

Second Order ODEs (H.1)

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Types of First Order ODEs

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

Separable Equations

$$\frac{dy}{dx} = g_1(x)g_2(y)$$

$$y' = g_1(x)g_2(y)$$

$$y = f(x)$$

Linear Equations

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$a_1(x)y' + a_0(x)y = g(x)$$

$$y = f(x)$$

Exact Equations

$$M(x, y)dx + N(x, y)dy = 0$$

$$\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = 0$$

$$z = f(x, y)$$

Second Order ODEs

First Order Linear Equations

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$a_1(x)y' + a_0(x)y = g(x)$$

Second Order Linear Equations

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

Second Order Linear Equations with Constant Coefficients

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0y = g(x)$$

$$a_2y'' + a_1y' + a_0y = g(x)$$

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g(x)$$

$$ay'' + by' + cy = g(x)$$

Auxiliary Equation

Homogeneous Second Order DEs with Constant Coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$a y'' + b y' + c y = 0$$

try a solution $y = e^{mx}$

$$a \frac{d^2}{dx^2} \{e^{mx}\} + b \frac{d}{dx} \{e^{mx}\} + c \{e^{mx}\} = 0$$

$$a \{e^{mx}\}'' + b \{e^{mx}\}' + c \{e^{mx}\} = 0$$

$$a \{m^2 e^{mx}\} + b \{m e^{mx}\} + c \{e^{mx}\} = 0$$

$$a \{m^2 e^{mx}\} + b \{m e^{mx}\} + c \{e^{mx}\} = 0$$

$$(a m^2 + b m + c) \cdot e^{mx} = 0$$

$$(a m^2 + b m + c) \cdot e^{mx} = 0$$

auxiliary equation

$$(a m^2 + b m + c) = 0$$

$$(a m^2 + b m + c) = 0$$

Roots of the Auxiliary Equation

Homogeneous Second Order DEs with Constant Coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$a y'' + b y' + c y = 0$$

try a solution $y = e^{mx}$



$$(a m^2 + b m + c) = 0$$

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$

$$y_1 = e^{m_1 x}, \quad y_2 = e^{m_2 x}$$



(A) $b^2 - 4ac > 0$ Real, distinct m_1, m_2

$$y_1 = e^{m_1 x} = y_2 = e^{m_1 x}$$



(B) $b^2 - 4ac = 0$ Real, equal m_1, m_2

$$y_1 = e^{m_1 x}, \quad y_2 = e^{m_2 x}$$



(C) $b^2 - 4ac < 0$ Conjugate complex m_1, m_2

Linear Combination of Solutions

DEQ

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$y_1$$

$$y_2$$

$$C_1 y_1 + C_2 y_2$$

$$a y_1'' + b y_1' + c y_1 = 0$$

$$a y_2'' + b y_2' + c y_2 = 0$$

$$a(y_1'' + y_2'') + b(y_1' + y_2') + c(y_1 + y_2) = 0$$

$$a(y_1 + y_2)'' + b(y_1 + y_2)' + c(y_1 + y_2) = 0$$

$$y_3 = y_1 + y_2$$

$$y_4 = y_1 - y_2$$

$$y_5 = y_3 + 2y_4$$

$$y_6 = y_3 - 2y_4$$

$$a(C_1 y_1'' + C_2 y_2'') + b(C_1 y_1' + C_2 y_2') + c(C_1 y_1 + C_2 y_2) = 0$$

$$a(C_1 y_1 + C_2 y_2)'' + b(C_1 y_1 + C_2 y_2)' + c(C_1 y_1 + C_2 y_2) = 0$$

Solutions of 2nd Order ODEs

DEQ

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$y_1$$

$$y_2$$

$$C_1 y_1 + C_2 y_2$$

$$\begin{cases} y_1 = e^{m_1 x} \\ y_2 = e^{m_2 x} \end{cases} \quad (D > 0)$$

$$\begin{cases} y_1 = e^{m_1 x} \\ y_2 = e^{m_1 x} \end{cases} \quad (D = 0)$$

$$\begin{cases} y_1 = e^{m_1 x} \\ y_2 = e^{m_2 x} \end{cases} \quad (D < 0)$$

$$\begin{cases} y = C_1 e^{m_1 x} + C_2 e^{m_2 x} & (D > 0) \\ y = C_1 e^{m_1 x} \quad ? & (D = 0) \\ y = C_1 e^{m_1 x} + C_2 e^{m_2 x} & (D < 0) \end{cases}$$

auxiliary equation

$$(a m^2 + b m + c) = 0$$

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$

Linear Independent Functions Example (2)

linearly dependent functions

$$y_1 = e^{2x} \quad y_2 = 3e^{2x}$$

$$y = c_1 e^{2x} + c_2 \cdot 3e^{2x}$$

$$y = (c_1 + 3c_2)e^{2x} = C e^{2x}$$

~~general solution~~

linearly independent functions

$$y_1 = e^{2x} \quad y_2 = x e^{2x}$$

$$y = c_1 e^{2x} + c_2 x e^{2x}$$

general solution

$$y'' - 4y' + 4y = 0 \quad m^2 - 4m + 4 = 0$$

$$(m - 2)^2 = 0$$

$$y_1 = e^{2x}$$

$$y_1' = 2e^{2x}$$

$$y_1'' = 4e^{2x}$$

$$4y_1 = 4e^{2x}$$

$$-4y_1' = -8e^{2x}$$

$$y_1'' = 4e^{2x}$$

0

$$y_2 = x e^{2x}$$

$$y_2' = e^{2x} + 2x e^{2x}$$

$$y_2'' = 2e^{2x} + 2e^{2x} + 4x e^{2x}$$

$$4y_2 = 4x e^{2x}$$

$$-4y_2' = -4e^{2x} - 8x e^{2x}$$

$$y_2'' = 4e^{2x} + 4x e^{2x}$$

0

Fundamental Set of Solutions

Second Order EQ

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

y_1

y_2

Functions y_1 and y_2 are either

- linearly independent functions or
- linearly dependent functions

$$C_1 y_1 + C_2 y_2$$

$$\{y_1, y_2\}$$

Second Order

there can be at most two linearly independent functions

$$y = C_1 e^{\alpha x} + C_2 e^{\alpha x}$$

$$y = C_1 e^{\alpha x} + C_2 x e^{\alpha x}$$

$$y = C_1 e^{\alpha x} + C_2 e^{\alpha x} = C_1 e^{(\alpha+\eta)x} + C_2 e^{(\alpha-\eta)x}$$

any n linearly independent solutions of the homogeneous linear n -th order differential equation

Fundamental Set of Solutions

(A) Real Distinct Roots Case

Homogeneous Second Order DEs with Constant Coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$a y'' + b y' + c y = 0$$

try a solution $y = e^{mx}$



$$(a m^2 + b m + c) = 0$$

auxiliary equation

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$

$$y_1 = e^{m_1 x}$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$

$$y_2 = e^{m_2 x}$$

$$b^2 - 4ac > 0 \quad \text{Real, distinct } m_1, m_2$$

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$b^2 - 4ac = 0 \quad \text{Real, equal } m_1, m_2$$

$$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$$

$$b^2 - 4ac < 0 \quad \text{Conjugate complex } m_1, m_2$$

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + \beta i)x} + C_2 e^{(\alpha - \beta i)x}$$

(B) Repeated Real Roots Case

Homogeneous Second Order DEs with Constant Coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$a y'' + b y' + c y = 0$$

try a solution $y = e^{mx}$



$$(a m^2 + b m + c) = 0$$

auxiliary equation

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$



$$b^2 - 4ac = 0$$



$$m_1 = -b/2a$$

$$e^{m_1 x} = e^{m_2 x} = e^{-\frac{b}{2a}x}$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$

$$m_2 = -b/2a$$

$$b^2 - 4ac > 0 \quad \text{Real, distinct } m_1, m_2$$

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$b^2 - 4ac = 0 \quad \text{Real, equal } m_1, m_2$$

$$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$$

$$b^2 - 4ac < 0 \quad \text{Conjugate complex } m_1, m_2$$

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + \beta i)x} + C_2 e^{(\alpha - \beta i)x}$$

(C) Complex Roots of the Auxiliary Equation

Homogeneous Second Order DEs with Constant Coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

$$a y'' + b y' + c y = 0$$

try a solution $y = e^{mx}$

$$(am^2 + bm + c) = 0$$

auxiliary equation

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$

$$m_1 = (-b + \sqrt{4ac - b^2} i)/2a$$

$$y_1 = e^{m_1 x}$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$

$$m_2 = (-b - \sqrt{4ac - b^2} i)/2a$$

$$y_2 = e^{m_2 x}$$

$$b^2 - 4ac > 0 \quad \text{Real, distinct } m_1, m_2$$

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$b^2 - 4ac = 0 \quad \text{Real, equal } m_1, m_2$$

$$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$$

$$b^2 - 4ac < 0 \quad \text{Conjugate complex } m_1, m_2$$

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

Complex Exponential Conversion

Homogeneous Second Order DEs with Constant Coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

$$a y'' + b y' + c y = 0$$

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a \Rightarrow m_1 = (-b + \sqrt{4ac - b^2} i)/2a = \alpha + i\beta$$

$$y_1 = e^{m_1 x}$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a \Rightarrow m_2 = (-b - \sqrt{4ac - b^2} i)/2a = \alpha - i\beta$$

$$y_2 = e^{m_2 x}$$

$$b^2 - 4ac < 0 \quad \text{Conjugate complex } m_1, m_2$$

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

Pick **two** homogeneous solution

$$y_1 = [e^{(\alpha + i\beta)x} + e^{(\alpha - i\beta)x}]/2 = e^{\alpha x} \cos(\beta x) \quad (C_1 = +1/2, \quad C_2 = +1/2)$$

$$y_2 = [e^{(\alpha + i\beta)x} - e^{(\alpha - i\beta)x}]/2i = e^{\alpha x} \sin(\beta x) \quad (C_1 = +1/2i, \quad C_2 = -1/2i)$$

$$y = C_3 e^{\alpha x} \cos(\beta x) + C_4 e^{\alpha x} \sin(\beta x) = e^{\alpha x} (C_3 \cos(\beta x) + C_4 \sin(\beta x))$$

Advanced Eng Math in plain view

ODE

2nd Order ODE : 2P.pdf

: Linear Equation (1A.pdf)

: Reduction of Order (....)

