumptions cannot be made about a **union of their values**.

a **heterogeneous list** using a **type hider**  
**type hider'** functions as a **wrapper type**  
which guarantees certain facilities  
by implying a **predicate** or **constraint** on the permissible **types**.

the purpose of **forall** is to **impose type constraint**  
on the permissible types within a **type declaration**  
guaranteeing certain facilities with such types.

```
data ShowBox = forall s. Show s => SB s
heteroList :: [ShowBox]
heteroList = [SB (), SB 5, SB True]
```

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# Summary of heterogeneous list examples (1)

An **existential datatype**

```
data T = forall a. MkT a
```

This defines a **polymorphic constructor**,  
or a family of constructors for **T**

```
MkT :: forall a. (a -> T)
```

**Pattern matching** on our existential constructor

```
foo (MkT x) = ... -- what is the type of x?
```

Constructing the **heterogeneous list**

```
heteroList = [MkT 5, MkT (), MkT True, MkT map]
```

```
data ShowBox = forall s. Show s => SB s
```

```
heteroList :: [ShowBox]
```

```
heteroList = [SB (), SB 5, SB True]
```

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# Summary of heterogeneous list examples (2)

A new existential data type, with a **class constraint**

```
data T' = forall a. Show a => MkT' a
```

```
data T = forall a. MkT a
```

Using our new heterogeneous setup

```
heteroList' = [MkT' 5, MkT' (), MkT' True, MkT' "Sartre"]
```

```
main = mapM_ (\(MkT' x) -> print x) heteroList'
```

```
{- prints:
```

```
5
```

```
()
```

```
True
```

```
"Sartre"
```

```
-}
```

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

runST

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

# State and ST monads

the **ST** monad is essentially

a more powerful version of the **State** monad:

It was originally written to provide Haskell with **IO**.

**IO** is basically just a **State** monad

with an environment of all the information about the real world.

In fact, inside GHC at least, **ST** is used,

and the environment is a **type** called **RealWorld**.

To get out of the **State** / **ST** monad,

use **runState** / **runST**

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# runST – rank-2 polymorphism

```
runST :: forall a. (forall s. ST s a) -> a
```

This is actually an example of **rank-2 polymorphism**

a **forall** appearing within the **left-hand side** of (->)  
cannot be moved up, and therefore forms **another level or rank**  
therefore, there are **2 levels** of universal quantification.

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# runST – initial state

```
runST :: forall a. (forall s. ST s a) -> a
```

there is **no parameter** for the **initial state** ... **s**

Indeed, **ST** uses a different notion of state to **State**;

**State** allows you to **get** and **put** the *current state*,

**ST** provides an **interface to references**

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# runST – reference interfaces

To create references of the type **STRef**

**newSTRef** :: a -> ST s (STRef s a)

To provide an **initial value**

**readSTRef** :: STRef s a -> ST s a

To manipulate them.

**writeSTRef** :: STRef s a -> a -> ST s ()

**runST** :: forall a. (forall s. ST s a) -> a

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)



# runST – a mapping

the **internal environment** of a **ST** computation

is not one specific value,

but a **mapping** from **references** to **values**.

... (STRef s a)

**newSTRef** :: a -> ST s (STRef s a)

No need to provide an **initial state** to **runST**,

as the **initial state** is just the **empty mapping**

... ()

containing **no references**.

**runST** :: forall a. (forall s. ST s a) -> a

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# runST – no specific references

It is not allowed

to create a **reference** in one **ST computation**,  
then to use the created **reference** in another **ST computation**.  
for reasons of **thread-safety**

because no ST computation should be allowed  
to assume that the **initial internal environment**  
contains any specific references.

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# runST

```
runST :: forall a. (forall s. ST s a) -> a
newSTRef :: a -> ST s (STRef s a)
readSTRef :: STRef s a -> ST s a
```

Example: Bad ST code

```
let v = runST (newSTRef True) ... one ST computation
in runST (readSTRef v) ... another ST computation
```

Example: Briefer bad ST code

```
... runST (newSTRef True) ...
```

```
newSTRef True :: ST s (STRef s a)
runST (newSTRef True) :: STRef s a
v :: STRef s a
```

```
readSTRef v :: ST s a
runST (readSTRef v) :: a
```

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# runST

Example: Bad ST code

```
let v = runST (newSTRef True)
in runST (readSTRef v)
```

