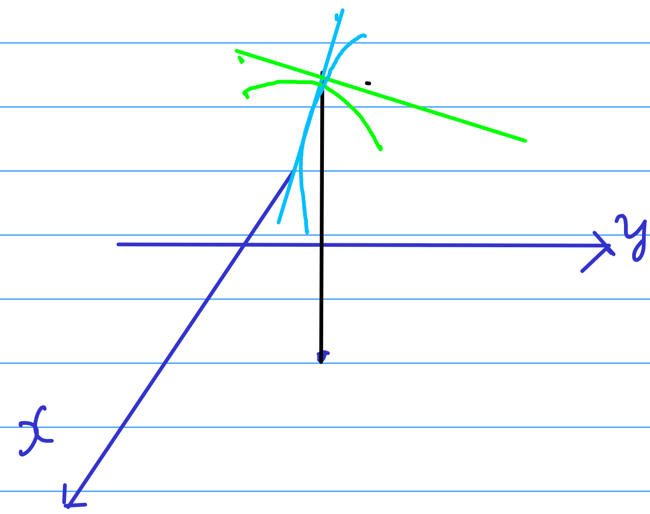
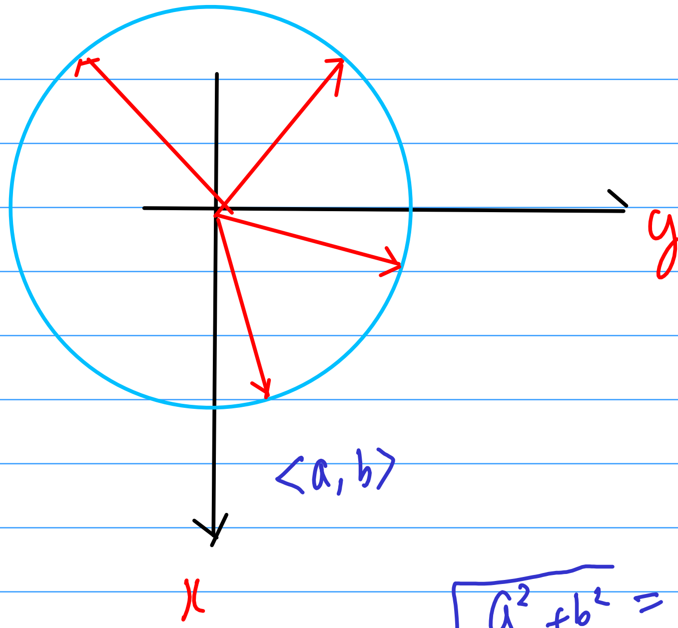
Vector Calculus (H.1)
Line Integrals

20150922

Directional Derivatives





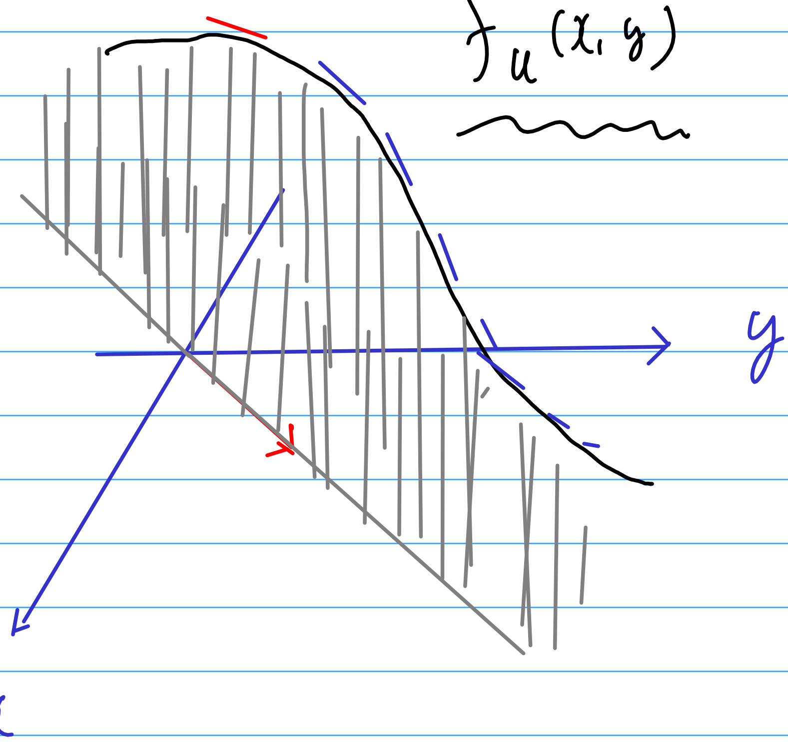
$\vec{u} = \langle a, b \rangle$

unit directional Vector

magnitude = length = 1

$\sqrt{a^2 + b^2} = 1$ $\quad a^2 + b^2 = 1$

$f_u(x, y)$



$$D_{\vec{u}}\, f(x,y) = \lim_{h \to 0} \frac{f(x+ah,\ y+bh) - f(x,y)}{h}$$

$$D_{\vec{u}}\, f(x,y) = a\, D_x\, f(x,y) + b\, D_y\, f(x,y)$$

$$= a\, \boxed{f_x\,(x,y)} + b\, \boxed{f_y\,(x,y)}$$

$\vec{u} = \langle a, b \rangle$

$$\langle \boxed{f_x\,(x,y)},\ \boxed{f_y\,(x,y)} \rangle \cdot \langle a, b \rangle$$

true vector

$f(x,y)$

scalar

∂x ∂y

partial derivative

$$\frac{\partial}{\partial x} f(x,y)$$

partial derivative

$$\frac{\partial}{\partial y} f(x,y)$$

$\vec{u} = \langle a, b, c \rangle$

$D_{\vec{u}} f(x,y,z) = a \cdot f_x(x,y,z) + b \cdot f_y(x,y,z) + c \cdot f_z(x,y,z)$

$= \langle f_x, f_y, f_z \rangle \bullet \langle a, b, c \rangle$

dot product

$f \rightarrow$ 3 variables x, y, z

$\downarrow$

consider this as a vector. $\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

gradient (vector) of f

del

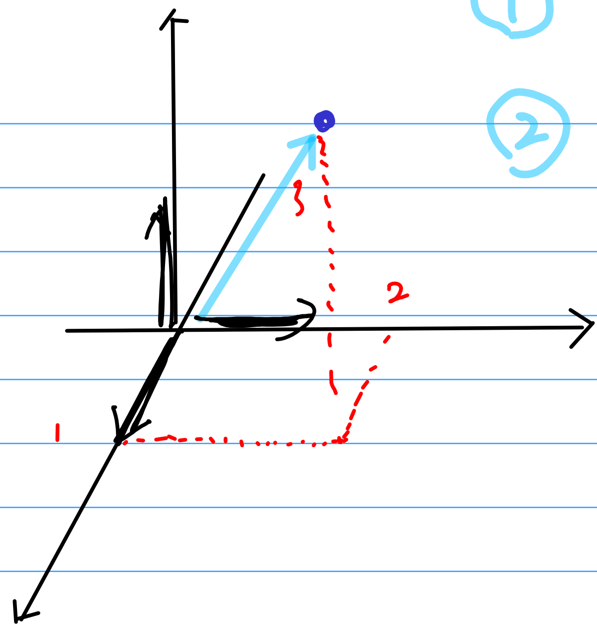
$\nabla f \triangleq \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