Complex numbers

Zill & Wright Sec 3.3, 3.2

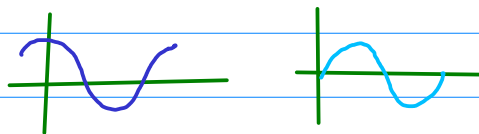
Complex Number Background

Taylor Series

Euler Formula

Taylor Series

$$f(x) = \cos x$$



$$f'(x) = -\sin x$$

$$f'(0) = -\sin 0 = 0$$

$$f^{(2)}(x) = -\cos x$$

$$f^{(2)}(0) = -\cos 0 = -1$$

$$f^{(3)}(x) = +\sin x$$

$$f^{(3)}(0) = +\sin 0 = 0$$

$$f^{(4)}(x) = +\cos x$$

$$f^{(4)}(0) = +\cos 0 = +1$$

$$f^{(5)}(x) = -\sin x$$

$$f^{(5)}(0) = -\sin 0 = 0$$

$$f^{(6)}(x) = -\cos x$$

$$f^{(6)}(0) = -\cos 0 = -1$$

$$f^{(7)}(x) = +\sin x$$

$$f^{(7)}(0) = +\sin 0 = 0$$

$$f^{(8)}(x) = +\cos x$$

$$f^{(8)}(0) = +\cos 0 = +1$$

⋮

⋮

$x=0$ 지점에서 Taylor series

$$f(x) = f(0) + \frac{f^{(1)}(0)}{1!} x^1 + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \dots$$

$$\cos x = +1 + \frac{0}{1!} x^1 + \frac{-1}{2!} x^2 + \frac{0}{3!} x^3 + \frac{-1}{4!} x^4 + \dots$$

$$f'(0) = -\sin 0 = 0$$

$$f^{(2)}(0) = -\cos 0 = -1$$

$$f^{(3)}(0) = +\sin 0 = 0$$

$$f^{(4)}(0) = +\cos 0 = +1$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \dots$$

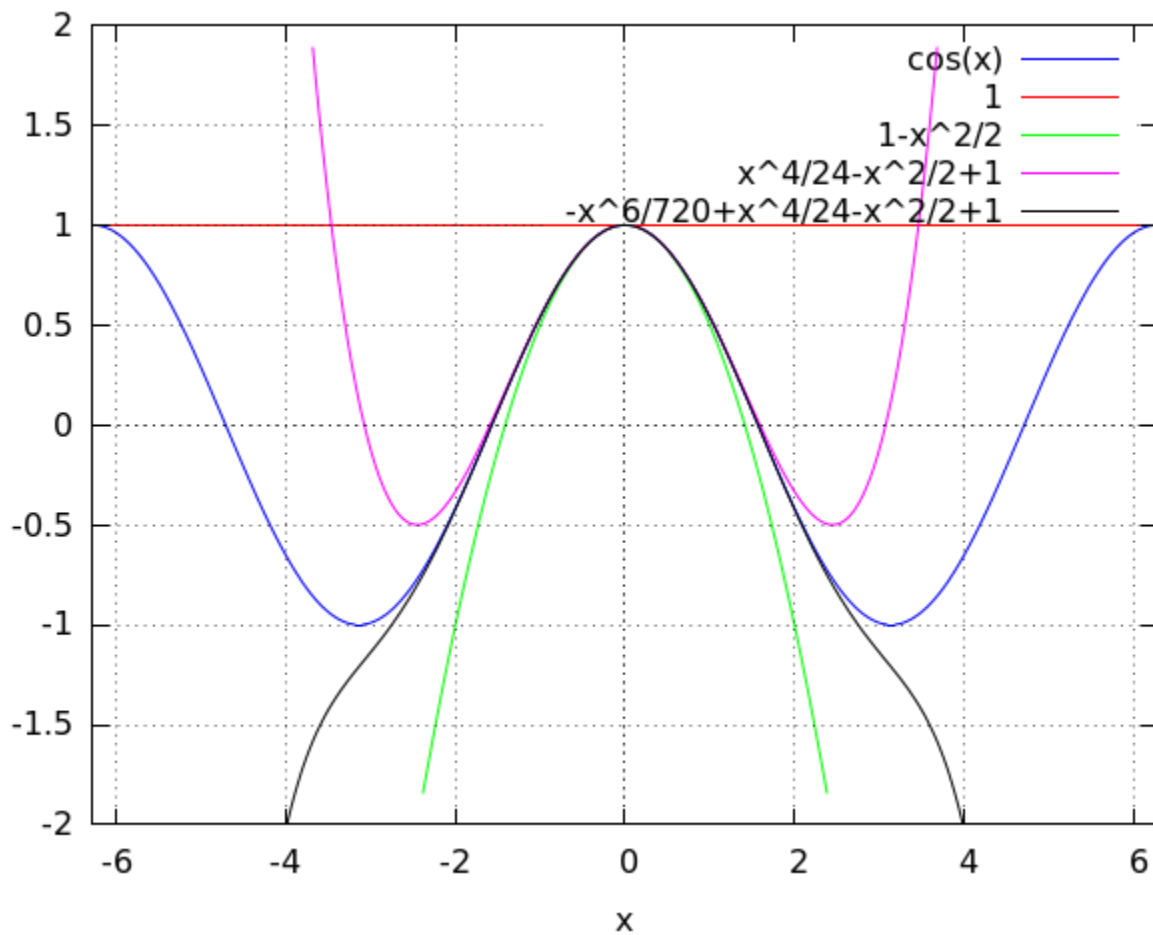
0 지점에서의

Series (2/4)

```

(%i1) f1(x) := 1;
(%o1) f1(x):=1
(%i2) f2(x) := f1(x) - x^2/factorial(2);
(%o2) f2(x):=f1(x)- $\frac{x^2}{2!}$ 
(%i3) f2(x);
(%o3)  $1-\frac{x^2}{2}$ 
(%i4) f3(x) := f2(x) + x^4 / factorial(4);
(%o4) f3(x):=f2(x)+ $\frac{x^4}{4!}$ 

```



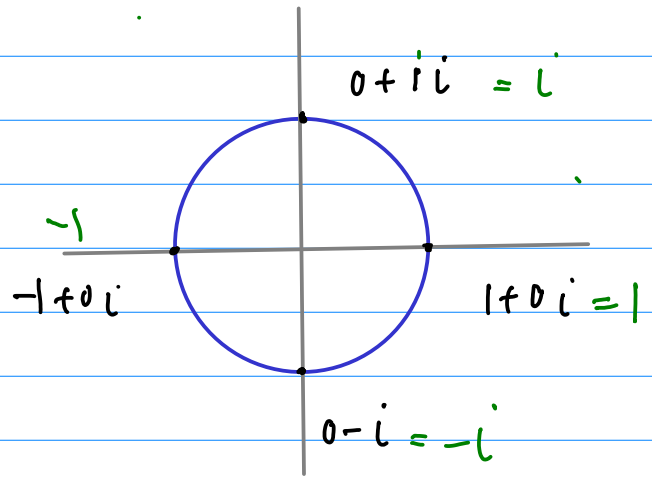
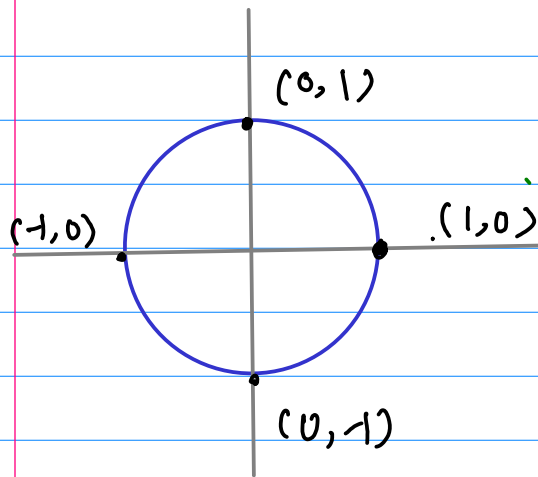
Taylor Series Example

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

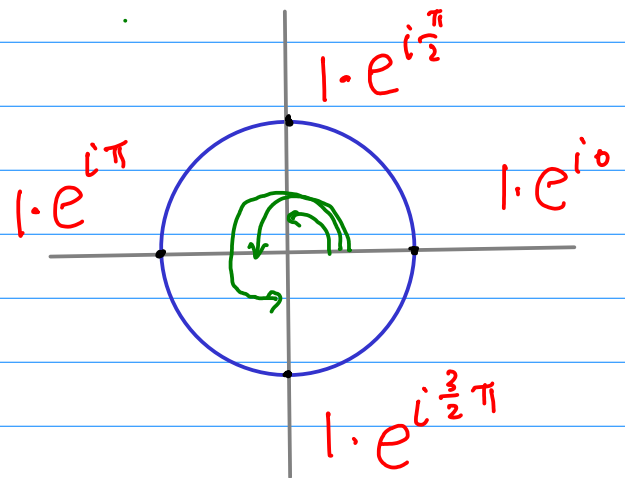
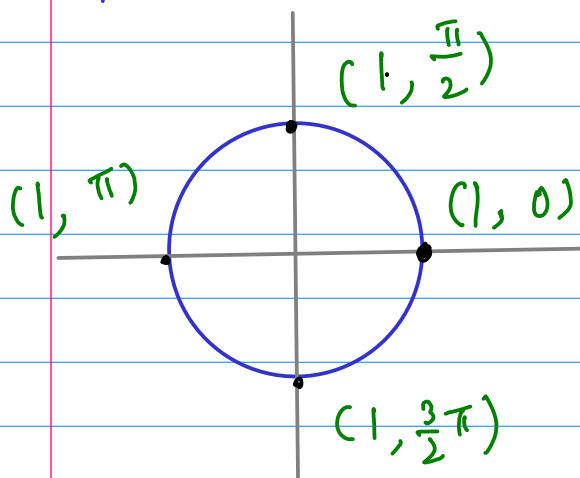
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

