Laurent Series and z-Transform - Geometric Series Properties A

20220105 Wed

Unshifted Geometric Sequences

Causal

$\frac{1}{1-\square z} \Rightarrow u(n)$

Anti-causal

$\frac{1}{1-\square z^{-1}} \Rightarrow u(-n)$

Positive Exponent

$az,\ a^{-1}z^{-1} \Rightarrow a^{n}$

Negative Exponent

$a^{-1}z,\ az^{-1} \Rightarrow a^{-n}$

Positive Exponent

o-including (1)

$\frac{1}{1-az}$ $|z|<a^{-1}$ $a^{n}\ u(n)$

o-including (3)

$-\frac{1}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ $a^{n}\ u(-n)$

o-execluding (5)

$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ $a^{n}\ u(-n-1)$

o-excluding (7)

$\frac{az}{1-az}$ $|z|<a^{-1}$ $a^{n}\ u(n-1)$

Negative Exponent

o-including (2)

$\frac{1}{1-a^{-1}z}$ $|z|<a^{-1}$ $a^{-n}\ u(n)$

o-including (4)

$-\frac{1}{1-az^{-1}}$ $|z|>a$ $a^{-n}\ u(-n)$

o-excluding (6)

$-\frac{az^{-1}}{1-az^{-1}}$ $|z|>a^{-1}$ $a^{-n}\ u(-n-1)$

o-excluding (8)

$\frac{a^{-1}z}{1-a^{-1}z}$ $|z|<a^{-1}$ $a^{-n}\ u(n-1)$

Decoding Geometric Series

Positive Exponent

$az, a^{-1}z^{-1} \Rightarrow a^{n}$

Negative Exponent

$a^{-1}z, az^{-1} \Rightarrow a^{-n}$

Causal

$\square z \Rightarrow u(n), u(n-1)$

Anti-causal

$\square z^{-1} \Rightarrow u(-n), u(-n-1)$

$\frac{1}{1-\square z} \Rightarrow u(n)$

$\frac{\square z}{1-\square z} \Rightarrow u(n-1)$

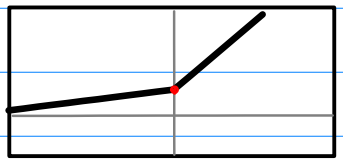
$\frac{1}{1-\square z^{-1}} \Rightarrow u(-n)$

$\frac{\square z^{-1}}{1-\square z^{-1}} \Rightarrow u(-n-1)$

Symmetric exponents and symmetric ranges

Positive Exponent

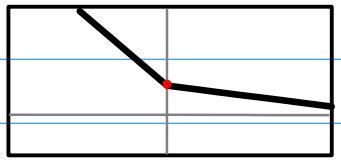
az, $a^{-1}z^{-1}$ ➡



a^n

Negative Exponent

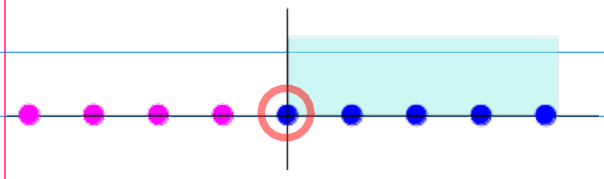
$a^{-1}z$, az^{-1} ➡



a^{-n}

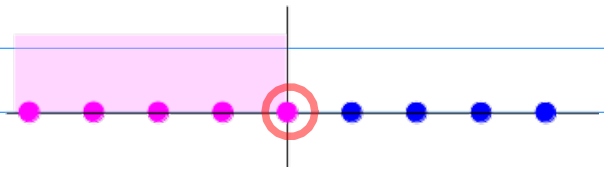
Causal **including the origin**

$\frac{1}{1-\square z}$ ➡ $u(n)$



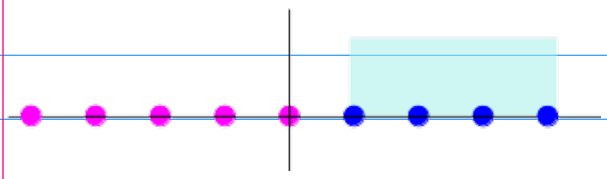
Anti-causal **including the origin**

$\frac{1}{1-\square z^{-1}}$ ➡ $u(-n)$



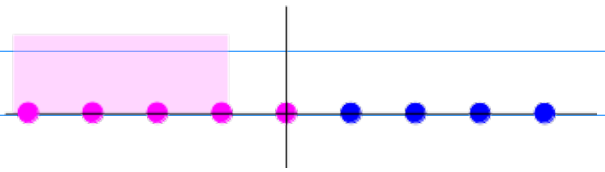
Causal **excluding the origin**

$\frac{\square z}{1-\square z}$ ➡ $u(n-1)$



Anti-causal **excluding the origin**

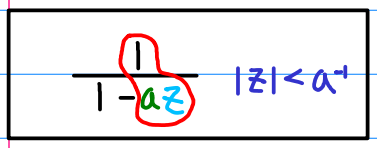
$\frac{\square z^{-1}}{1-\square z^{-1}}$ ➡ $u(-n-1)$



Decoding examples

Positive Exponent

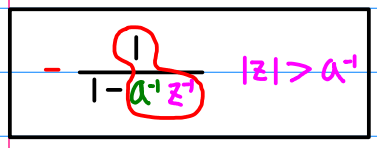
o-including (1)