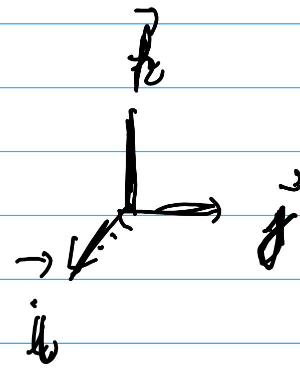
x 방향의 접선 기울기 | y 방향의 접선 기울기 | z 방향의 접선 기울기

$= \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$

① $\langle 1, 2, 3 \rangle$

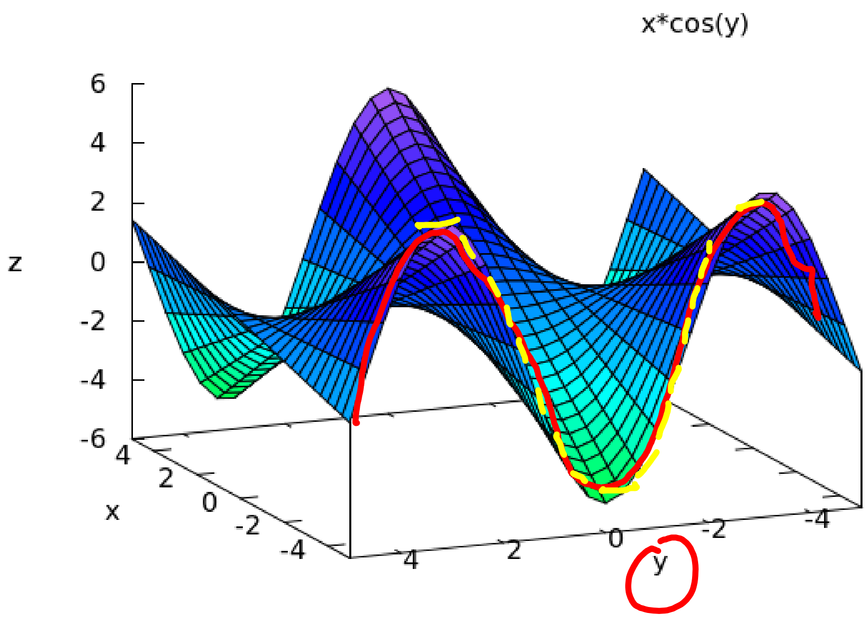
② $\underline{1}\,\vec{i} + \underline{2}\,\vec{j} \quad \underline{3}\,\vec{k}$

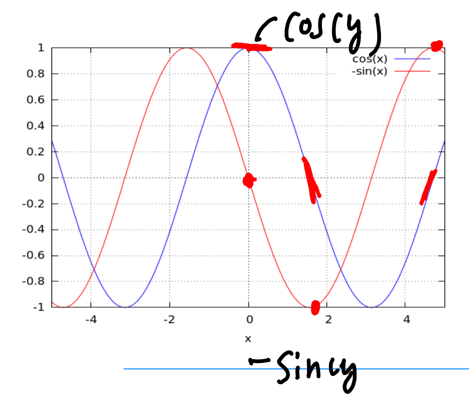




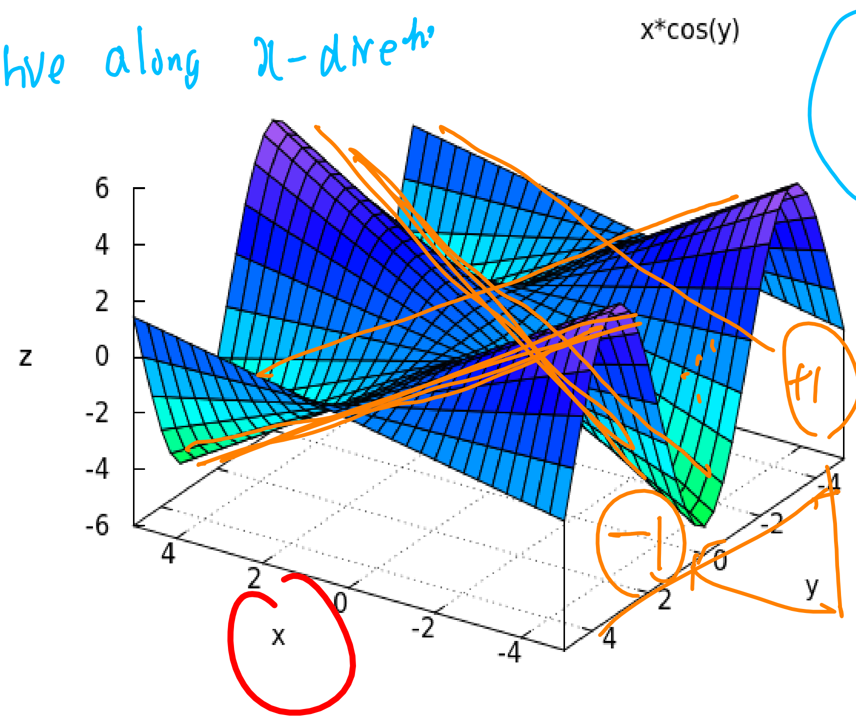
$f(x, y) = x \cdot \cos(y)$

$\frac{\partial}{\partial y} f(x, y) = -x\sin(y)$





derivative along x-directi



$\frac{\partial}{\partial x} f(x, y) = \cos(y)$

$+1 \geq \cos(y) \geq -1$

기울기가 상수

y 고정!!

$$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$= \langle \cos(y), -x\sin(y) \rangle$$

gradient vector of f

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \longleftarrow (x, y)$$

$$\nabla \triangleq \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}$$

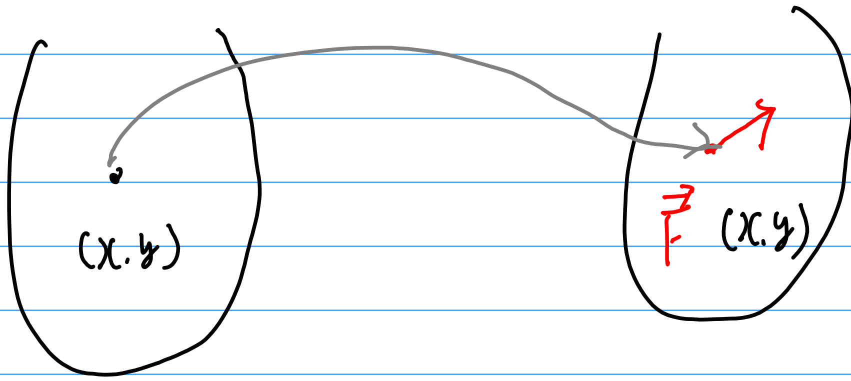
$$\nabla f(x,y) \triangleq \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} \equiv \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