직교좌표



극좌표



$$x + yi = r e^{i\theta}$$

$$\begin{aligned}
 e^{i0} &= \cos 0 & + i \sin 0 \\
 e^{i\frac{\pi}{2}} &= \cos \frac{\pi}{2} & + i \sin \frac{\pi}{2} \\
 e^{i\pi} &= \cos \pi & + i \sin \pi \\
 e^{i\frac{3}{2}\pi} &= \cos \frac{3}{2}\pi & + i \sin \frac{3}{2}\pi
 \end{aligned}$$

Euler Formula

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

hw 20150626

Paul's online math note

Differential Equation

Second Order :

Two distinct real roots

Complex conjugate roots

repeated roots

Reduction of Orders

Undetermined Coefficients

$$a y'' + b y' + c y = 0$$

homogeneous eq



y_h : homogeneous solution

$$a y'' + b y' + c y = \underbrace{g(x)}_{\text{let's try}}$$

y_p : particular solution

① 3.4 Undetermined Coefficients ✓

② 3.5 Variation of Parameters ✓

③ . Green's function.

finding y_p : undetermined coefficients

$$(e^x)' = e^x$$

$$(e^x)'' = e^x$$

$$(e^x)'' - 2(e^x)' + (e^x) = 0$$

for all x

$$y'' - 2y' + y = 0$$

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1$$

$$y_h = c_1 e^x + c_2 x e^x$$

\downarrow repeated root \uparrow

$$y'' - 2y' + y = e^{3x}$$

$$y = A e^{3x}$$

$$y' = 3A e^{3x} \quad y'' = 9A e^{3x}$$

$$9A e^{3x} - 2 \cdot 3A e^{3x} + A e^{3x} = e^{3x}$$

$$(9A - 6A + A) e^{3x} = e^{3x}$$

$$4A = 1 \quad A = \frac{1}{4}$$

$$y_p = \frac{1}{4} e^{3x}$$

When the multiplication rule must be considered

$$a y'' + b y' + c y = e^{kt}$$

$$a m^2 + b m + c = 0$$

real distinct m_1, m_2

$$e^{m_1 t}$$

$$e^{m_2 t}$$

$$A e^{kt}$$

$$k = m_1, k = m_2$$

if e^{kt} is already in y_h

When the multiplication rule must be considered

$$a y'' + b y' + c y = e^{\alpha t} \cos(\beta t)$$

complex conjugate roots
 $\{ e^{m_1 t}, e^{m_2 t} \}$

$$\{ e^{\alpha t + i\beta t}, e^{\alpha t - i\beta t} \}$$

$$\{ e^{\alpha t} \cos(\beta t), e^{\alpha t} \sin(\beta t) \}$$

$$e^{\alpha t} \sin(\beta t)$$

already in y_h

$$\alpha = 0$$

$$a y'' + b y' + c y = \cos(\beta t)$$

complex conjugate roots
 $\{ e^{m_1 t}, e^{m_2 t} \}$

$$\{ e^{i\beta t}, e^{-i\beta t} \}$$

$$\{ \cos(\beta t), \sin(\beta t) \}$$

$$\sin(\beta t)$$

already in y_h

When the multiplication rule must be considered

Zill & Wright 3.4 Ex 2.13, 2.15

$$y'' + 4y = 3 \sin 2x$$

$$A \sin 2x + B \cos 2x \quad \times$$

$$m^2 + 4 = 0$$

$$m = \pm 2i \quad (\alpha=0)$$

$$c_1 e^{2ix} + c_2 e^{-2ix}$$

$$c_3 \cos(2x) + c_4 \sin(2x)$$

$$y_p = x(A \sin 2x + B \cos 2x)$$

~~$Cx + D$~~

$$y'' + y = 3x \sin x$$

$$m^2 + 1 = 0$$

$$m = +i, -i$$

$$c_1 \cos x + c_2 \sin x$$

(1차 다항식) \times ($\sin x$)

$$(Ax + B) \sin x + (Cx + D) \cos x$$

already in y_h

$$x \sin x$$

$$\underline{\sin x}$$

$$x \cos x$$

$$\underline{\cos x}$$

$$x(Ax + B) \sin x + x(Cx + D) \cos x$$

$$4y'' + 6y' + 17y = e^{-2t} \sin\left(\frac{t}{2}\right) + 6t \cos\left(\frac{t}{2}\right)$$

$$4y'' + 6y' + 17y = 0$$

$$4m^2 + 2 \cdot 8m + 17 = 0$$

$$m = \frac{-8 \pm \sqrt{64 - 68}}{4}$$

$$m = -2 \pm \frac{1}{2}i \iff = \frac{-8 \pm \sqrt{-4}}{4} = \frac{-8 \pm \sqrt{4}i}{4}$$

$$y_c = c_1 e^{(-2 + \frac{1}{2}i)t} + c_2 e^{(-2 - \frac{1}{2}i)t}$$

$$= c_3 e^{-2t} \cos\left(\frac{1}{2}t\right) + c_4 e^{-2t} \sin\left(\frac{1}{2}t\right)$$

$$4y'' + 6y' + 17y = e^{-2t} \sin\left(\frac{t}{2}\right) + 6t \cos\left(\frac{t}{2}\right)$$

$$y_p = e^{-2t} \left[A \sin\left(\frac{t}{2}\right) + B \cos\left(\frac{t}{2}\right) \right] + (Ct + D) \sin\left(\frac{t}{2}\right) + (Et + F) \cos\left(\frac{t}{2}\right)$$

already in y_h

$$y_p = t e^{-2t} \left[A \sin\left(\frac{t}{2}\right) + B \cos\left(\frac{t}{2}\right) \right] + (Ct + D) \sin\left(\frac{t}{2}\right) + (Et + F) \cos\left(\frac{t}{2}\right)$$

$$4y'' + 6y' + 17y = \underline{e^{-2t} \sin\left(\frac{t}{2}\right)} + \underline{6t \cos\left(\frac{t}{2}\right)}$$

$$y_p = \underline{e^{-2t} \left[A \sin\left(\frac{t}{2}\right) + B \cos\left(\frac{t}{2}\right) \right]} + (ct + D) \sin\left(\frac{t}{2}\right) + (Et + F) \cos\left(\frac{t}{2}\right)$$

already in y_h

$$\left\{ \begin{array}{lll} \underline{e^{-2t} \sin\left(\frac{t}{2}\right)} & t \sin\left(\frac{t}{2}\right) & \sin\left(\frac{t}{2}\right) \\ \underline{e^{-2t} \cos\left(\frac{t}{2}\right)} & t \cos\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\right) \end{array} \right\}$$

$$\begin{aligned} y_c &= c_1 e^{(-2 + \frac{1}{2}i)t} + c_2 e^{(-2 - \frac{1}{2}i)t} \\ &= e^{-2t} \left(c_1 e^{(\frac{1}{2}i)t} + c_2 e^{(-\frac{1}{2}i)t} \right) \\ &= e^{-2t} \left(c_3 \cos\left(\frac{1}{2}t\right) + c_4 \sin\left(\frac{1}{2}t\right) \right) \end{aligned}$$

$$y_p = \underline{t e^{-2t} \left[A \sin\left(\frac{t}{2}\right) + B \cos\left(\frac{t}{2}\right) \right]} + (ct + D) \sin\left(\frac{t}{2}\right) + (Et + F) \cos\left(\frac{t}{2}\right)$$

$$\left\{ \begin{array}{lll} \underline{t e^{-2t} \sin\left(\frac{t}{2}\right)} & t \sin\left(\frac{t}{2}\right) & \sin\left(\frac{t}{2}\right) \\ \underline{t e^{-2t} \cos\left(\frac{t}{2}\right)} & t \cos\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\right) \end{array} \right\}$$

$$y_p = t e^{-2t} \left[A \sin\left(\frac{t}{2}\right) + B \cos\left(\frac{t}{2}\right) \right] + (Ct + D) \sin\left(\frac{t}{2}\right) + (Et + F) \cos\left(\frac{t}{2}\right)$$

$$y_p: \left\{ \begin{array}{l} t e^{-2t} \sin\left(\frac{t}{2}\right), \\ t e^{-2t} \cos\left(\frac{t}{2}\right), \end{array} \quad \begin{array}{l} t \sin\left(\frac{t}{2}\right), \\ t \cos\left(\frac{t}{2}\right), \end{array} \quad \begin{array}{l} \sin\left(\frac{t}{2}\right), \\ \cos\left(\frac{t}{2}\right) \end{array} \right\}$$

$$y_h: \left\{ \begin{array}{l} e^{-2t} \sin\left(\frac{t}{2}\right), \\ e^{-2t} \cos\left(\frac{t}{2}\right) \end{array} \right\}$$

$$y_c = c_1 \overset{y_1}{\boxed{e^{(-2 + \frac{1}{2}i)t}}} + c_2 \overset{y_2}{\boxed{e^{(-2 - \frac{1}{2}i)t}}}$$

$$\boxed{4y'' + 16y' + 17y = 0} \begin{matrix} \leftarrow y_1 \\ \leftarrow y_2 \end{matrix}$$

$$\begin{cases} y_3 = \frac{1}{2} y_1 + \frac{1}{2} y_2 \\ y_4 = \frac{1}{2i} y_1 - \frac{1}{2i} y_2 \end{cases}$$

if y_1 & y_2 are solutions, then so are y_3 & y_4

$$\begin{aligned} y_3 &= \frac{1}{2} e^{(-2 + \frac{1}{2}i)t} + \frac{1}{2} e^{(-2 - \frac{1}{2}i)t} && \text{a solution} \\ &= e^{2t} \left(\frac{e^{i\frac{t}{2}} + e^{-i\frac{t}{2}}}{2} \right) = \underline{e^{2t} \cos\left(\frac{t}{2}\right)} \end{aligned}$$

$$\begin{aligned} y_4 &= \frac{1}{2i} e^{(-2 + \frac{1}{2}i)t} - \frac{1}{2i} e^{(-2 - \frac{1}{2}i)t} && \text{a solution} \\ &= e^{2t} \left(\frac{e^{i\frac{t}{2}} - e^{-i\frac{t}{2}}}{2i} \right) = \underline{e^{2t} \sin\left(\frac{t}{2}\right)} \end{aligned}$$

$$y_c = c_3 \overset{y_3}{\boxed{e^{2t} \cos\left(\frac{t}{2}\right)}} + c_4 \overset{y_4}{\boxed{e^{2t} \sin\left(\frac{t}{2}\right)}}$$