$az \Rightarrow a^n$

$1, az \Rightarrow u(n)$

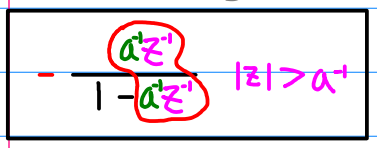
o-including (3)



$a^{-1}z^{-1} \Rightarrow a^n$

$1, a^{-1}z^{-1} \Rightarrow u(-n)$

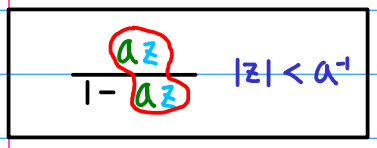
o-execluding (5)



$a^{-1}z^{-1} \Rightarrow a^n$

$a^{-1}z^{-1}, a^{-1}z^{-1} \Rightarrow u(-n-1)$

o-excluding (7)

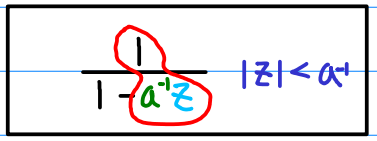


$az \Rightarrow a^n$

$az, az \Rightarrow u(n-1)$

Negative Exponent

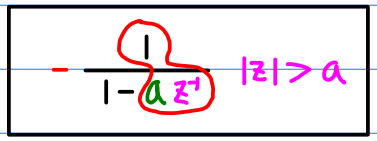
o-including (2)



$a^{-1}z \Rightarrow a^{-n}$

$1, a^{-1}z \Rightarrow u(n)$

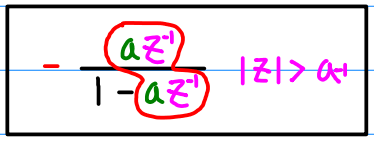
o-including (4)



$az^{-1} \Rightarrow a^{-n}$

$1, az^{-1} \Rightarrow u(-n)$

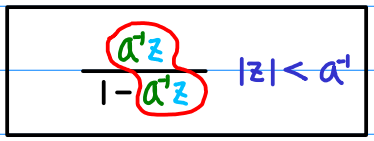
o-excluding (6)



$az^{-1} \Rightarrow a^{-n}$

$az^{-1}, az^{-1} \Rightarrow u(-n-1)$

o-excluding (8)