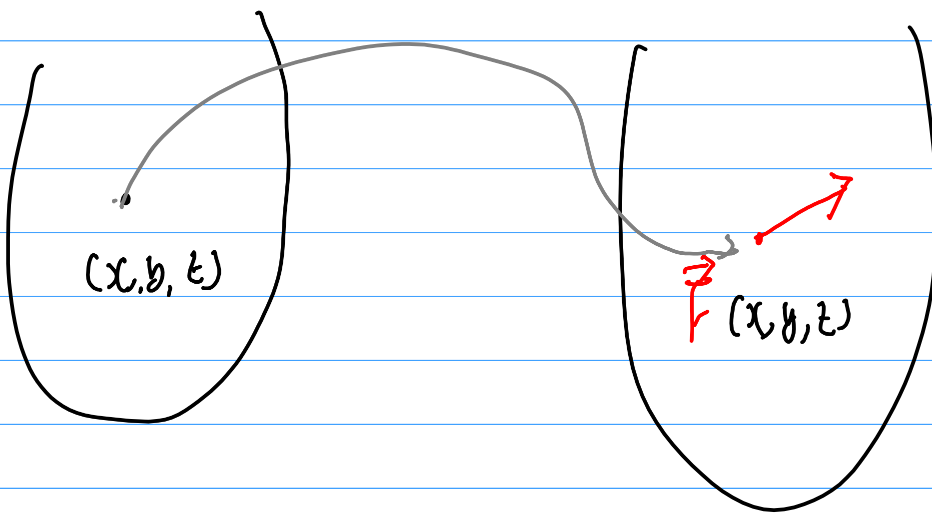
<Vector field> ..... vector valued function

at each point

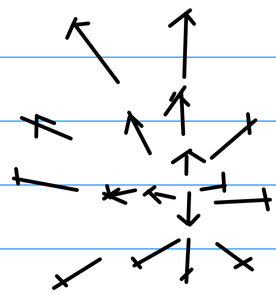
2-d (x, y) a vector (2-d) is assigned

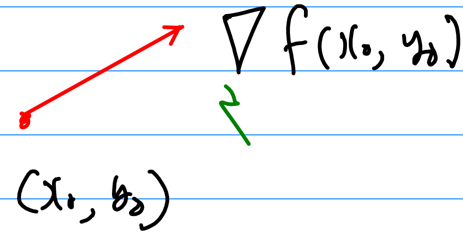
3-d (x, y, z) a vector (3-d) is assigned



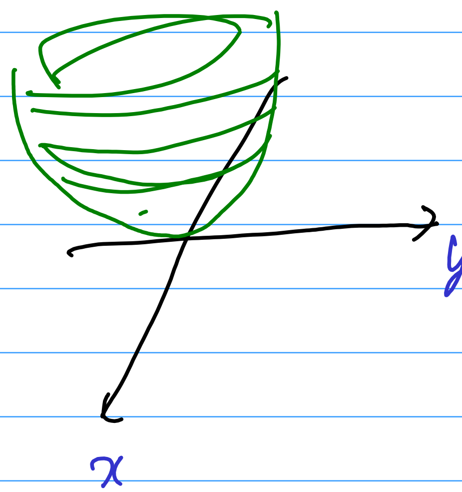


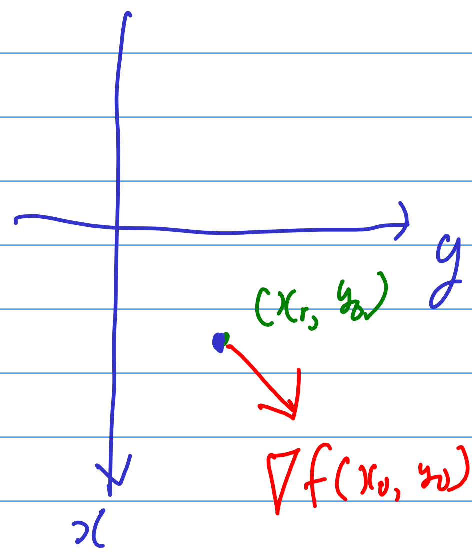
gradient vector field



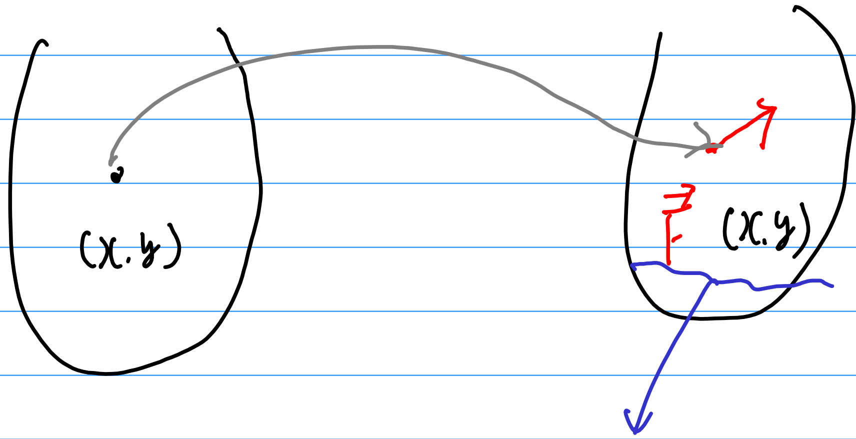


$z = f(x, y)$: 3차원 곡선





2-d vector field



at a 2-d point (x, y)

the value of a function $\vec{F}$

: a 2-d vector.

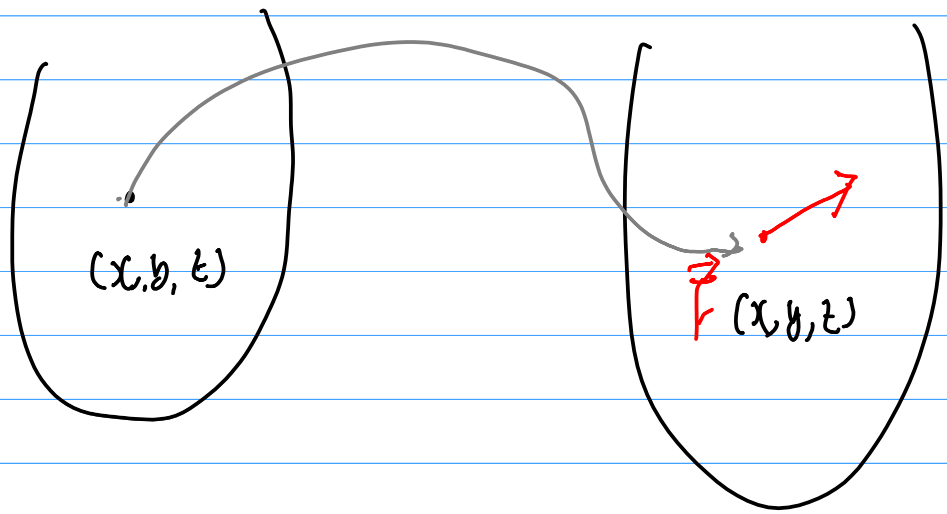
2 component

$$\langle P(x,y), Q(x,y) \rangle$$

$$= P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$$

scalar function

Vector valued function



at a 3-d point (x, y)   the value of a function $\vec{F}$

: a 3-d vector.