Linear 2nd order with constant coefficients

$$* ay'' + by' + cy = g(x)$$

general solution

$$\underline{y_h} + y_p$$

Finding y_h

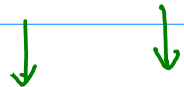
$$ay'' + by' + cy = 0$$

$$am^2 + bm' + cm^0 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$D > 0$

$$m_1, m_2$$



$$e^{m_1 x}, e^{m_2 x}$$



$$c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

linear combination

$D = 0$

$$m_1 = m_2$$



$$e^{m_1 x}, x e^{m_2 x}$$



$$c_1 e^{m_1 x} + c_2 x e^{m_2 x}$$

linear combination

Finding y_h

$$a y'' + b y' + c y = 0$$

$$a m^2 + b m + c = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$D < 0$

$$\begin{array}{cc} m_1, & m_2 \\ \downarrow & \downarrow \\ e^{m_1 x}, & e^{m_2 x} \\ \underbrace{\hspace{10em}} \\ \Downarrow \end{array}$$

$$C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

linear combination

$$m_1 = \alpha + i\beta \quad m_2 = \alpha - i\beta$$

$$\textcircled{1} C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

$$\textcircled{2} e^{\alpha x} [C_1 e^{i\beta x} + C_2 e^{-i\beta x}]$$

$$\textcircled{3} e^{\alpha x} [C_3 \cos(\beta x) + C_4 \sin(\beta x)]$$

Finding y_p Example

$$(a) \quad \underline{\underline{y'' + 3y' - 28y = 0}}$$

$$m^2 + 3m - 28 = (m+7)(m-4) = 0$$

$$m = -7, 4$$

$$\downarrow$$
$$c_1 e^{-7t} + c_2 e^{4t}$$

$$(b) \quad \underline{\underline{y'' - 100y = 0}}$$

$$m^2 - 100m^0 = 0$$

$$m^2 - 100 = 0$$

$$m = 10 \quad -10$$

$$c_1 e^{+10t} + c_2 e^{-10t}$$

$$(c) \quad \underline{\underline{4y'' + y = 0}}$$

$$4m^2 + m^0 = 0$$

$$4m^2 + 1 = 0$$

$$m^2 = -\frac{1}{4}$$

$$m = \frac{i}{2}$$

$$-\frac{i}{2}$$

$$c_1 e^{\frac{i}{2}t} + c_2 e^{-\frac{i}{2}t}$$

$$c_3 \underline{\cos\left(\frac{t}{2}\right)} + c_4 \underline{\sin\left(\frac{t}{2}\right)}$$

$$(d) \quad \underline{\underline{4y'' + 16y' + 17y = 0}}$$

$$4m^2 + 16m + 17 = 0 \quad \begin{array}{cc} 2 & 17 \\ 2 & 2 \end{array}$$

$$4(m^2 + 4m + 4) + 1 = 0$$

$$4(m+2)^2 + 1 = 0$$

$$(m+2)^2 = -\frac{1}{4}$$

$$(m+2) = \pm \frac{i}{2}$$

$$m = -2 + \frac{i}{2} \quad \cdot \quad -2 - \frac{i}{2}$$

$$(1) \quad C_1 e^{(-2 + \frac{i}{2})t} + C_2 e^{(-2 - \frac{i}{2})t}$$

$$C_1 e^{-2t} e^{+\frac{i}{2}t} + C_2 e^{-2t} e^{-\frac{i}{2}t}$$

$$(2) \quad e^{-2t} \left(C_1 e^{+\frac{i}{2}t} + C_2 e^{-\frac{i}{2}t} \right)$$

$$(3) \quad e^{-2t} \left(C_3 \cos\left(\frac{t}{2}\right) + C_4 \sin\left(\frac{t}{2}\right) \right)$$

$$(c) \quad \underline{\underline{y'' + 8y' + 16y = 0}}$$

$$m^2 + 8m + 16 = (m+4)^2 = 0$$

$$m = -4$$

↓

$$c_1 \underbrace{e^{-4t}} + c_2 t \cdot \underbrace{e^{-4t}}$$

3.4. Undetermined ...

Exercise

9

$$y'' - y' = -3$$

$$m^2 - m = 0$$

$$m(m-1) = 0$$

$$m=0,$$

$$1$$

$$e^{0 \cdot x}$$

$$e^x$$

$$y_h = C_1 + C_2 e^x$$

$$y_p = A$$

Already in y_h

$$y_p = xA$$

$$y' = A \quad y'' = 0$$

$$-A = -3$$

$$y_p = -3x$$

$$\textcircled{1} \quad y'' - y' + \frac{1}{4}y = 3 + e^{\frac{x}{2}}$$

$$m^2 - m + \frac{1}{4} = 0$$

$$m^2 - 2 \cdot \frac{1}{2}m + \left(\frac{1}{2}\right)^2 = 0$$

$$\left(m - \frac{1}{2}\right)^2 = 0$$

$$m = \frac{1}{2}$$

↓
 $e^{\frac{1}{2}x}$

$$y_p = A + B e^{\frac{x}{2}}$$

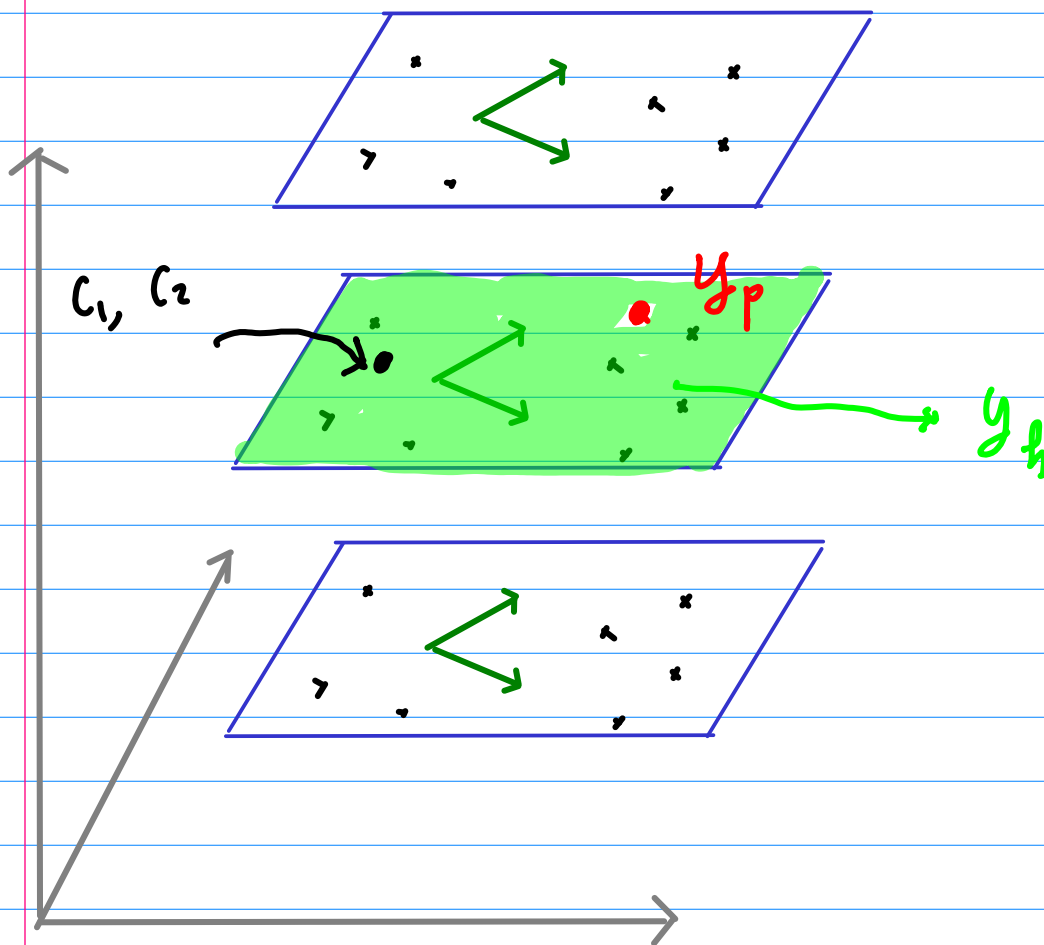
$$y_p = A + B x e^{\frac{x}{2}}$$

$$y_p = A + B x^2 e^{\frac{x}{2}}$$

$$y = C_1 e^{\frac{1}{2}x} + C_2 x e^{\frac{1}{2}x}$$

$$a y'' + b y' + c y = 0$$

$$y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

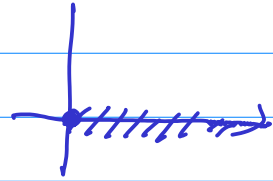


to determine c_1 & c_2

we need two additional conditions

IVP (Initial Value Problem)