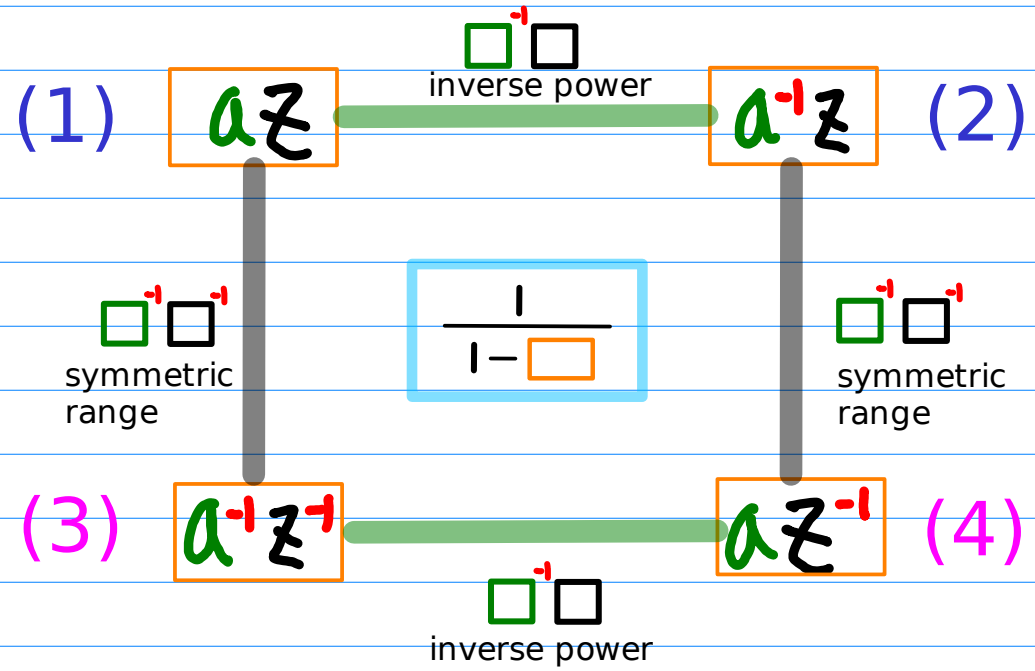
$a^{-1}z \Rightarrow a^{-n}$

$a^{-1}z, a^{-1}z \Rightarrow u(n-1)$

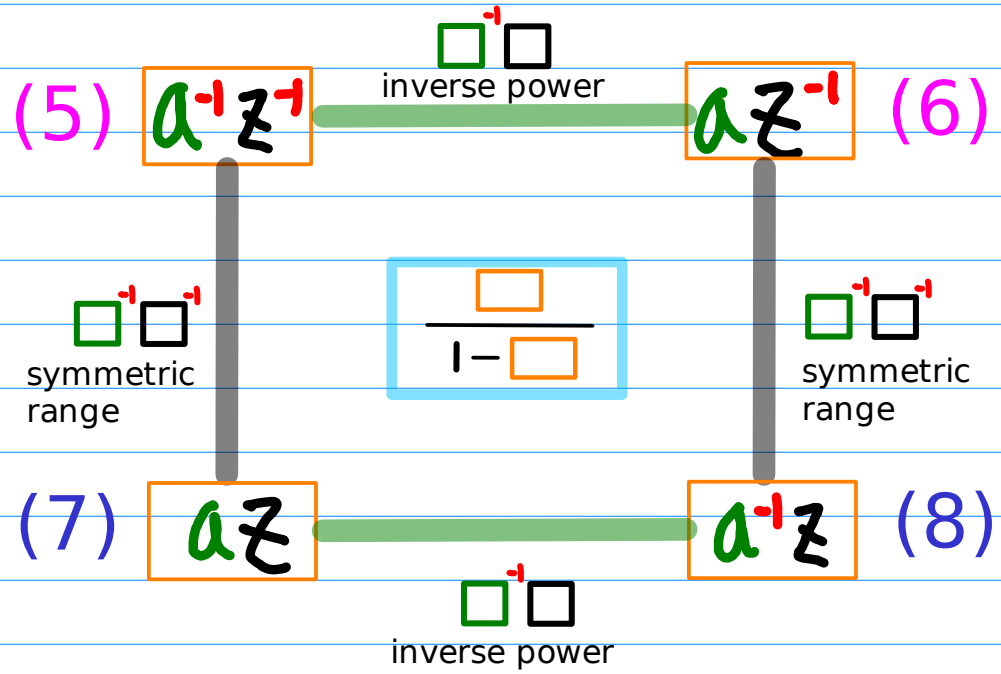
Geometric sequence cases - (1) CR view

inverse power, symmetric range relations

origin including geometric sequences



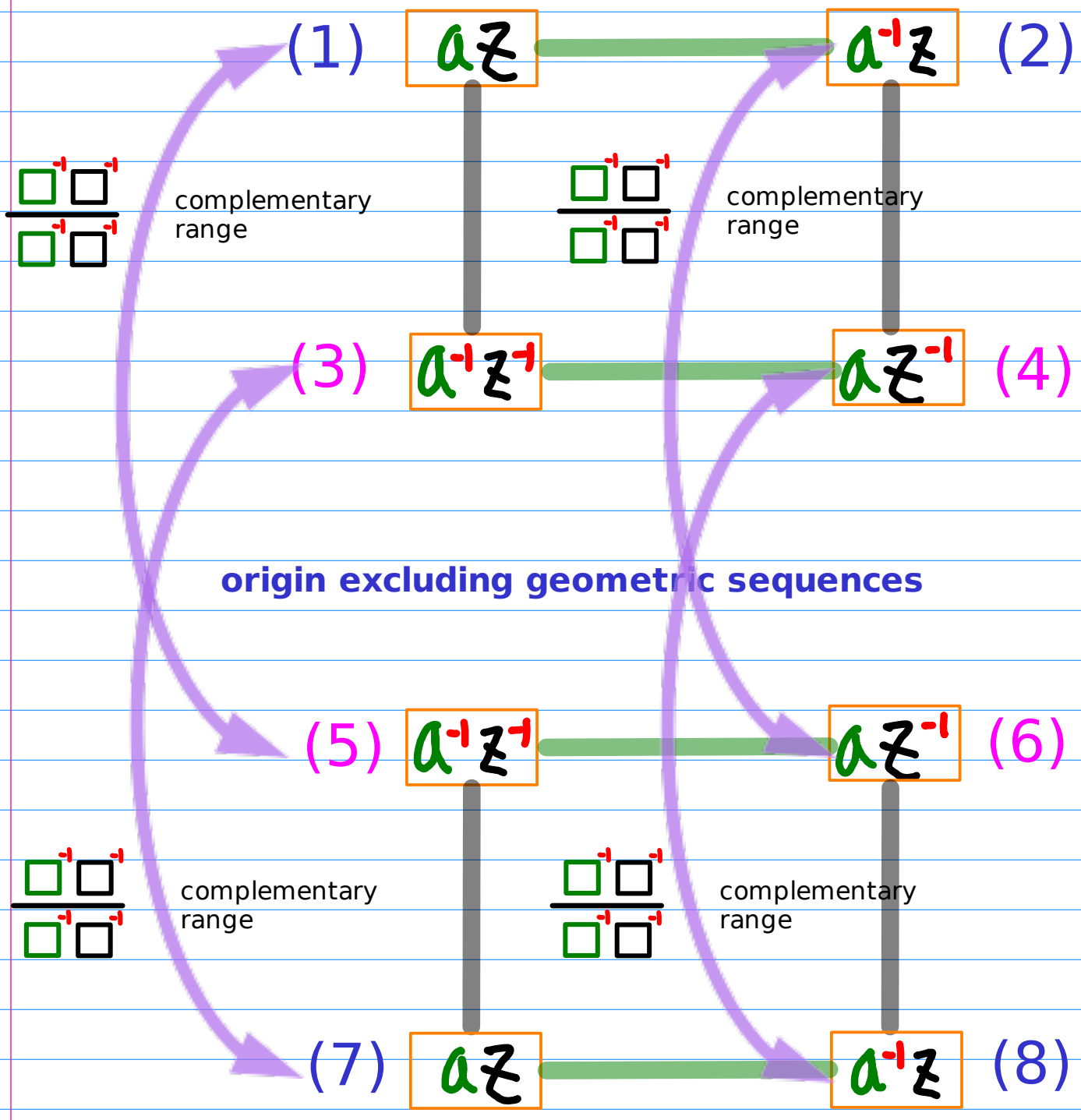
origin excluding geometric sequences



Geometric sequence cases - (1) CR view

complementary range relations

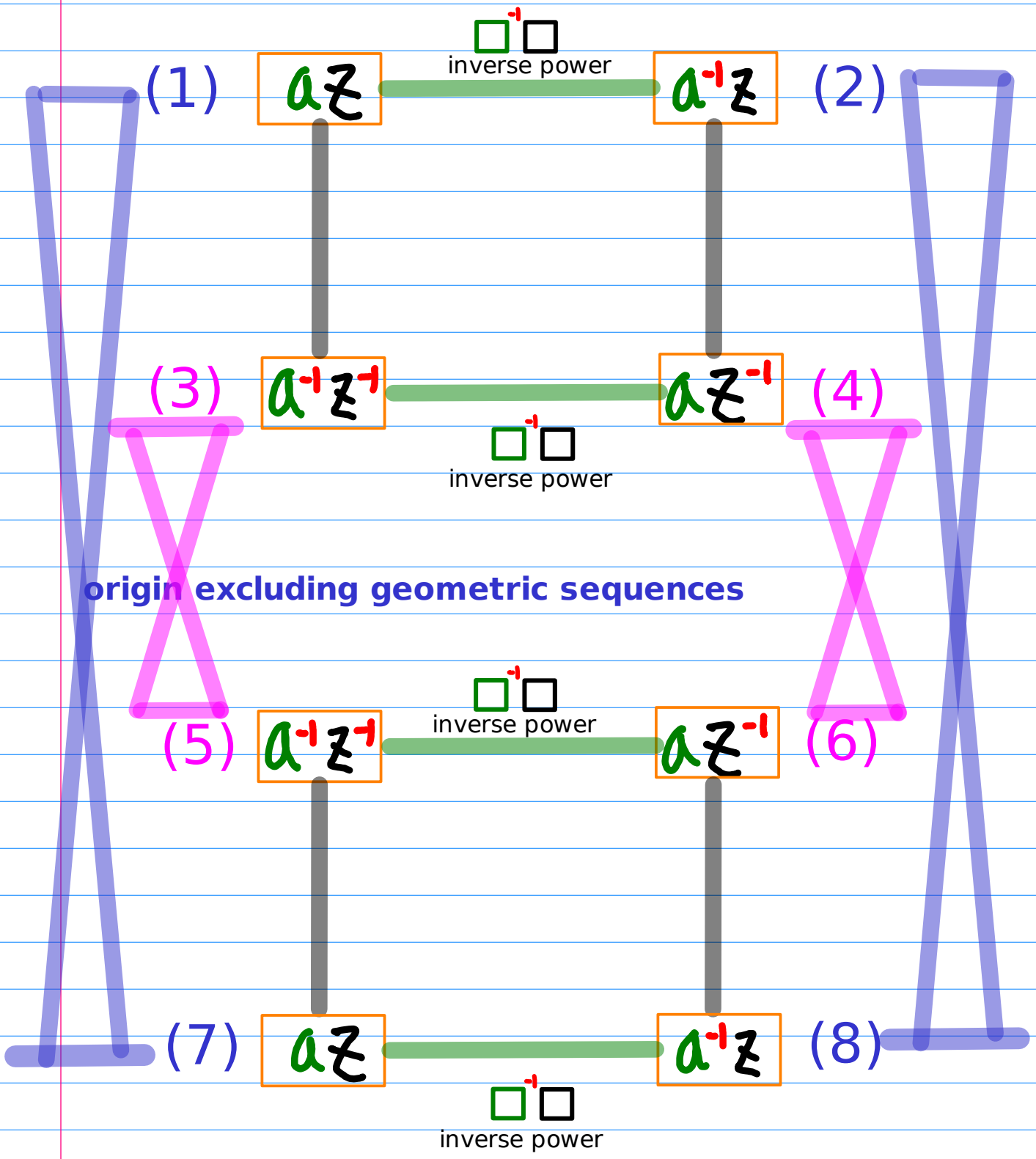