3 component

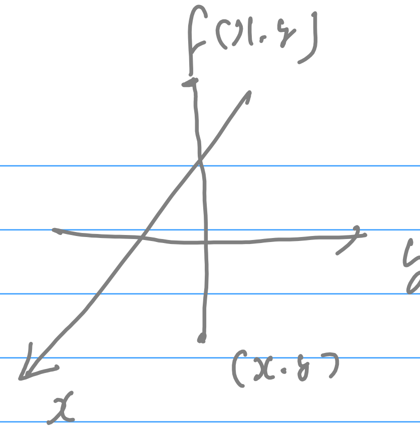
$$\langle P(x,y),\ Q(x,y),\ R(x,y) \rangle$$

$$= P(x,y)\,\vec{i} + Q(x,y)\,\vec{j} + R(x,y)\,\vec{k}$$

scalar function

Vector valued function

$f(x.y) = x^2 + y^2$

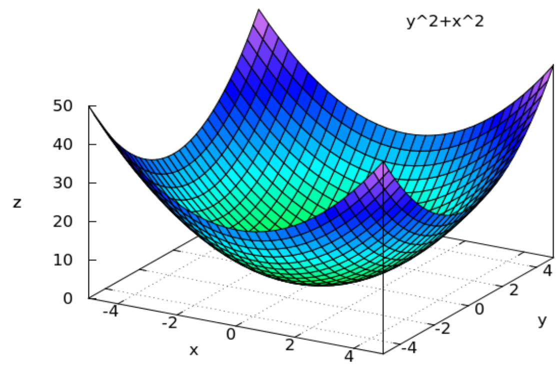


at a 2-d point (x, y),

a value is assigned x^2+y^2

Scalar function $f(x.y)$

~~Vector~~ function

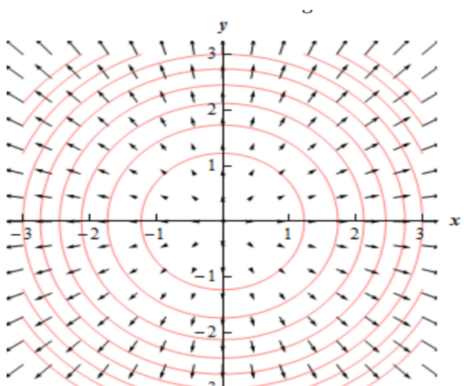


gradient vector

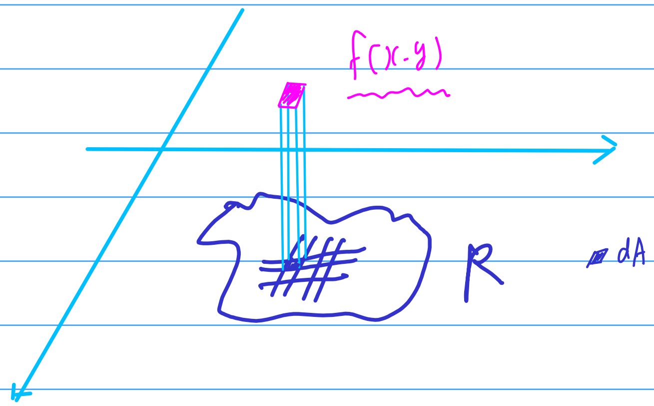
$$\vec{F}(x,y) = \left\langle \frac{\partial f}{\partial x}(x.y),\ \frac{\partial f}{\partial y}(x.y) \right\rangle$$

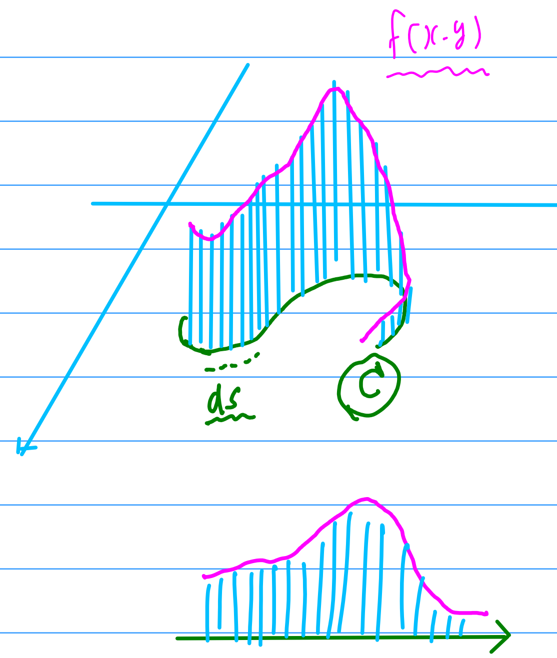
at a 2-d point (x, y),

a vector is assigned $\left\langle \frac{\partial f}{\partial x}(x.y),\ \frac{\partial f}{\partial y}(x.y) \right\rangle$

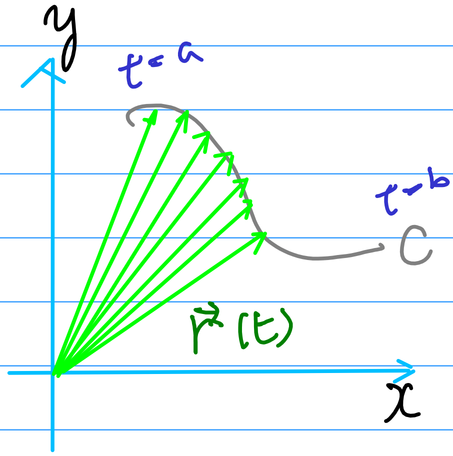


$$\iint_R f(x,y)\,dA \qquad\qquad \int_C f(x,y)\,ds$$





t : parameter time

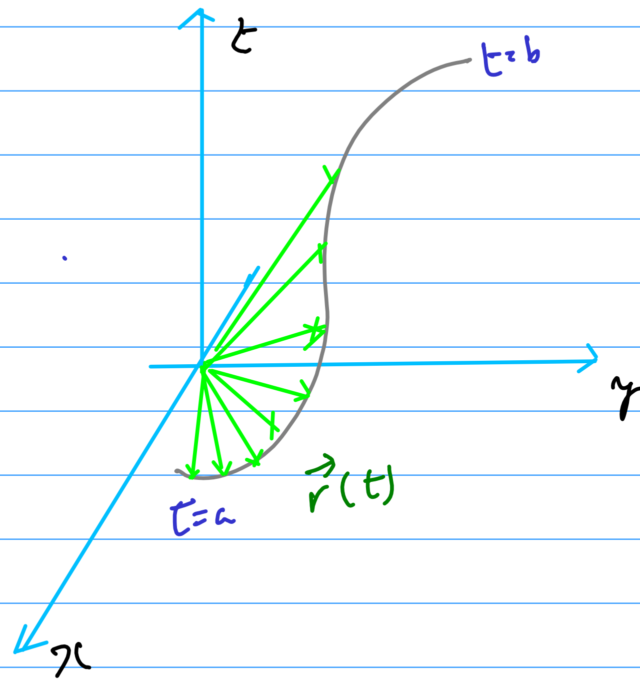


$\vec{r}(t) = \langle m(t), n(t) \rangle$

related to each other by a parameter t

$x = m(t)$.