$$a y'' + b y' + c y = g(x)$$



initial
value
(conditions)

$$\begin{cases} y(0) = k_0 \\ y'(0) = k_1 \end{cases}$$

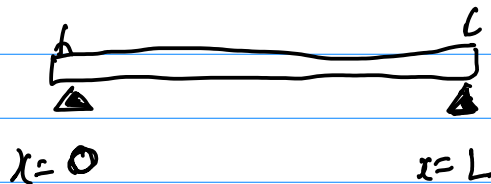
$$x > 0$$

$$\begin{cases} y(x_0) = k_0 \\ y'(x_0) = k_1 \end{cases}$$

$$x > x_0$$

BVP (Boundary Value Problem)

$$a y'' + b y' + c y = g(x)$$



Boundary
value

$x=0$

$$\left\{ \begin{array}{l} y(0) \\ y'(0) \end{array} \right.$$

$$\left\{ \begin{array}{l} y(0) \\ y'(0) \end{array} \right.$$

$$\left\{ \begin{array}{l} y(0) \\ y'(0) \end{array} \right.$$

$$\left\{ \begin{array}{l} y'(0) \\ y'(0) \end{array} \right.$$

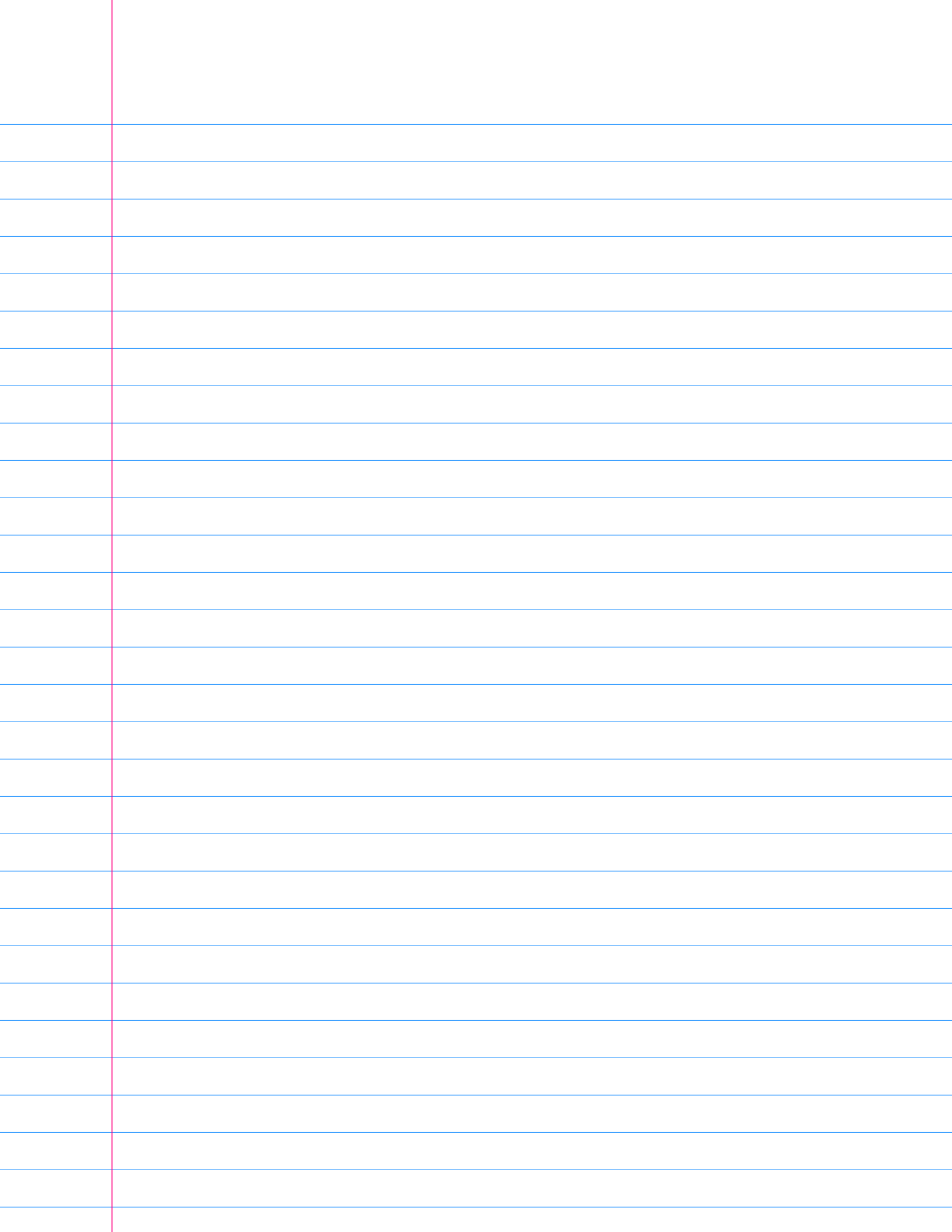
$x=L$

$$\left\{ \begin{array}{l} y(L) \\ y'(L) \end{array} \right.$$

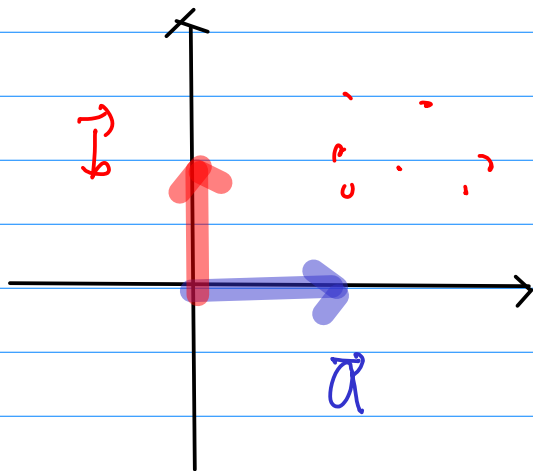
$$\left\{ \begin{array}{l} y(L) \\ y'(L) \end{array} \right.$$

$$\left\{ \begin{array}{l} y'(L) \\ y'(L) \end{array} \right.$$

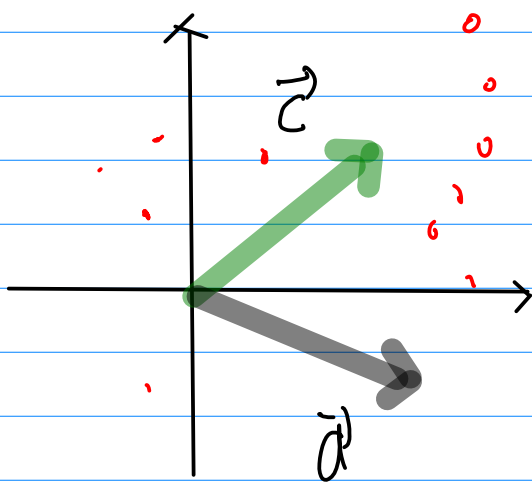
$$\left\{ \begin{array}{l} y'(L) \\ y'(L) \end{array} \right.$$



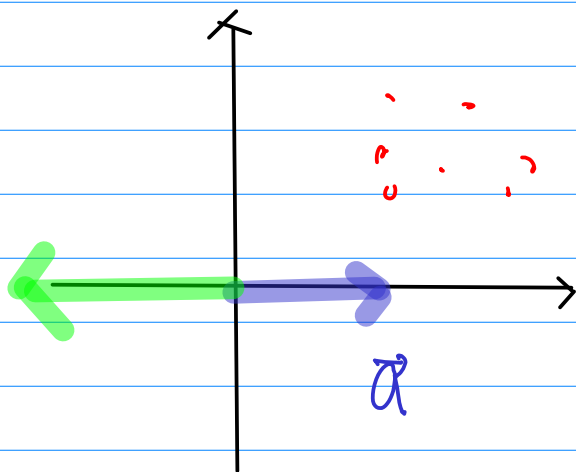
Linear Independent Vectors



$$c_1 \vec{a} + c_2 \vec{b}$$

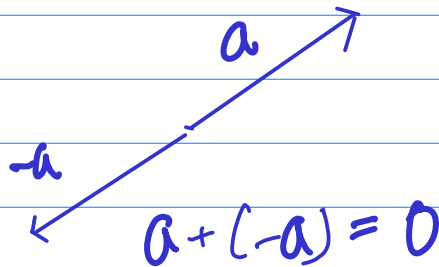
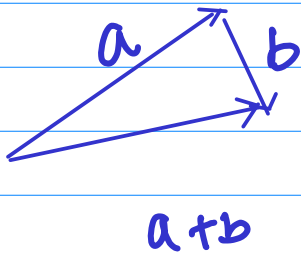


$$c_3 \vec{c} + c_4 \vec{d}$$



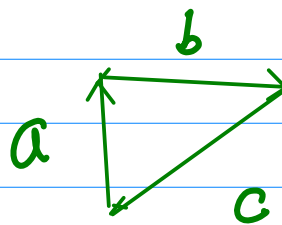
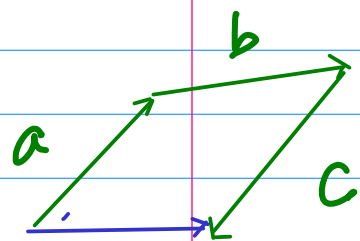
Second Order Differential Equation (2.P.pdf)

- Linear Independence
- Wronskian



2개의 Vec
= 다른 2 vector

Linear
Comb.