origin including geometric sequences



Geometric sequence cases - (1) CR view

Shifting relations

origin including geometric sequences

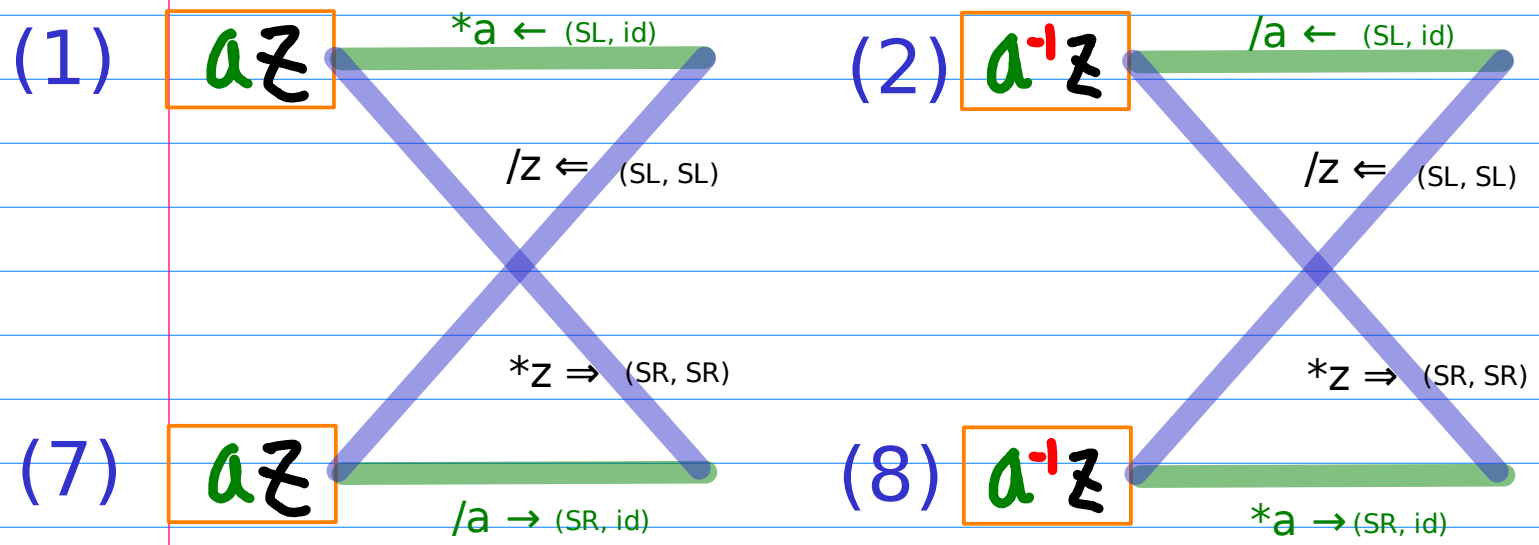


origin excluding geometric sequences

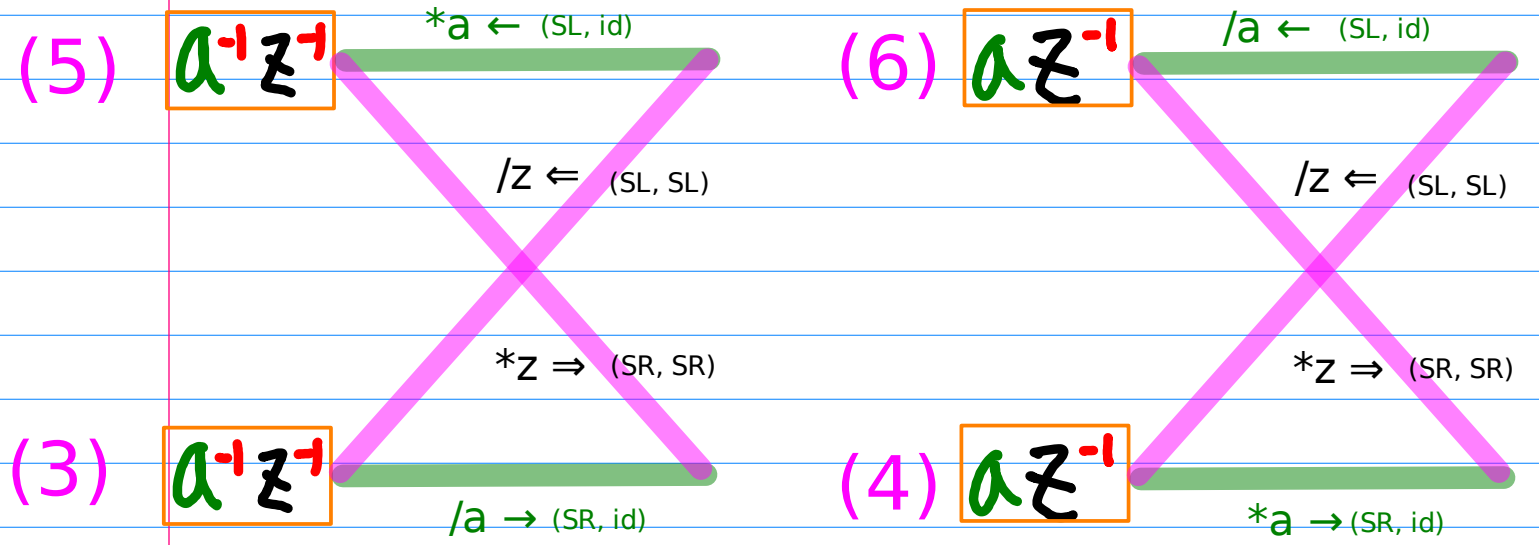
Geometric sequence cases - (1) CR view

Shifting relations

origin including geometric sequences

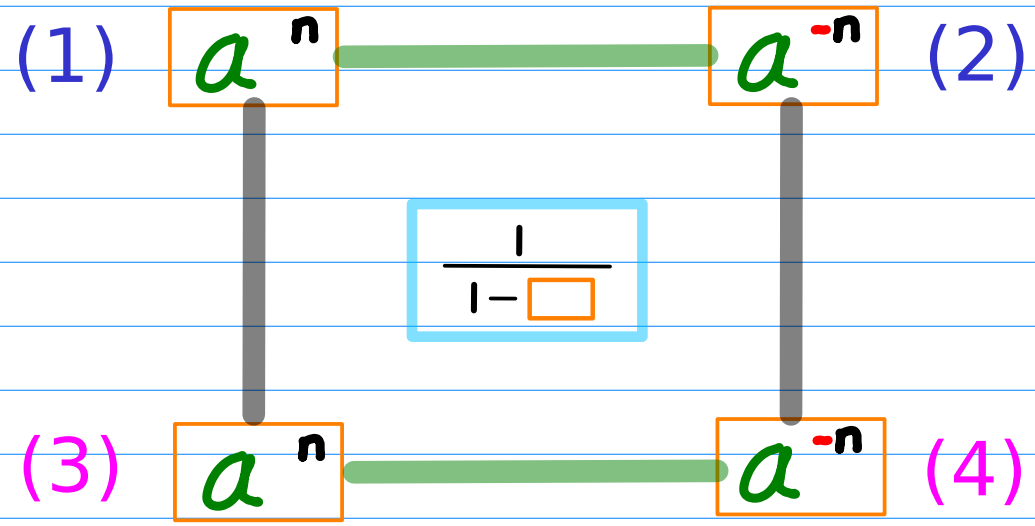


origin excluding geometric sequences