$y = n(t)$



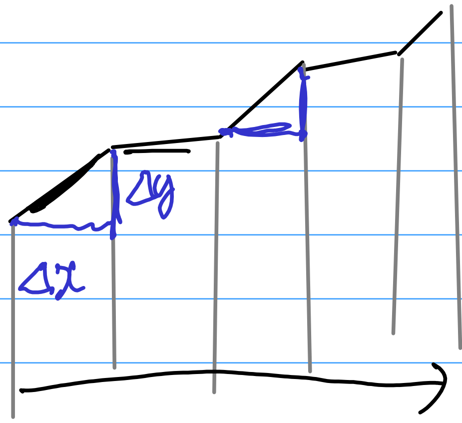
$\vec{r}(t) = \langle m(t), n(t), k(t) \rangle$

parameter

$x = m(t)$

$y = n(t)$

$z = k(t)$



$$\sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\Delta y = f'(x)\,\Delta x$$

$$dy = f'(x)\,dx$$

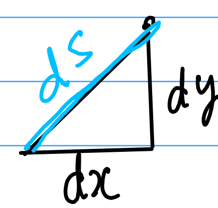
$$\sqrt{(dx)^2 + \left(f'(x)\right)^2 (dx)^2}$$

$$= \sqrt{1 + [f'(x)]^2}\; dx$$

$$\int \sqrt{1 + [f'(x)]^2}\; dx = L$$

$$\int \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\; dx = L$$

$$\int ds = L$$



$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx$$

$$\Rightarrow \sqrt{(dx)^2 + (dy)^2}$$

$$\sqrt{(dx)^2 + (dy)^2} = ds$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx = ds$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = ds$$

$x = f(t)$ $\qquad dx = \frac{df}{dt} dt$

$y = g(t)$ $\qquad dy = \frac{dg}{dt} dt$

$$\sqrt{\left(\frac{df}{dt}\right)^2 (dt)^2 + \left(\frac{dg}{dt}\right)^2 (dt)^2} = \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2} \, dt = ds$$

arc length L

$$L = \int_C ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Line Integral

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_a^b f(x(t), y(t))\, \|\vec{r}'(t)\|^2\, dt$$

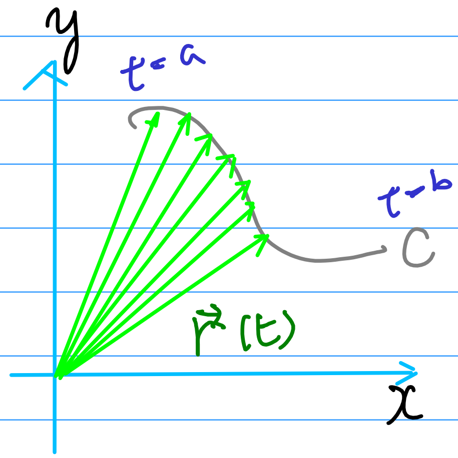
$$\int_C f(x,y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$

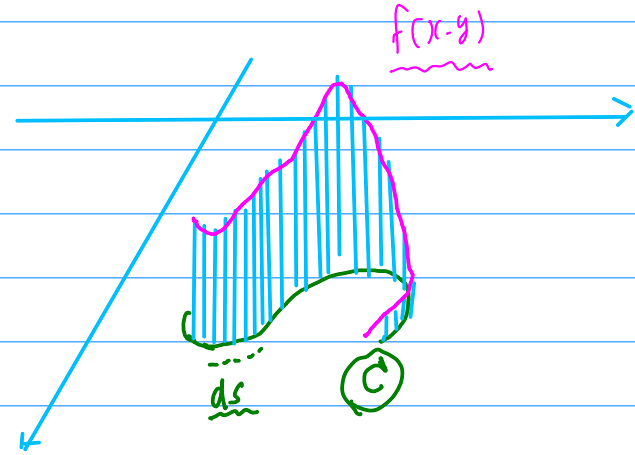
$$\int_C f(x,y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$$

$$\boxed{\int_{-C} f(x,y)\,ds = \int_{C} f(x,y)\,ds}$$

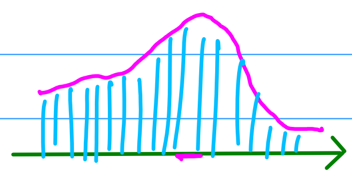
$$\boxed{\begin{aligned}\int_{-C} f(x,y)\,dx &= -\int_{C} f(x,y)\,dx\\ \int_{-C} f(x,y)\,dy &= -\int_{C} f(x,y)\,dy\end{aligned}}$$

t : parameter time





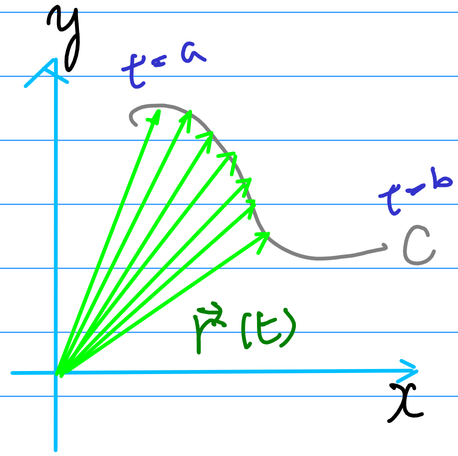
$\vec{r}(t) = \langle m(t), n(t) \rangle$

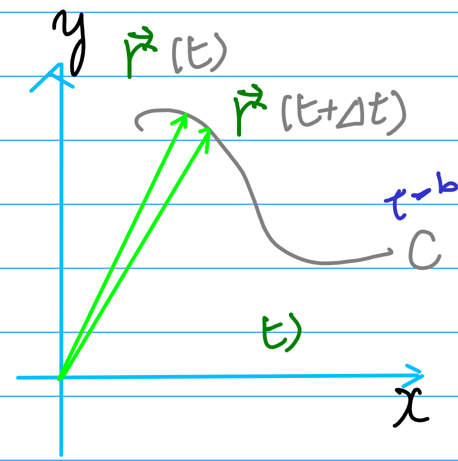


$$\int_C f(x,y)\, ds$$

$$= \int_C f(m(t), n(t))\, ds$$

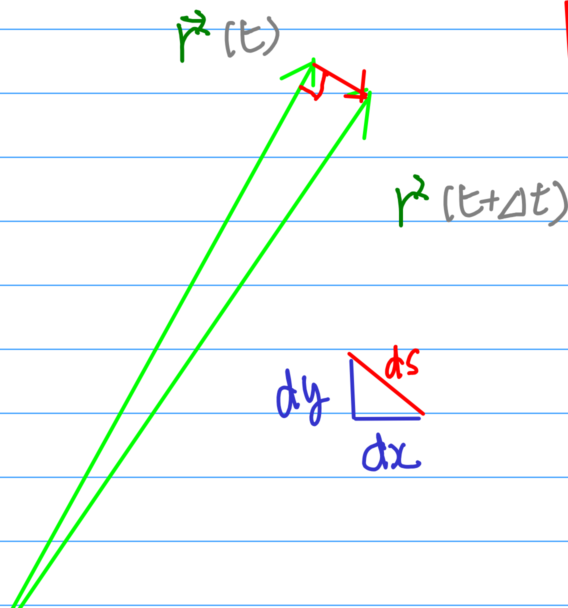
$$= \int_a^b f(m(t), n(t)) \sqrt{\left(\frac{dm}{dt}\right)^2 + \left(\frac{dn}{dt}\right)^2}\, dt$$