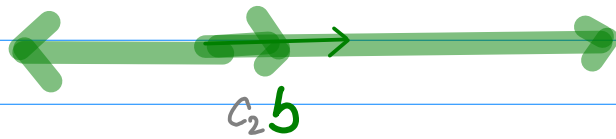
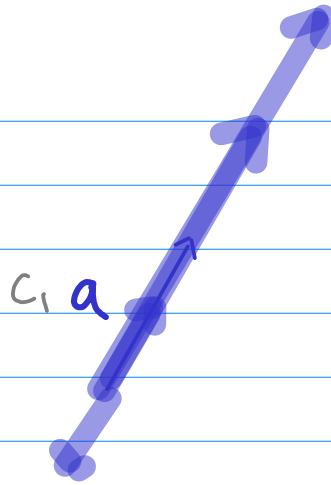
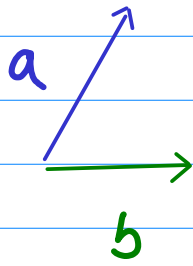
$$\begin{aligned} a &= -b - c \\ b &= -a - c \\ c &= -a - b \end{aligned}$$

$$0 \quad 0 \quad 0$$

$$1 \quad 1$$

$$|a + |b + |c = 0$$

~~Linearly
indep~~



$$c_1 \vec{a} + c_2 \vec{b}$$

a, b 의 선형결합
linear combination

\vec{a} 는 \vec{b} 를 만들 수 없고

\vec{b} 도 \vec{a} 를 만들 수 없다

a 와 b 는 서로 독립

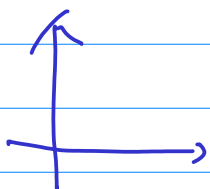
다른 종류의 벡터들이야

$$c_1 \vec{a} + c_2 \vec{b} = \vec{0} \text{ 일}$$

만족시키는데 c_1, c_2 는 $c_1 = 0, c_2 = 0$ all zero case

\Leftrightarrow \vec{a} 와 \vec{b} 는 서로 linear independent

2차원 평면의 다른 모든 vector들은 $c_1 \vec{a} + c_2 \vec{b}$ 로 표현 가능



n 개 벡터의 linear independent

\mathbb{R}^n

n 개 $\left\langle \begin{array}{l} \text{함수} \\ x \text{의 식} \end{array} \right\rangle$ linear independent \rightarrow

n -th order
ODE

2 차 미분 방정식

$$y'' = g(x, y, y')$$

\rightarrow max 2 개 linear independent solution을 찾아야 함

3 차 미분 방정식

$$y^{(3)} = g(x, y, y', y'')$$

\rightarrow max 3 개 linear independent solution을 찾아야 함

4 차 미분 방정식

$$y^{(4)} = g(x, y, y', y'', y^{(3)})$$

\rightarrow max 4 개 linear independent solution을 찾아야 함

* Wronskian : 함수가 linearly independent ?

$$y_1 = e^{3x}$$

$$y_2 = e^{-3x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & e^{-3x} \\ (e^{3x})' & (e^{-3x})' \end{vmatrix} = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix}$$
$$= -3 - 3 = -6 \neq 0$$

linearly independent

$$y_1 = e^x, \quad y_2 = e^{2x}, \quad y_3 = e^{3x}$$

$$W = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} \neq 0$$

$y_1 = e^x, \quad y_2 = e^{2x}, \quad y_3 = e^{3x}$ are linearly independent

$$y_h = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

한글 y_1, y_2, y_3 linearly independent?

$y_1(x), y_2(x), y_3(x)$

Form a square Matrix

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix}$$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$W \neq 0$: y_1, y_2, y_3 linearly independent !

$W = 0$: y_1, y_2, y_3 linearly ~~independent~~ !

p 4 n 3

determinant

matrix

$$(11) \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}$$

$$(12) \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & -\frac{4}{3} \end{bmatrix}$$

$$(13) \begin{bmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{bmatrix}$$

$$(14) \begin{bmatrix} -3-\lambda & -4 \\ -2 & 5-\lambda \end{bmatrix}$$

determinant

$$(11) \begin{vmatrix} 3 & 5 \\ -1 & 4 \end{vmatrix}$$

$$(12) \begin{vmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & -\frac{4}{3} \end{vmatrix}$$

$$(13) \begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix}$$

$$(14) \begin{vmatrix} -3-\lambda & -4 \\ -2 & 5-\lambda \end{vmatrix}$$

$$(15) \begin{vmatrix} 0 & 2 & 0 \\ 3 & 0 & 1 \\ 0 & 5 & 8 \end{vmatrix}$$

$$(16) \begin{vmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

p 495 예제 1

$$3x_1 + 2x_2 + x_3 = 7$$

$$x_1 - x_2 + 3x_3 = 3$$

$$5x_1 + 4x_2 - 2x_3 = 1$$

1469 Ex 11.21

$$\begin{pmatrix} 3x_1 + 2x_2 + x_3 = 7 \\ 1x_1 - 1x_2 + 3x_3 = 3 \\ 5x_1 + 4x_2 - 2x_3 = 1 \end{pmatrix} \quad \text{a unique solution}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & 3 \\ 5 & 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

①

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & 3 \\ 5 & 4 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

using
inverse matrix

②

Cramer's rule

$$\Delta = \begin{vmatrix} 3 & 2 & 1 \\ 1 & -1 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 13$$

$$\frac{1}{\Delta} \begin{vmatrix} 7 & 2 & 1 \\ 3 & -1 & 3 \\ 1 & 4 & -2 \end{vmatrix} = \frac{1}{13} (-39) = x_1$$

$$\frac{1}{\Delta} \begin{vmatrix} 3 & 7 & 1 \\ 1 & 3 & 3 \\ 5 & 1 & -2 \end{vmatrix} = \frac{1}{13} (18) = x_2$$

$$\frac{1}{\Delta} \begin{vmatrix} 3 & 2 & 7 \\ 1 & -1 & 3 \\ 5 & 4 & 1 \end{vmatrix} = \frac{1}{13} (52) = x_3$$

* 2nd Order, Linear, Homogeneous, constant coefficients

$$a[y''] + b[y'] + c[y] = 0$$

$$am^2 + bm + c = 0$$

$$y = e^{mx}$$

$$D < 0$$

$$m_1 = \alpha + i\beta$$

$$m_2 = \alpha - i\beta$$

$$c_1 e^{(\alpha + i\beta)t} + c_2 e^{(\alpha - i\beta)t}$$

$$W(e^{(\alpha + i\beta)t}, e^{(\alpha - i\beta)t}) \neq 0$$

linearly independent. \rightarrow Fundamental Set of solution

$c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$ of an solution이 된다

$$\frac{1}{2} \left(e^{(\alpha + i\beta)t} + e^{(\alpha - i\beta)t} \right) = e^{\alpha t} \cos(\beta t)$$

$c_1 = \frac{1}{2i}$, $c_2 = \frac{-1}{2i}$ of an solution이 된다

$$\frac{1}{2i} \left(e^{(\alpha + i\beta)t} - e^{(\alpha - i\beta)t} \right) = e^{\alpha t} \sin(\beta t)$$

$$C_1 e^{(\alpha+i\beta)t} + C_2 e^{(\alpha-i\beta)t}$$

General Solution

$C_1 = \frac{1}{2}$, $C_2 = \frac{1}{2}$ 일 때 solution 이 된다

$$\frac{1}{2} \left(e^{(\alpha+i\beta)t} + e^{(\alpha-i\beta)t} \right) = e^{\alpha t} \cos(\beta t)$$

$C_1 = \frac{1}{2i}$, $C_2 = \frac{-1}{2i}$ 일 때 solution 이 된다

$$\frac{1}{2i} \left(e^{(\alpha+i\beta)t} - e^{(\alpha-i\beta)t} \right) = e^{\alpha t} \sin(\beta t)$$

$e^{\alpha t} \cos(\beta t)$ 은 solution 이다

$e^{\alpha t} \sin(\beta t)$ 은 solution 이다

$C_3 e^{\alpha t} \cos(\beta t) + C_4 e^{\alpha t} \sin(\beta t)$ 은 solution 이다.

$$W(e^{\alpha t} \cos(\beta t), e^{\alpha t} \sin(\beta t)) \neq 0$$

linearly independent. \rightarrow Fundamental set of solution

$$C_3 e^{\alpha t} \cos(\beta t) + C_4 e^{\alpha t} \sin(\beta t)$$

General Solution

$$\begin{aligned}
 & \begin{vmatrix} e^{(\alpha+i\beta)x} & e^{(\alpha-i\beta)x} \\ (e^{(\alpha+i\beta)x})' & (e^{(\alpha-i\beta)x})' \end{vmatrix} \\
 &= \begin{vmatrix} e^{(\alpha+i\beta)x} & e^{(\alpha-i\beta)x} \\ (\alpha+i\beta)e^{(\alpha+i\beta)x} & (\alpha-i\beta)e^{(\alpha-i\beta)x} \end{vmatrix} \\
 &= (\alpha-i\beta)e^{2\alpha x} - (\alpha+i\beta)e^{2\alpha x} \\
 &= (-i2\beta)e^{2\alpha x} \neq 0
 \end{aligned}$$

these are
fundamental
solution

$$w \neq 0$$

these also
fundamental
solution

$$w \neq 0$$

$$\begin{aligned}
 & \begin{vmatrix} e^{\alpha x} \cos(\beta x) & e^{\alpha x} \sin(\beta x) \\ (e^{\alpha x} \cos(\beta x))' & (e^{\alpha x} \sin(\beta x))' \end{vmatrix} = \begin{vmatrix} e^{\alpha x} \cos(\beta x) & e^{\alpha x} \sin(\beta x) \\ (\alpha e^{\alpha x} \cos(\beta x) - \beta e^{\alpha x} \sin(\beta x)) & (\alpha e^{\alpha x} \sin(\beta x) + \beta e^{\alpha x} \cos(\beta x)) \end{vmatrix} \\
 &= e^{\alpha x} \cos(\beta x) (\alpha e^{\alpha x} \sin(\beta x) + \beta e^{\alpha x} \cos(\beta x)) - e^{\alpha x} \sin(\beta x) (\alpha e^{\alpha x} \cos(\beta x) - \beta e^{\alpha x} \sin(\beta x)) \\
 &= e^{\alpha x} \cos(\beta x) \beta e^{\alpha x} \cos(\beta x) + e^{\alpha x} \sin(\beta x) \beta e^{\alpha x} \sin(\beta x) = \beta e^{2\alpha x} (\cos^2(\beta x) + \sin^2(\beta x)) = \beta e^{2\alpha x} \neq 0
 \end{aligned}$$

$$\beta < 0 \Rightarrow \beta \neq 0$$

wikiversity.org

ch4

Second Order Differential Equation
Variation of Parameters (4.A.pdf)