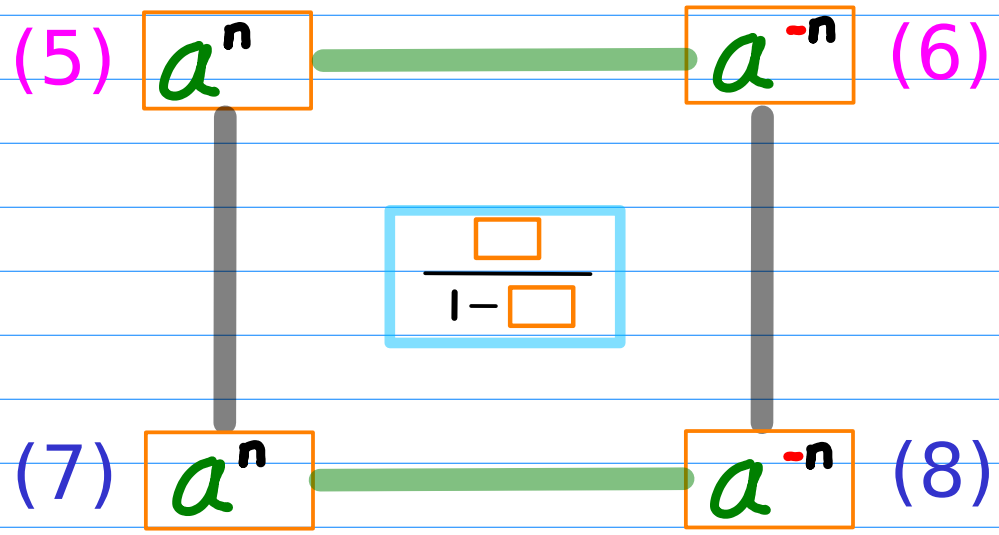


Geometric sequence cases - (2) Exponent view

origin including geometric sequences

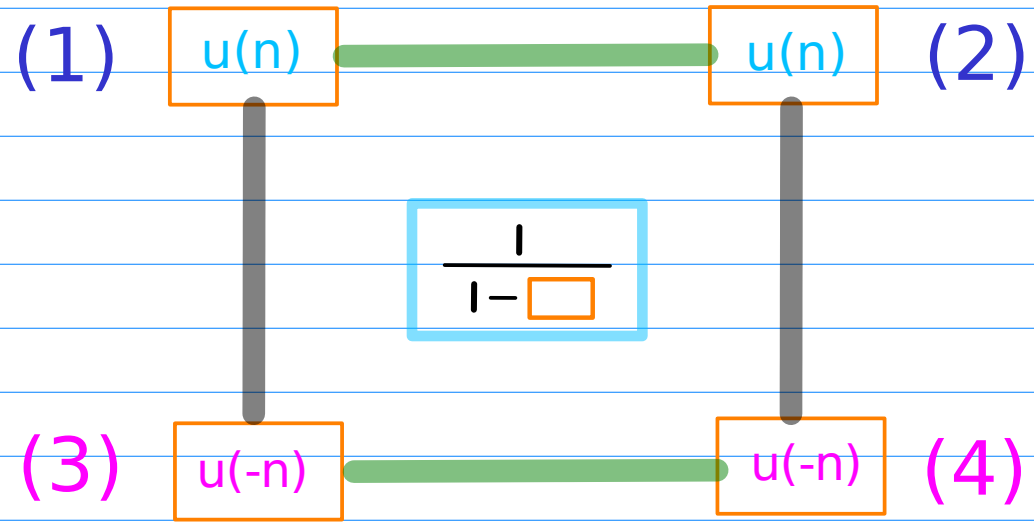


origin excluding geometric sequences

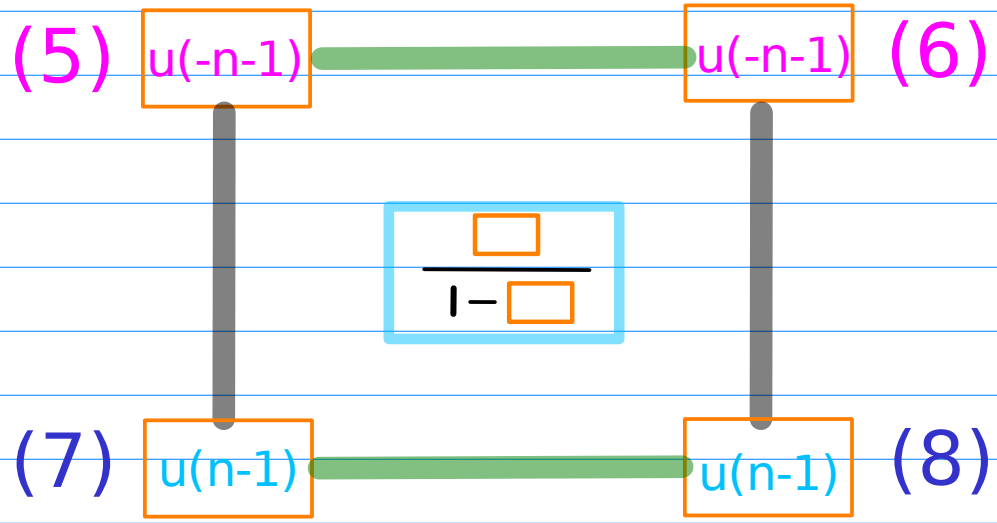


Geometric sequence cases - (3) Range view

origin including geometric sequences



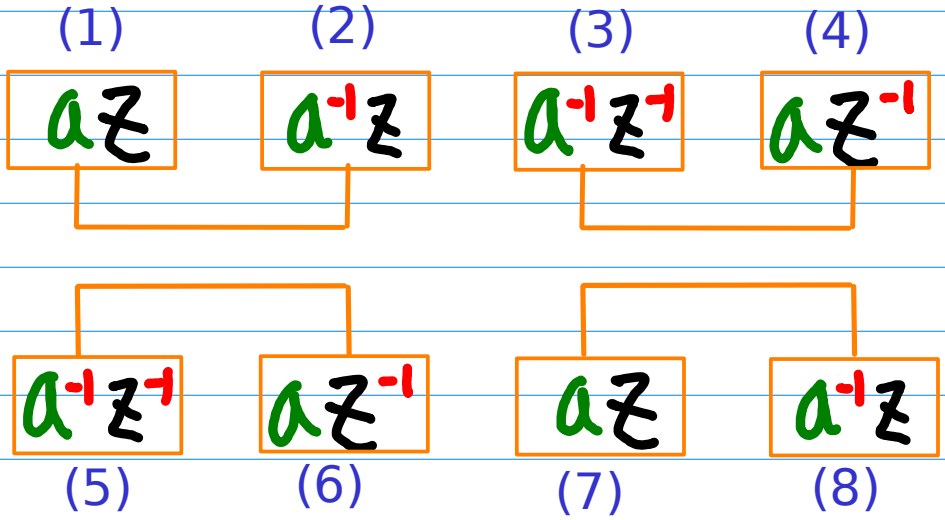
origin excluding geometric sequences



Inverse Power Relations