$\vec{r}(t) = \langle m(t), n(t) \rangle$

$\vec{r}(t) = \langle m(t), n(t) \rangle$



$\boxed{\vec{r}(t+\Delta t) - \vec{r}(t)}$

$$\lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \vec{r}\,'(t)$$

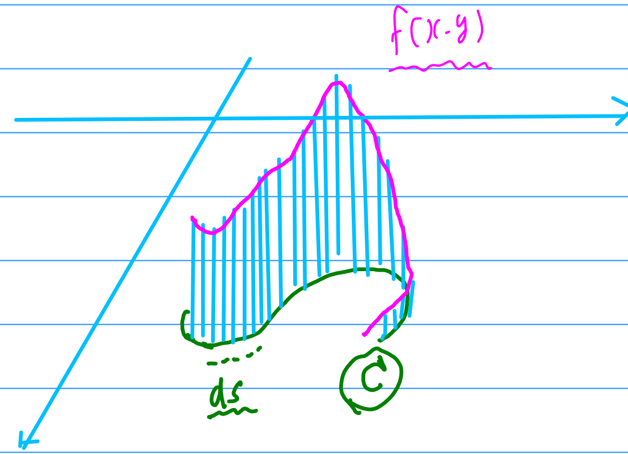
a vector orthogonal to $\vec{r}(t)$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$ds = \|\vec{r}\,'(t)\|\, dt$$

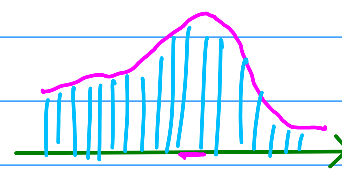
$$\int \|\vec{r}\,'(t)\|\, dt = L$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \left\|\frac{d\vec{r}}{dt}\right\|$$



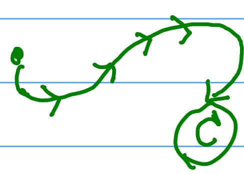
$$\int_C f(x,y)\, ds$$

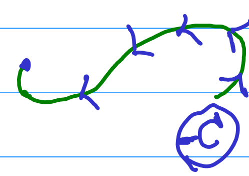
$$= \int_C f(m(t), n(t))\, ds$$



$$= \int_a^b f(m(t), n(t)) \sqrt{\left(\frac{dm}{dt}\right)^2 + \left(\frac{dn}{dt}\right)^2}\, dt$$

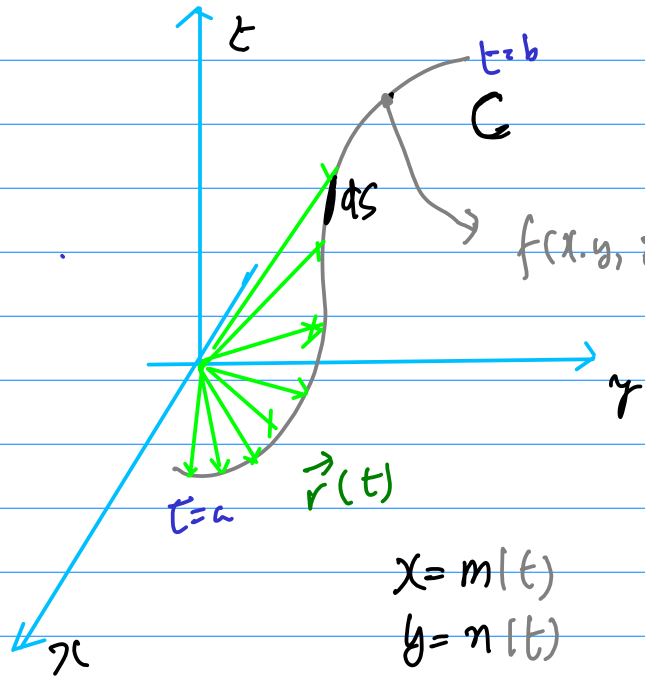
$$= \int_a^b f(m(t), n(t)) \left\| \frac{d\vec{r}}{dt} \right\| dt$$





$$\int_C f(x,y)\, ds = \int_{-C} f(x,y)\, ds$$

$$\int_C f(x,y,z)\,dx$$



f(x,y,z)는 (x,y,z) 점에서 정의되는 함수

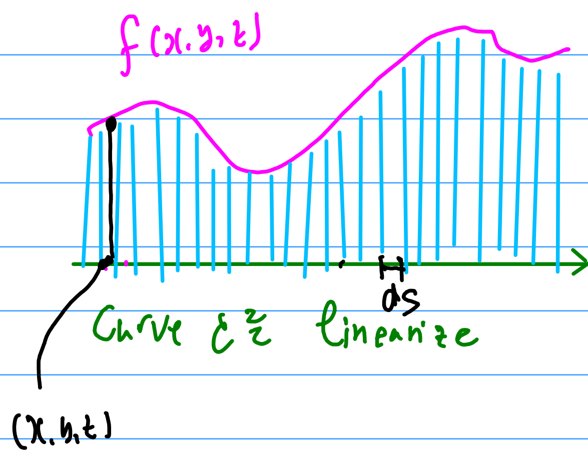
3변수 함수

plot → 4차원이 필요

f(x,y,z) 값을 3차원에 나타내니 그릴수 없다.

$x = m(t)$
$y = n(t)$
$z = k(t)$

$$\vec{r}(t) = \langle \underbrace{m(t),\ n(t),\ k(t)}_{\text{parameter}} \rangle$$



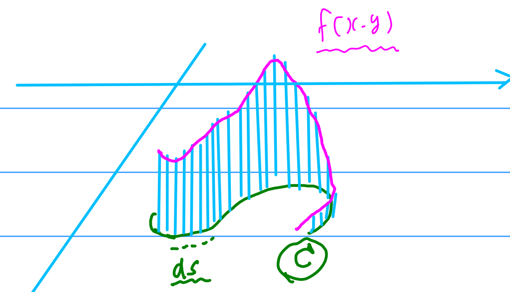
$$\int_C f(x,y,z)\,ds$$

$$= \int_C f(m(t), n(t), k(t))\,ds$$

$$= \int_a^b f(m(t), n(t), k(t)) \sqrt{\left(\frac{dm}{dt}\right)^2 + \left(\frac{dn}{dt}\right)^2 + \left(\frac{dk}{dt}\right)^2}\, dt$$

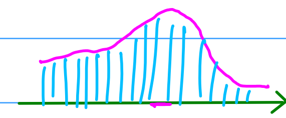
$$= \int_a^b f(m(t), n(t), k(t)) \left\| \frac{d\vec{r}}{dt} \right\| dt$$

$$\int_C f(x,y)\, ds$$

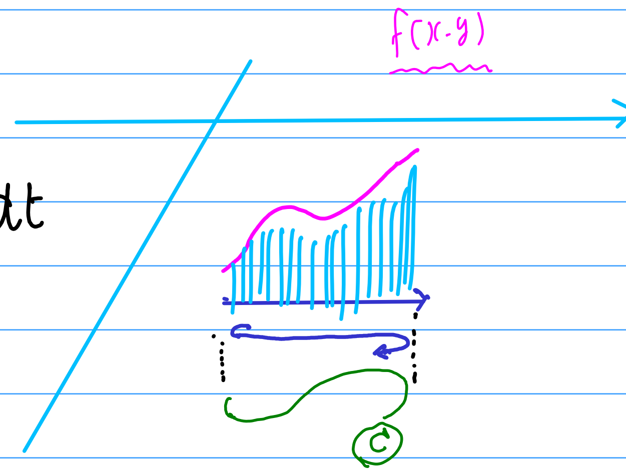


$$= \int_C f(m(t), n(t))\, ds$$

$$= \int_a^b f(m(t), n(t)) \sqrt{\left(\frac{dm}{dt}\right)^2 + \left(\frac{dn}{dt}\right)^2}\, dt$$