- ch1

Linear Algebra
Determinants (A.pdf)

3x3 matrix invert

Paul's online math note

158p-160p examples

→ Variation of parameters

Reduction of Orders

Undertermined Coefficients [7]

Finding a second solution from y_1

$$a y'' + b y' + c y = 0$$

2nd Order \rightarrow there are 2 fundamental solutions
(i.e. linearly independent)

* If we know one solution y_1 ,
then the other solution can be found
by assuming $y_2 = u(x) \cdot y_1$

* Repeated root ($D=0$)
we know $y_1 = e^{-\frac{b}{2a}x}$ solution.
then the other solution can be found
by assuming $y_2 = u(x) \cdot y_1$
 \Downarrow
 $y_2 = x \cdot y_1$

Finding y_p from y_h

particular
solution

homogeneous
solution

$$a \boxed{y''} + b \boxed{y'} + c \boxed{y} = g(x)$$

* $(y_1), (y_2) \dots$ homogeneous solutions known

finding a particular solution (y_p)

$$\left. \begin{aligned} a y_1'' + b y_1' + c y_1 &= 0 \leftrightarrow y_1 \\ a y_2'' + b y_2' + c y_2 &= 0 \leftrightarrow y_2 \end{aligned} \right\} \underline{y_h = c_1 y_1 + c_2 y_2}$$

then the particular solution y_p can be found

by assuming $\underline{y_p = a_1 y_1 + a_2 y_2}$

* Finding a second solution : Reduction of Orders

given $y_1 = e^x$ \Rightarrow $y_2 = u(x) \cdot e^x$ Suppose $u(x)?$

$$y'' - y = 0$$

$m^2 - 1 = 0$
 $m = \pm 1$
 e^{+x}, e^{-x}

$$y_2' = u' e^x + u \cdot e^x$$

$$y_2'' = u'' e^x + u' e^x + u' e^x + u e^x$$

$$= u'' e^x + 2u' e^x + u e^x$$

$$y_2'' - y_2 = (u'' e^x + 2u' e^x + u e^x) - (u e^x)$$

$$= u'' e^x + 2u' e^x = 0$$

$$e^x (u'' + 2u') = 0$$

$e^x \neq 0$ $u'' + 2u' = 0$

y_2 solution exist condition $\rightarrow u(x)$.

Reduction of Orders

\Downarrow \Downarrow \Downarrow
 $u'' + 2u' = 0$
 \Downarrow \Downarrow \Downarrow
 $(w)' + 2(w) = 0$

Reduction of Orders

* $u'' + 2u' + u = 0$

\Downarrow \Downarrow \Downarrow

$(w)' + 2(w) + ? = 0$

Reduction of Orders

given $y_1 = e^x$

$$y'' - y = 0$$



$y_2 = u(x) \cdot e^x$
 $u(x)?$

Suppose

z.zr $u'' + 2u' = 0$

l.zr $(w)' + 2(w) = 0$

$$w' = -2w$$

$$\frac{w'}{w} = -2$$

$$\int \frac{1}{w} \left(\frac{dw}{dx} dx \right) = \int -2 dx$$

$$\int \frac{1}{w} dw = \int -2 dx$$

$$\ln |w| = -2x + c$$

$$w = u'$$

$$|w| = e^{-2x+c}$$
$$= c e^{-2x}$$

$$w = c e^{-2x}$$

$$w = c e^{-2x}$$

$$u' = c e^{-2x}$$

$$u = \int c e^{-2x} dx$$

$$= -\frac{c_1}{2} e^{-2x} + c_2$$

$$y_1 = e^x$$

$$y_2 = e^{-x}$$

$$u = -\frac{c_1}{2} e^{-2x} + c_2$$

$$y_2 = u \cdot y_1$$

$$= \left(\frac{c_1}{2} e^{-2x} + c_2 \right) e^x$$

$$= \frac{c_1}{2} e^{-x} + c_2 e^x$$

$$y = c_1 e^x + c_2 e^{-x}$$

= 0 a, b, c
homogeneous, constant coefficients

* Finding a second solution

y_1 → y_2 ? y_1 라 다른 solution!

$y_2 = u y_1$ u(x) x's formula

$= \left(\frac{1}{x^4} + C \right) y_1$

$y_2 = \frac{1}{x^4} y_1 + C y_1$

$\neq y_1$

exclude!

$D = 0$ repeating solution

Finding another solution y_2 from the known y_1

Second Order EQ

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$y_1 = f(x)$ known solution
 $y_2 = u(x)f(x)$ another solution to be found

$$b^2 - 4ac = 0 \quad m_1 = m_2 = -\frac{b}{2a}$$

We know one solution

$$y_1(x) = e^{m_1 x} = e^{m_2 x} = e^{-\frac{b}{2a} x}$$

Suppose the other solution

$$y_2(x) = u(x)y_1(x) = u(x)e^{m_1 x}$$

Condition for $y_2(x)$ to be a solution

→ Find $u(x)$

$$e^{-\frac{b}{2a} x} \rightarrow u(x) \cdot e^{-\frac{b}{2a} x} \quad u(x) = ? \quad \dots \rightarrow \text{OK}$$

$$y'' + 4y' + 4y = 0$$

$$m^2 + 4m + 4 = (m+2)^2 = 0 \quad m = -2$$

$$y_1 = e^{-2x}$$

$$\Rightarrow y_2 = u \cdot e^{-2x} \quad \text{Assume}$$

$$y_2' = u' \cdot e^{-2x} - 2u \cdot e^{-2x}$$

$$y_2'' = u'' \cdot e^{-2x} - 2u' \cdot e^{-2x} - 2u' \cdot e^{-2x} + 4u \cdot e^{-2x}$$

$$= u'' \cdot e^{-2x} - 4u' \cdot e^{-2x} + 4u \cdot e^{-2x}$$

$$y'' + 4y' + 4y$$

$$= \left(u'' \cdot e^{-2x} - 4u' \cdot e^{-2x} + 4u \cdot e^{-2x} \right)$$

$$+ 4 \left(u' \cdot e^{-2x} - 2u \cdot e^{-2x} \right)$$

$$+ 4 \left(u \cdot e^{-2x} \right)$$

$$4u \cdot e^{-2x} (1 - 2 + 1) = 0$$

$$y'' + 4y' + 4y = u'' \cdot e^{-2x} - 4u' \cdot e^{-2x} + 4u' \cdot e^{-2x}$$

$$= u'' \cdot e^{-2x} = 0$$

$$u'' = 0 \Rightarrow u' = C_1 \Rightarrow u = C_1 x + C_2$$

$$y_1 = e^{-2x}$$

$$\begin{aligned} y_2 &= u \cdot e^{-2x} = (C_1 x + C_2) e^{-2x} \\ &= \underbrace{C_1 x e^{-2x}}_{y_2} + \underbrace{C_2 e^{-2x}}_{y_1} \end{aligned}$$

$$y_1 = e^{-2x}$$

$$y_2 = x \cdot e^{-2x}$$

* $(y_1), (y_2) \dots$ homogeneous solution given

Finding a particular solution

$$y_p = \underbrace{u_1 y_1 + u_2 y_2}$$

$$u_1(x) = ?$$

$$u_2(x) = ?$$

homogeneous solution y_1, y_2 and the particular

$$y_p = \left(\begin{array}{l} \boxed{\text{////}} + C_1 \\ \boxed{\text{////}} + C_2 \end{array} \right) \begin{array}{l} y_1 + \\ y_2 \end{array}$$

$$= \begin{array}{l} \boxed{\text{////}} y_1 + C_1 y_1 \\ \boxed{\text{////}} y_2 + C_2 y_2 \end{array} \begin{array}{l} y_1 \\ y_2 \end{array}$$

$\downarrow \quad \downarrow$
 $y_1 \quad y_2$

$$a y'' + b y' + c y = 0$$



Linear Eq with constant coefficient

$$a m^2 + b m + c = 0 \quad \text{aux}$$

$$m = m_1, m_2$$

$$c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

$$a(t) y'' + b(t) y' + c(t) y = 0$$

$$~~a m^2 + b m + c = 0~~$$

$$t y'' - (t+1) y' + y = t^2$$

$$\begin{cases} a(t) = t \\ b(t) = -t-1 \\ c(t) = 1 \end{cases}$$

~~constant coefficients~~

examples) . Variable coefficients

Cauchy-Euler Eq

$$(ax^2)y'' + (bx)y' + cy = 0 \quad (\text{Sec 3.6})$$

Bessel Eq.

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

Legendre Eq

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$y'' + p(t)y' + q(t)y = 0$$

$y_1, y_2 \dots$ linearly indep.