Inverse power relations $\square^{-1}\square$ replace

inverse power, same range

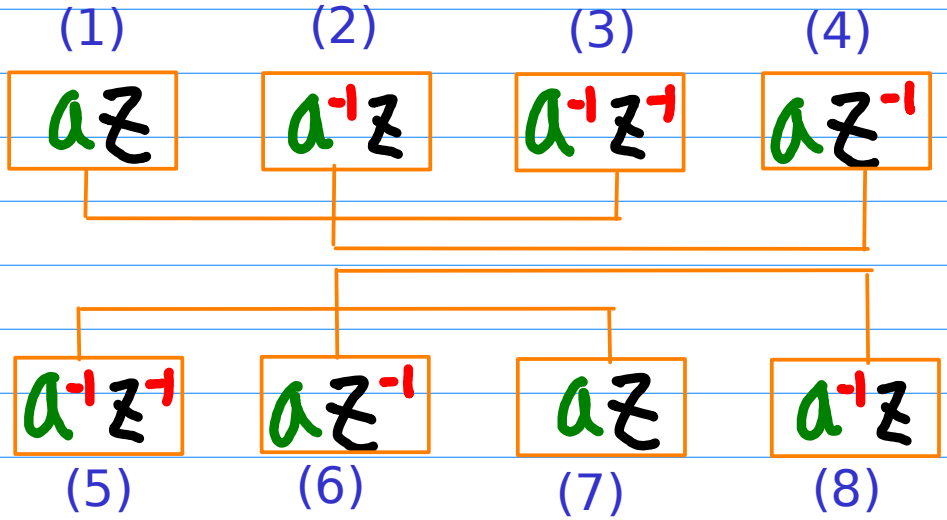


Symmetric and Complementary Relations

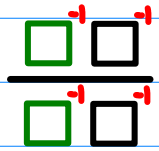
Symmetric range relations

replace

same power, symmetric range

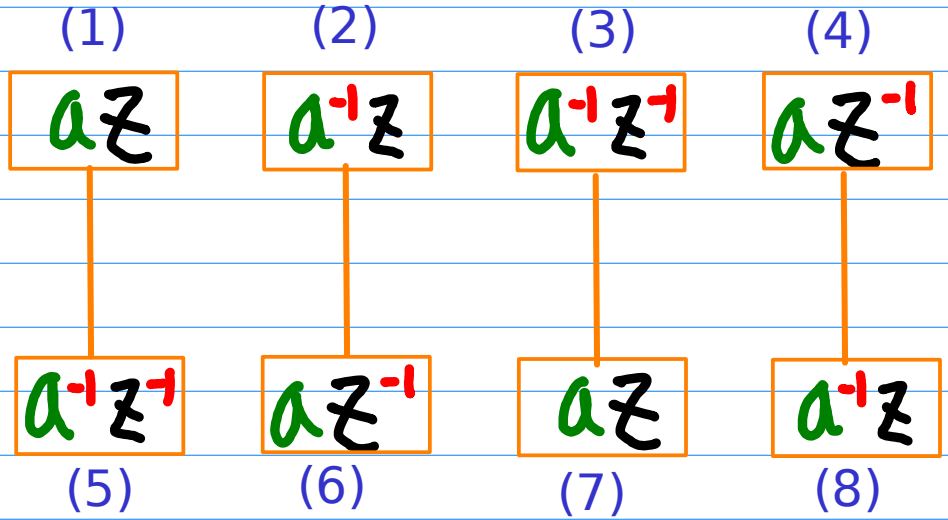


Complementary range relations



multiply

same power, complement range



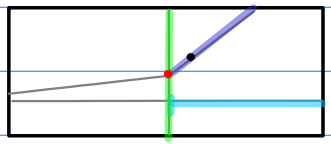