**runST** :: forall a. (forall s. ST s a) -> a

the **rank-2 polymorphism** in **runST**'s type  
to constrain the scope of the **type variable s**  
to be within the first parameter (the left hand side of ->)

if the **type variable s** appears in the first parameter  
it cannot also appear in the second.  
(the right hand side of ->)

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# runST

Example: Briefer bad ST code

```
... runST (newSTRef True) ...
```

Example: The compiler's typechecking stage

```
newSTRef True :: forall s. ST s (STRef s Bool)
```

```
runST :: forall a. (forall s. ST s a) -> a
```

```
runST (newSTRef True) ::
```

```
(forall s. ST s (STRef s Bool)) -> STRef s Bool
```

```
runST :: forall a. (forall s. ST s a) -> a
```

```
newSTRef :: a -> ST s (STRef s a)
```

```
readSTRef :: STRef s a -> ST s a
```

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# forall

The importance of the forall in the first bracket is that we can change the name of the s.

`runST (newSTRef True) ::`

`(forall s. ST s (STRef s Bool)) -> STRef s Bool`

Example: A type mismatch!

`(forall s'. ST s' (STRef s' Bool)) -> STRef s Bool`

This is similar to  $\forall x . x > 5$  is precisely the same as  $\forall y . y > 5$  giving the variable a different label.

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# forall

Example: A type mismatch!

```
(forall s'. ST s' (STRef s' Bool)) -> STRef s Bool
```

Notice that as the `forall` does not scope over the return type of `runST`, `STRef s Bool` we don't rename the `s` there as well.

But suddenly, we've got a **type mismatch!**

The result type of the ST computation in the **first parameter** must match the **result type** of `runST`, but now it doesn't!

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# forall

```
(forall s'. ST s' (STRef s' Bool)) -> STRef s Bool
```

The key feature of the **existential** is that it allows the compiler to **generalise** the **type** of the **state** in the **first parameter**, and so the **result type** cannot depend on it.

This neatly sidesteps our **dependence problems**, 'compartmentalises' each call to **runST** into **its own little heap**, with **references** not being able to be shared between different **calls**.

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)



# Quantified types as products and sums

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

# Quantified Types as Products and Sums

A **universally** quantified type may be interpreted as an **infinite product** of types.

a **polymorphic function** can be understood as a **product**, or a **tuple**, of **individual functions**, one per every possible **type a**.

To construct a **value** of such **type**, we have to provide all the **components** of the **tuple** at once.

-- one formula generating an **infinity** of functions

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# Quantified Types as Products and Sums

Example: Identity function

**id :: forall a. a -> a**

**id a = a**

a **polymorphic function** can be understood  
as a **product**, or a **tuple**, of **individual functions**,  
one per every possible **type a**.

**Int -> Int, Double -> Double, ...**

**Char -> Char, [Char] -> [Char], ...**

...

...

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# Quantified Types as Products and Sums

To construct a **value** of such **type**, we have  
to provide all the **components** of the **tuple** at once.

in case of **numeric types**, one **numeric constant**  
may be used to initialize **many types** at once.

Example: Polymorphic value

```
x :: forall a. Num a => a
```

```
x = 0
```

x may be conceptualized as a **tuple** consisting  
of an **Int value**, a **Double value**, etc.

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# Quantified Types as Products and Sums

Similarly, an **existentially quantified type** may be interpreted as an **infinite sum**.

Example: Existential type

```
data ShowBox = forall s. Show s => SB s
```

may be conceptualized as a **sum**:

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# Quantified Types as Products and Sums

Example: Existential type

```
data ShowBox = forall s. Show s => SB s
```

Example: Sum type

```
data ShowBox = SBUnit | SBInt Int | SBBool Bool | SBIntList [Int] | ...
```

to construct a **value** of this **type**,  
we only have to pick one of the constructors.

A **polymorphic constructor SB**

combines all those constructors into one.

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# Quantification as a primitive

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

# Pair type example (1)

**Universal quantification** is useful

for defining data types that aren't already defined.

Suppose there was no such thing as **pairs** built into haskell.