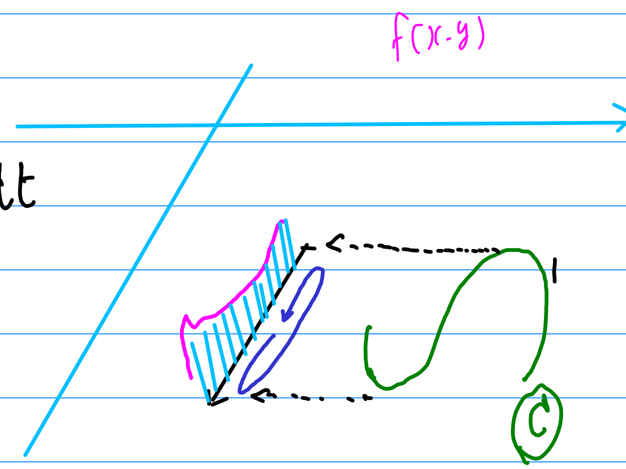
$$= \int_a^b f(m(t), n(t)) \left\| \frac{d\vec{r}}{dt} \right\| dt$$



$$\int_a^b f(m(t), n(t)) \sqrt{\left(\frac{dm}{dt}\right)^2 + \cancel{\left(\frac{dn}{dt}\right)^2}}\, dt$$

$$\int_a^b f(m(t), n(t)) \sqrt{\left(\frac{dm}{dt}\right)^2}\, dt$$

$$\Rightarrow \int_a^b f(m(t), n(t))\, m'(t)\, dt \overset{\Delta}{=} \int_C f(x,y)\, dx$$



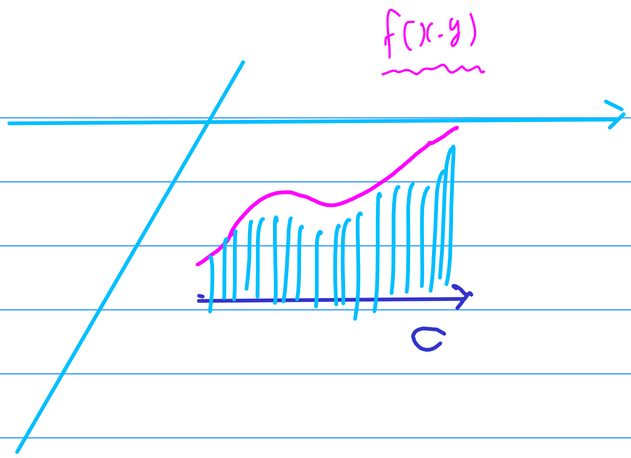
$$\int_a^b f(m(t), n(t)) \sqrt{\cancel{\left(\frac{dm}{dt}\right)^2} + \left(\frac{dn}{dt}\right)^2}\, dt$$

$$\int_a^b f(m(t), n(t)) \sqrt{\left(\frac{dn}{dt}\right)^2}\, dt$$

$$\Rightarrow \int_a^b f(m(t), n(t))\, n'(t)\, dt \overset{\Delta}{=} \int_C f(x,y)\, dy$$

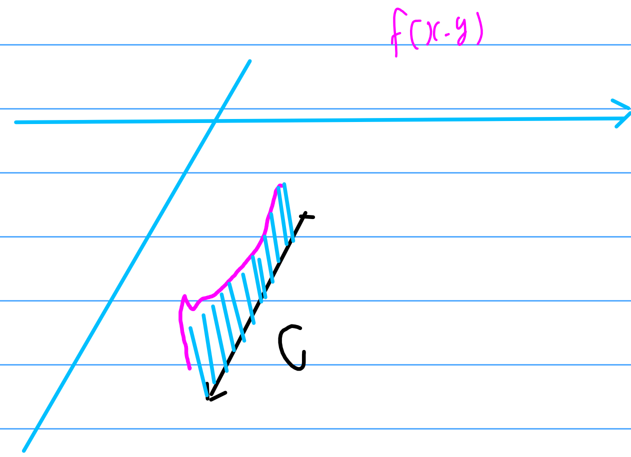
$$\int_C \boxed{f(x,y)}\, dx$$

$$\overset{\Delta}{=} \int_a^b f(m(t), n(t))\, m'(t)\, dt$$



$$\int_C \boxed{f(x,y)}\, dy$$

$$\overset{\Delta}{=} \int_a^b f(m(t), n(t))\, n'(t)\, dt$$



$$\int_C P(x,y)\, dx + \int_C Q(x,y)\, dy$$

$$\overset{\Delta}{=} \int_C P\, dx + Q\, dy$$

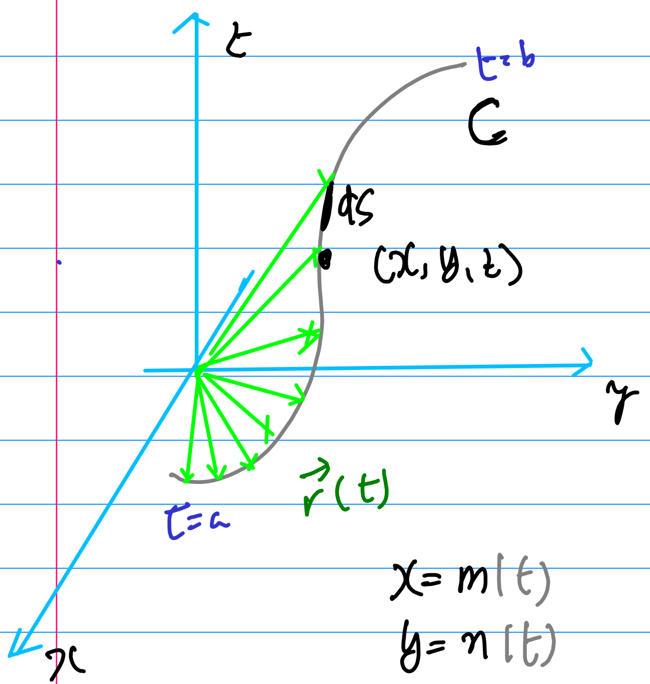
$$\int_C f(x,y)\,dx = -\int_{-C} f(x,y)\,dx$$

$$\int_C f(x,y)\,dy = -\int_{-C} f(x,y)\,dy$$

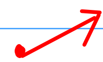
$$\int_C P\,dx + Q\,dy = -\int_{-C} P\,dx + Q\,dy$$

$$\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$$

Line Integral with Vector field

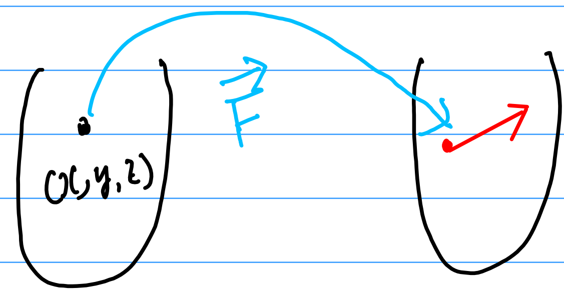


$$\int_C f(x, y, z)\, dx$$



· (x, y, z) point 에서
함수 값으로 Vector를 할당

$x = m(t)$
$y = n(t)$
$z = k(t)$

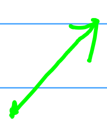


$$\vec{r}(t) = \langle m(t), n(t), k(t) \rangle$$

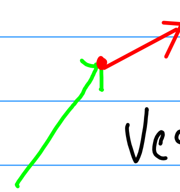
$$\vec{F}(x, y, z) = \quad = \langle P, Q, R \rangle$$

$$= \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$$

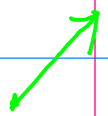
$$= P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$$