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

$$y = C_1 y_1(t) + C_2 y_2(t)$$

$t = t_0$

$$y_3 = C_1 y_1(t) + C_2 y_2(t)$$

$$y(t_0) = 0 \quad y'(t_0) = 1$$

C_1 & C_2 를
구할 수 있다

$$y_4 = C_1 y_1(t) + C_2 y_2(t)$$

$$y(t_0) = 1 \quad y'(t_0) = 0$$

C_1 & C_2 를
구할 수 있다

y_3, y_4 linear indep

$$\begin{vmatrix} y_3 & y_4 \\ y_3' & y_4' \end{vmatrix} \neq 0$$

3.2

exercise

11

$$x y'' + y' = 0 \quad y_1 = \ln x$$

15

$$(1-2x-x^2)y'' + 2(1+x)y' - 2y = 0$$

$$y_1 = x+1$$

y_1

↓

$$y_2 = u y_1$$

$$a y'' + b y' + c y = g(x)$$

$$y'' + \frac{b}{a} y' + \frac{c}{a} y = \frac{g}{a}$$

$$| y'' + P y' + Q y = R(x)$$

$$y_2 = y_1 \cdot \int \frac{e^{-\int P dx}}{y_1^2} dx$$

$$e^{\ln a} = e^{\log_e a} = a$$

$$\textcircled{11} \quad x y'' + y' = 0$$

$$y_1 = \ln x$$

$$y'' + \frac{1}{x} y' = 0$$

$$y_2 = \ln x$$

$$-\int p dx = -\int \frac{1}{x} dx = -\ln|x| + C$$

$$e^{-\ln|x| + C} = e^{-\ln|x|} \cdot e^C$$

$$= C_1 e^{-\ln \frac{1}{|x|}}$$

$$= C_1 \frac{1}{|x|}$$

$$= \pm C_1 \frac{1}{x}$$

$$= C_2 \frac{1}{x}$$

$$y_2 = y_1 \cdot \int \frac{e^{-\int p dx}}{y_1^2} dx$$

$$= \ln x \int \frac{C_2 \frac{1}{x}}{(\ln x)^2} dx$$

$$= \ln x \int \frac{C_2}{(\ln x)^2} (\ln x)' dx$$

$$= \cancel{(\ln x)} \cdot \frac{-C_2}{\cancel{(\ln x)}} = \textcircled{C}$$

$$y_h = C_1 y_1 + C_2 y_2$$

$$= C_1 \ln x + C_2 \cdot 1$$

$$(15) \quad (1-2x-x^2)y'' + 2(1+x)y' - 2y = 0$$

$$/ \cdot y'' + \frac{2(1+x)}{(1-2x-x^2)} y' - \frac{2}{(1-2x-x^2)} y = 0$$

$$y_1 = x+1$$

$$y_2 = y_1 \cdot \int \frac{e^{-\int p dx}}{y_1^2} dx$$

$$-\int p dx = -\int \frac{2(1+x)}{1-2x-x^2} dx$$

$$\begin{aligned} (1-2x-x^2)' &= -2-2x \\ &= -2(1+x) \end{aligned}$$

$$\Rightarrow +\int \frac{(1-2x-x^2)'}{1-2x-x^2} dx$$

$$= \ln |1-2x-x^2| + C$$

$$e^{-\int p dx} = e^{\ln |1-2x-x^2| + C} = C_1 (1-2x-x^2)$$

$$y_2 = (x+1) \int C_1 \frac{(1-2x-x^2)}{(x+1)^2} dx = C_1 (x+1) \int \frac{2-(x+1)^2}{(x+1)^2} dx$$

$$= \left[(x+1) \int \left[\frac{2}{(x+1)^2} - 1 \right] dx \right] C_1$$

$$= \left[\frac{-2}{x+1} - x(x+1) \right] C_1$$

$$= (-2 - x^2 - x) C_1 = C (x^2 + x + 2)$$

$$(1-2x-x^2)y'' + 2(1+x)y' - 2y = 0$$

$$y_1 = x+1$$

$$y_2 = y_1 \cdot \int \frac{e^{-\int p dx}}{y_1^2} dx$$

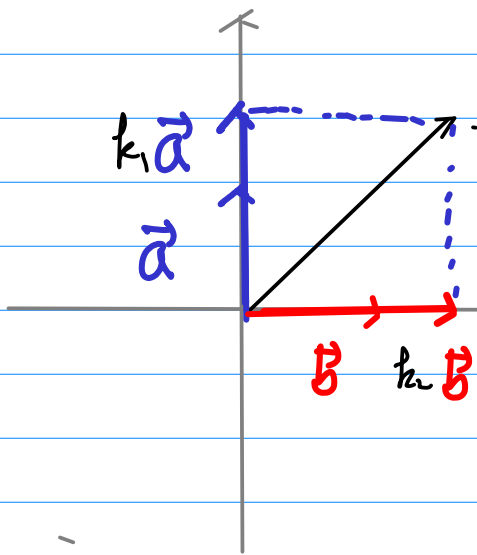
$$y_2 = C(x^2 + x + 2)$$

$$y_h = C_1 y_1 + C_2 y_2$$

$$= C_1(x+1) + C_2(x^2 + x + 2)$$

$$= C_1(x+1) + C_2(x^2 - x - 2)$$

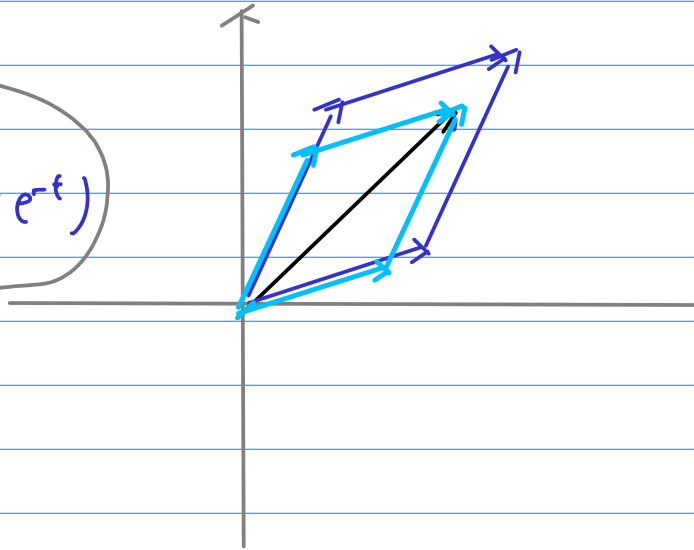
basis vector



$$\vec{x} = k_1 \vec{a} + k_2 \vec{b}$$

$$c_1 e^{3t} + c_2 e^{-t}$$

$$c_1 \left(-\frac{1}{2} e^{3t} + \frac{3}{2} e^{-t} \right) + c_2 \left(-\frac{1}{2} e^{3t} + \frac{1}{2} e^{-t} \right)$$



$$y'' - 4y' - 12y = 3e^{5t} + \sin(2t) + te^{4t}$$

$$y'' - 4y' - 12y = 0 \quad \leftarrow y_h$$

$$y'' - 4y' - 12y = 3e^{5t} \quad \leftarrow y_{p1}$$

$$y'' - 4y' - 12y = \sin(2t) \quad \leftarrow y_{p2}$$

$$y'' - 4y' - 12y = te^{4t} \quad \leftarrow y_{p3}$$

$$y = y_h + y_{p1} + y_{p2} + y_{p3}$$

2.2 Separable equation

2.3 Linear equation

2.4 Exact equation

2.5 Substitution

3.2 Reduction of orders

3.3 Homogeneous, Constant coefficients

3.4 Undetermined Coefficients

3.5 Variation of Parameters

Substitution + Homogeneous Function

$$(x^2 + y^2) dx + (x^2 - xy) dy = 0$$

$$\begin{array}{cc} \parallel & \parallel \\ M(x, y) & N(x, y) \end{array}$$

$$M(x, y) = x^2 + y^2 = x^2 \left(1 + \left(\frac{y}{x}\right)^2 \right) = x^2 M\left(1, \frac{y}{x}\right)$$

$$N(x, y) = x^2 - xy = x^2 \left(1 - \left(\frac{y}{x}\right) \right) = x^2 N\left(1, \frac{y}{x}\right)$$

$$x^2 \left(1 + \left(\frac{y}{x}\right)^2 \right) dx + x^2 \left(1 - \left(\frac{y}{x}\right) \right) dy = 0$$

$$u = \frac{y}{x}$$

$$y = ux$$

$$dy = \frac{\partial}{\partial u}(ux) du + \frac{\partial}{\partial x}(ux) dx$$

$$dy = x du + u dx$$

$$\left(1 + u^2 \right) dx + (1 - u)(x du + u dx) = 0$$

$$\left(1 + u^2 \right) dx + (1 - u)x du + (1 - u)u dx = 0$$

$$(1 + \cancel{u^2} + u - \cancel{u^2}) dx + x(1 - u) du = 0$$

$$(1 + u) dx + x(1 - u) du = 0$$

$$dx + x \frac{(1 - u)}{(1 + u)} du = 0$$

$$\frac{dx}{x} + \frac{(1 - u)}{(1 + u)} du = 0$$

$$\frac{dx}{x} + \left[\frac{-1 - u + 2}{1 + u} \right] du = 0$$

$$\frac{dx}{x} + \left[-1 + \frac{2}{1 + u} \right] du = 0$$

Bernoulli: EG

$$x \frac{dy}{dx} + y = x^2 y^2$$

$$\frac{dy}{dx} + \frac{y}{x} = x y^2$$

$$\frac{dy}{dx} = x y^2 - \frac{y}{x}$$

$$\frac{1}{u^2} \frac{du}{dx} = \frac{x}{u^2} - \frac{1}{x u}$$

$$\frac{du}{dx} = -x + \frac{u}{x}$$

$$u' - \frac{1}{x} u = -x$$

$$\left(\frac{u}{x} \right)' = -1$$

$$u = \frac{1}{y}$$

$$y = \frac{1}{u}$$

$$\frac{dy}{dx} = \left(\frac{1}{u} \right)' \frac{du}{dx}$$

$$= \frac{-1}{u^2} \frac{du}{dx}$$