**Quantification** could be used to define them.

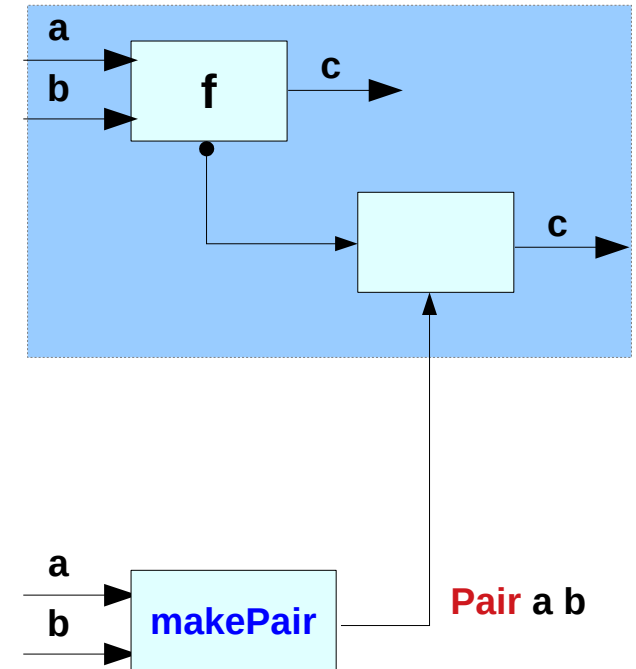
```
{-# LANGUAGE ExistentialQuantification, RankNTypes #-}
```

```
newtype Pair a b = Pair (forall c. (a -> b -> c) -> c)
```

```
makePair :: a -> b -> Pair a b
```

```
makePair a b = Pair $ \f -> f a b
```

**Pair** \$ \f -> f a b :: **Pair** a b



[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)



# Pair type example (2)

In GHCi

```
λ> :set -XExistentialQuantification
```

```
λ> :set -XrankNTypes
```

```
λ> newtype Pair a b = Pair {runPair :: forall c. (a -> b -> c) -> c}
```

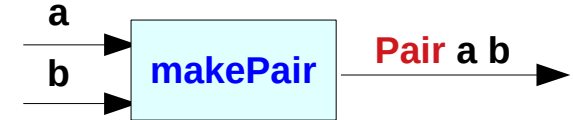
```
λ> makePair a b = Pair $ \f -> f a b
```

```
λ> pair = makePair "a" 'b'
```

```
Pair $ \f -> f "a" 'b'
```

$\backslash f$  : function itself      $f$  ::  $a \rightarrow b \rightarrow c$

$f$  "a" 'b' : function application result



[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# Pair type example (3)

```
λ> newtype Pair a b = Pair {runPair :: forall c. (a -> b -> c) -> c}
```

```
Pair {runPair}
```

```
Pair {forall c. (a -> b -> c) -> c}
```

```
pair :: Pair [Char] Char
```

```
pair = makePair "a" 'b'
```

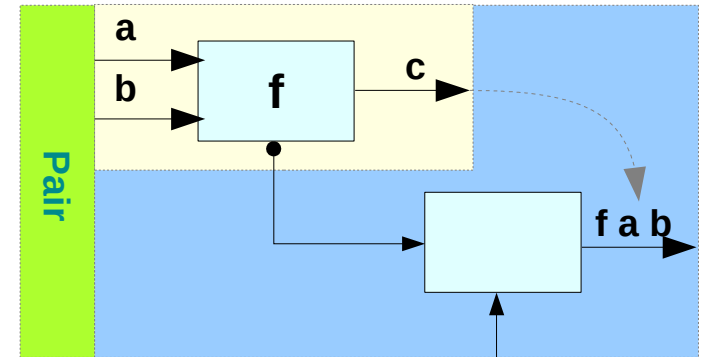
```
Pair $ \f -> f "a" 'b'
```

```
pair (λx y -> x) = Pair $ \f -> f "a" 'b' (λx y -> x)
```

```
runPair pair (λx y -> x)
```

```
runPair makePair "a" 'b' (λx y -> x)
```

```
Pair $ \f -> f a b :: Pair a b
```



```
pair (λx y -> x)
```

```
makePair "a" 'b' (λx y -> x)
```

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# Pair type example (4)