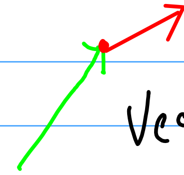
$$\vec{r}(t) = \langle x, y, z \rangle$$
$$= \langle m(t), n(t), k(t) \rangle$$



Vector field



$$\vec{r}(t) = \langle x, \ y, \ z \rangle$$
$$= \langle m(t), n(t), k(t) \rangle$$



Vector field

$$\vec{F}(x,y,z) = \quad = \langle P, \ Q, \ R \rangle$$

$$\vec{F}(\vec{r}(t)) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

$$\int_C f(x,y,z)\, ds \qquad (x,y,z)$$

$$= \int_C f(m(t), n(t), k(t))\, ds$$

$$= \int_a^b f(m(t), n(t), k(t)) \sqrt{\left(\frac{dm}{dt}\right)^2 + \left(\frac{dn}{dt}\right)^2 + \left(\frac{dk}{dt}\right)^2} \; dt$$

$$= \int_a^b f(m(t), n(t), k(t)) \left\| \frac{d\vec{r}}{dt} \right\| dt$$

Line Integrals of vector fields

Vector field $\langle x, y, z\rangle \longrightarrow \langle P, Q, R\rangle$

$P(x, y, z)$
$Q(x, y, z)$
$R(x, y, z)$

$$\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$$

curve C

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$= \int_a^b \vec{F}(x(t), y(t), z(t))\cdot\vec{r}\,'(t)\,dt$$

$$\int_{\vec{C}} \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \langle P, Q, R \rangle \cdot \langle x', y', z' \rangle\, dt$$

$$= \int_a^b P x' + Q y' + R z' \, dt$$

$$= \int_a^b P x'\, dt + \int_a^b Q y'\, dt + \int_a^b R z'\, dt$$

$$= \int_{\vec{C}} P\, dx + \int_{\vec{C}} Q\, dy + \int_{\vec{C}} R\, dz$$

$$= \int_{\vec{C}} P\, dx + Q\, dy + R\, dz$$

$$\int_{-\vec{C}} \vec{F} \cdot d\vec{r} = -\int_{\vec{C}} \vec{F} \cdot d\vec{r}$$

Fundamental Theorem

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$= \int_a^b \nabla f(\vec{r}(t)) \cdot d\vec{r}\,'(t)\,dt$$

$$= \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$$

$$= \int_a^b \left[\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right] dt$$

$$= \int_a^b \frac{d}{dt}\left[f(\vec{r}(t)) \right] dt$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

$\vec{F}$: a continuous vector field

1. $\vec{F}$: a conservative vector field

if there exists a function s.t. $\vec{F} = \nabla f$

f : a potential function for the vector field $\vec{F}$

$$\nabla \triangleq \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}$$

Gradient Vector Field

$$\nabla f(x,y) \triangleq \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

2. $\int_C \vec{F} \cdot d\vec{r}$: independent of path

if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$

for any two paths C_1 and C_2

with the same initial and final points

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \langle P, Q, R \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$$

$$= \int_a^b P(x,y,z)\frac{dx}{dt} + Q(x,y,z)\frac{dy}{dt} + R(x,y,z)\frac{dz}{dt}\, dt$$

$$= \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$$

$$= \int_a^b \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\, dt$$

$$= \int_a^b \frac{df}{dt}\, dt$$

3. a closed path: its initial and final points are the same point

4. a simple path: it doesn't cross itself

5. an open region: not include any of its boundary points

6. a connected region: can connect any two points with a path that lies completely in D

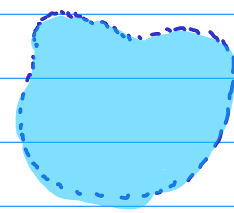
7. a simply-connected region: connected and containing no holes

1. $\int_C \nabla f \cdot d\vec{r}$ independent of path

2. $\vec{F}$: a conservative vector field

$\Rightarrow \int_C \vec{F} \cdot d\vec{r}$. independent of path

3. $\vec{F}$: a continuous vector field
on an open connected region D
and $\int_C \vec{F} \cdot d\vec{r}$: independent of path



$\Rightarrow \vec{F}$: a conservative vector field $\boxed{\vec{F} = \nabla f}$

4. $\int_C \vec{F} \cdot d\vec{r}$: independent of path

$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = 0$ for every closed path

5. $\oint_C \vec{F} \cdot d\vec{r} = 0$ for every closed path

$\Rightarrow \int_C \vec{F} \cdot d\vec{r}$: independent of path

1. $\int_C \nabla f \cdot d\vec{r}$ independent of path

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$$

$$= \int_a^b \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\, dt$$

$$= \int_a^b \frac{df}{dt}\, dt$$

2. $\vec{F}$: a conservative vector field

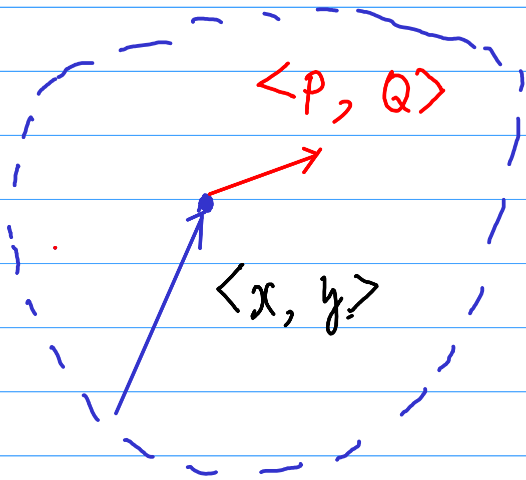
$\Rightarrow \int_C \vec{F} \cdot d\vec{r}$. independent of path

$\vec{F}$: a conservative vector field $\Rightarrow$ there exists $\boxed{\vec{F} = \nabla f}$

$$\int_C \vec{F} \cdot d\vec{r} = \underline{\int_C \nabla f \cdot d\vec{r}} \Rightarrow \text{independent of path}$$

In vector calculus a **conservative vector field** is a vector field that is the gradient of some function, known in this context as a scalar potential.[1] Conservative vector fields have the property that the line integral is path independent, i.e. the choice of integration path between any point and another does not change the result. Path independence of a line integral is equivalent to the vector field being conservative. A conservative vector field is also irrotational; in three dimensions this means that it has vanishing curl. An irrotational vector field is necessarily conservative provided that a certain condition on the geometry of the domain holds, i.e. the domain is simply connected.

Conservative vector fields appear naturally in mechanics: they are vector fields representing forces of physical systems in which energy is conserved.[2] For a conservative system, the work done in moving along a path in configuration space depends only on the endpoints of the path, so it is possible to define a potential energy independently of the path taken.



Open and simply connected region D

Vector field

$$\vec{F} = P\,\vec{i} + Q\,\vec{j}$$

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}} \Rightarrow \vec{F} : \text{conservative vector field}$$

$$\Rightarrow \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = P\,\vec{i} + Q\,\vec{j} = \vec{F}$$

$$\frac{\partial f}{\partial x} = P \qquad\qquad \frac{\partial f}{\partial y} = Q$$

$$f(x, y) = \int P(x, y)\, dx + g(y) \qquad\qquad f(x, y) = \int Q(x, y)\, dy + h(x)$$