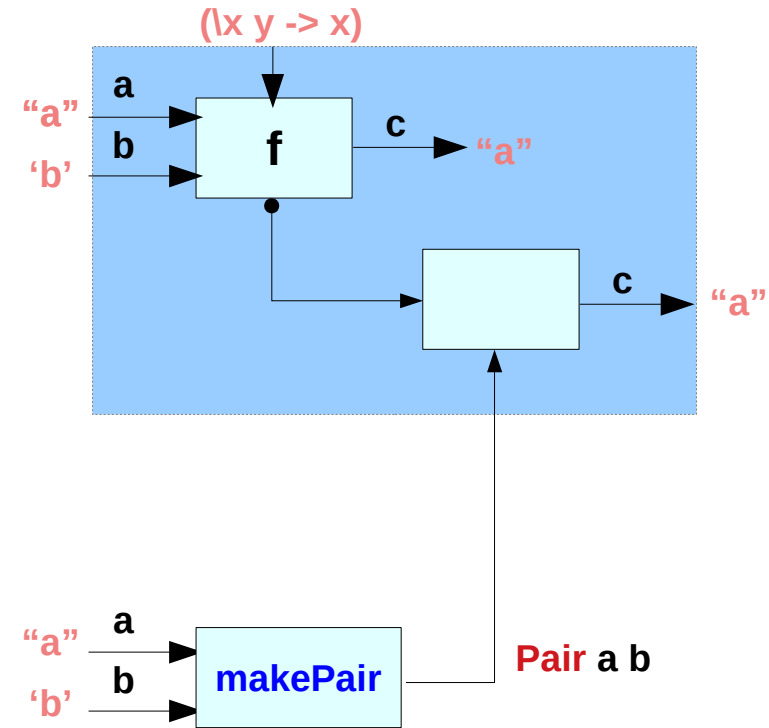
```
runPair pair (λx y -> x)
```

```
runPair makePair "a" 'b' (λx y -> x)
```

```
runPair Pair $ λf -> f "a" 'b' (λx y -> x)
```

"a"

**Pair** \$ λf -> f a b :: **Pair** a b



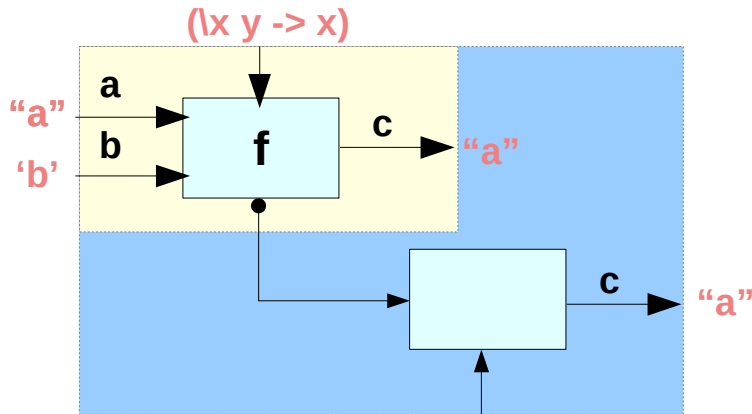
```
pair (λx y -> x)
```

```
makePair "a" 'b' (λx y -> x)
```

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# Pair type example (5)

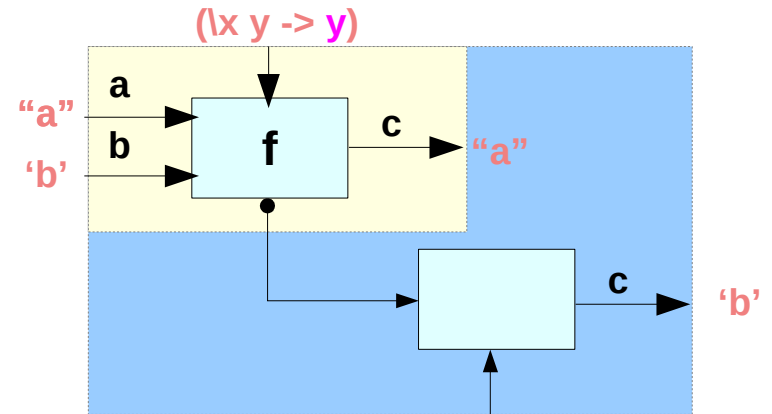
Pair \$ \!f \to f a b \!:: \text{Pair } a b



pair  $(\lambda x y \to x)$

makePair "a" 'b'  $(\lambda x y \to x)$

Pair \$ \!f \to f a b \!:: \text{Pair } a b



pair  $(\lambda x y \to y)$

makePair "a" 'b'  $(\lambda x y \to y)$

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# Pair type example (6)

```
λ> newtype Pair a b = Pair {runPair :: forall c. (a -> b -> c) -> c}
```

```
λ> makePair a b = Pair $ \f -> f a b
```

```
λ> pair = makePair "a" 'b'
```

```
λ> :t pair
```

```
pair :: Pair [Char] Char
```

```
λ> runPair pair (λx y -> x)           pair
```

```
"a"
```

```
λ> runPair pair (λx y -> y)
```

```
'b'
```

```
pair
```

```
makePair "a" 'b'
```

```
Pair $ \f -> f "a" 'b'
```

```
pair (λx y -> x)
```

```
Pair $ \f -> f "a" 'b'
```

```
f = (λx y -> x)
```

```
f "a" 'b' = "a"
```

```
Pair $ (λx y -> x) -> "a"
```

```
runPair :: forall c. (a -> b -> c) -> c
```

[https://en.wikibooks.org/wiki/Haskell/Existentially\\_quantified\\_types](https://en.wikibooks.org/wiki/Haskell/Existentially_quantified_types)

# forall – quantifier (1)

quantifier in predicate calculus

type quantifier polymorphic types

to encode a type in **type isomorphism**

**Isomorphism**

from . to = id

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

# forall – quantifier (2) type isomorphism

the class of **isomorphic types**, i.e. those which can be **cast** to each other without loss of information.

**type isomorphism** is an **equivalence relation** (**reflexive, symmetric, transitive**), but due to the limitations of the type system, only **reflexivity** is implemented for all types

**Isomorphism**

from . to = id

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

## forall – quantifier (3)

```
foo :: (forall a. a -> a) -> (Char, Bool)
bar :: forall a. ((a -> a) -> (Char, Bool))
```

To call **bar**, any **type a** can be chosen,  
and it is possible to pass a **function** from **type a** to **type a**.

the **function (+1)** or the **function reverse**.

the **forall** is considered to be as saying

"I get to pick the type now". (**instantiating**.)

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>



# forall – quantifier (4)

```
foo :: (forall a. a -> a) -> (Char, Bool)
```

```
bar :: forall a. ((a -> a) -> (Char, Bool))
```

The restrictions on calling **foo** are much more stringent:  
the argument to **foo** must be a **polymorphic function**.

With that type, **the only functions** that can be passed to **foo**  
are **id** or a **function** that always **diverges** or **errors**, like **undefined**.

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

# forall – quantifier (5)

```
foo :: (forall a. a -> a) -> (Char, Bool)
```

```
bar :: forall a. ((a -> a) -> (Char, Bool))
```

The reason is that with **foo**, the **forall** is to the **left of the arrow**, so as the **caller** of **foo** I don't get to pick what **a** is —rather it's the **implementation** of **foo** that gets to pick what **a** is.

Because **forall** is to the **left of the arrow**, rather than **above the arrow** as in **bar**, the **instantiation** takes place in the **body** of the **function** rather than at the **call** site.

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

# forall – quantifier (6) above, below, left

Jargon "**above**", "**below**", "**to the left of**".

nothing to do with the *textual ways* types are written  
everything to do with **abstract-syntax trees**.

In the **abstract syntax**,

- a **forall** takes the **name** of a **type variable**,  
and then there is a **full type** "**below**" the **forall**.
- an **arrow** takes **two types** (**argument** and **result type**)  
and forms a **new type** (the **function type**).
- the **argument type** is "**to the left of**" the **arrow**;
- it is the **arrow's left child** in the **abstract-syntax tree**.

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

# forall – quantifier (7)

**forall** a . [a] -> [a],

the **forall** is **above the arrow**;

what's to the **left of the arrow** is [a].

**forall** n f e x . (**forall** e x . n e x -> f -> Fact x f)

-> Block n e x -> f -> Fact x f

(**forall** e x . n e x -> f -> Fact x f)

the type in parentheses would be called

"a **forall** to the **left of an arrow**".

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

# foo :: a -> a (1)

**foo :: a -> a**

given this type signature, there is only one function  
that can satisfy this type and  
the identity function **id**.

**foo 5 = 6**

**foo True = False**

they both satisfy the above type signature,  
then why do Haskell folks claim  
that it is **id** alone which satisfies the type signature?

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

## foo :: a -> a (2)

That is because there is an implicit forall hidden in the type signature.

`id :: forall a. a -> a`

*Constraints liberate, liberties constrain*

A **constraint** at the **type level**,  
becomes a **liberty** at the **term level**

A **liberty** at the **type level**,  
becomes a **constraint** at the **term level**

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

## foo :: a -> a (3)

A **constraint** at the **type level**..

So putting a constraint on our type signature

```
foo :: (Num a) => a -> a
```

becomes a **liberty** at the term level gives us the liberty or flexibility to write all of these

```
foo 5 = 6
```

```
foo 4 = 2
```

```
foo 7 = 9
```

...

Same can be observed by constraining a with **any other typeclass** etc

A **constraint** at the **type level**,  
becomes a **liberty** at the **term level**

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

## foo :: a -> a (4)

foo :: (Num a) => a -> a translates to  
 $\exists a, \text{st } a \rightarrow a, \forall a \in \text{Num}$

### existential quantification

which translates to there exists some instances of **a**  
for which a function of **a -> a**  
and those instances all belong to the set of **Numbers**.

adding a **constraint** (**a** should belong to the set of **Nnumbers**),  
**liberates** the **term** level to have multiple possible implementations.

A **constraint** at the **type level**,  
becomes a **liberty** at the **term level**

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>



# foo :: a -> a (5)

the explanation of **forall**:

So now let us **liberate** the the **function** at the **type** level:

**foo :: forall a. a -> a** translates to:

**$\forall a, a -> a$**

the **implementation** of this type signature

should be such that it is **a -> a** for all circumstances.

A **liberty** at the **type level**, becomes  
a **constraint** at the **term level**

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

## foo :: a -> a (6)

So now this starts **constraining** us at the **term** level.

We can no longer write

**foo 5 = 7**

because this **implementation** would not satisfy  
when a **Bool** type value is passed to **foo**

this is because

**under all circumstances**  $\forall a, a \rightarrow a$

it should return something of the similar type.

**a** can be a **Char** or a **[Char]** or a custom datatype.

A **liberty** at the **type level**, becomes  
a **constraint** at the **term level**

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

# foo :: a -> a (7)

$\forall a, a \rightarrow a$  the **liberty** at the **type** level  
foo 5 = 7 a constraint at the **term** level  
(impossible implementation)

this **liberty** at the **type** level is what is known  
as **Universal Quantification**

the **only** **function** which can satisfy **foo :: forall a. a -> a**

foo a = a the **identity** function

A **liberty** at the **type** level, becomes  
a **constraint** at the **term** level

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

# foo :: a -> a (8)

*Runar Bjarnason titled "Constraints Liberate, Liberties Constrain".*

CONSTRAINTS LIBERATE,  
LIBERTIES CONSTRAIN

Its very important to digest and believe this statement

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

# RunST (1)

```
runST :: forall a. (forall s. ST s a) -> a
```

**runST** has to be able to produce a **value** of **type a**,  
no matter what **type** we give as **a**.

**runST** uses an **argument** of **type (forall s. ST s a)**  
which certainly must somehow produce the **a**.

**runST** must be able to produce a **value** of **type a**  
no matter what **type** the **implementation** of **runST**  
decides to give as **s**.

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

# RunST (2)

```
runST :: forall a. (forall s. ST s a) -> a
```

the benefit is that this puts a **constraint** on the **caller** of **runST** in that the **type a** cannot involve the **type s** at all.

you can't pass it a value of type **ST s [s]**, for example.

the implementation of **runST** is **free** to perform **mutation** with the value of **type s**.

The **type guarantees** that this **mutation** is **local** to the implementation of **runST**.

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

# RunST : rank-2 polymorphic type

```
runST :: forall a. (forall s. ST s a) -> a
```

The **type** of **runST** is an example of a **rank-2 polymorphic type** because the **type** of its **argument** contains a **forall** quantifier.

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

# Existential Quantification

```
-- test.hs
{-# LANGUAGE ExistentialQuantification #-}
data EQList = forall a. EQList [a]
eqListLen :: EQList -> Int
eqListLen (EQList x) = length x

ghci> :l test.hs
ghci> eqListLen $ EQList ["Hello", "World"]
2
```

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>



# Existential Quantification

```
ghci> :set -XRankNTypes
ghci> length (["Hello", "World"] :: forall a. [a])
  Couldn't match expected type 'a' against inferred type '[Char]'
  ...
```

With Rank-N-Types, forall a meant that your expression must fit all possible as. For example:

```
ghci> length ([] :: forall a. [a])
0
```

<https://stackoverflow.com/questions/3071136/what-does-the-forall-keyword-in-haskell-ghc-do>

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## References

- [1] <ftp://ftp.geoinfo.tuwien.ac.at/navratil/HaskellTutorial.pdf>
- [2] <https://www.umiacs.umd.edu/~hal/docs/daume02yaht.pdf>