A unit starting **origin including**

(1) $+\frac{1}{1-az}$ a^n u(n)

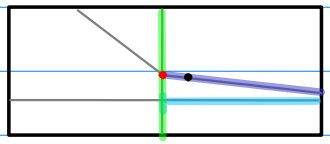
(2) $+\frac{1}{1-a^{-1}z}$ a^{-n} u(n)

(3) $-\frac{1}{1-a^{-1}z^{-1}}$ $-a^n$ u(-n)

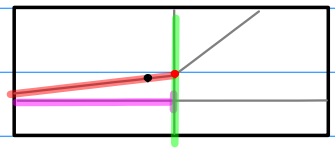
(4) $-\frac{1}{1-az^{-1}}$ $-a^{-n}$ u(-n)



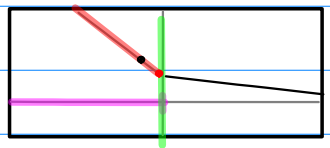
a^n u(n)



a^{-n} u(n)



a^n u(-n)



a^{-n} u(-n)

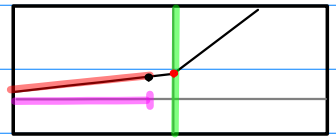
A CR starting **origin excluding**

(5) $$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$$ $-a^n\,u(-n-1)$

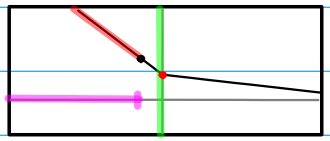
(6) $$-\frac{az^{-1}}{1-az^{-1}}$$ $-a^{-n}u(-n-1)$

(7) $$+\frac{az}{1-az}$$ $a^n\,u(n-1)$

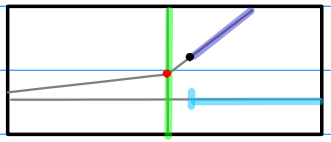
(8) $$+\frac{a^{-1}z}{1-a^{-1}z}$$ $a^{-n}\,u(n-1)$



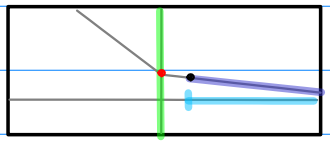
$a^n\,u(-n-1)$



$a^{-n}\,u(-n-1)$



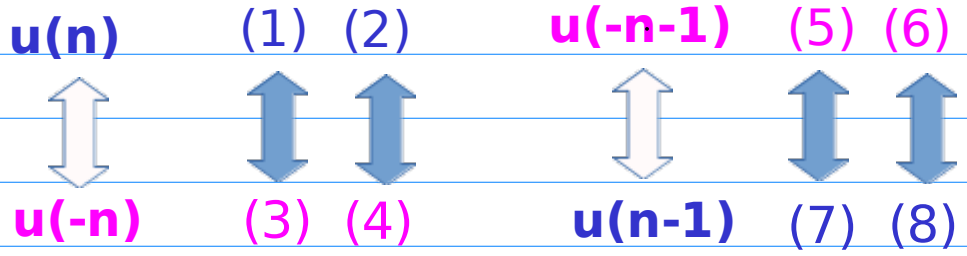
$a^n\,u(n-1)$

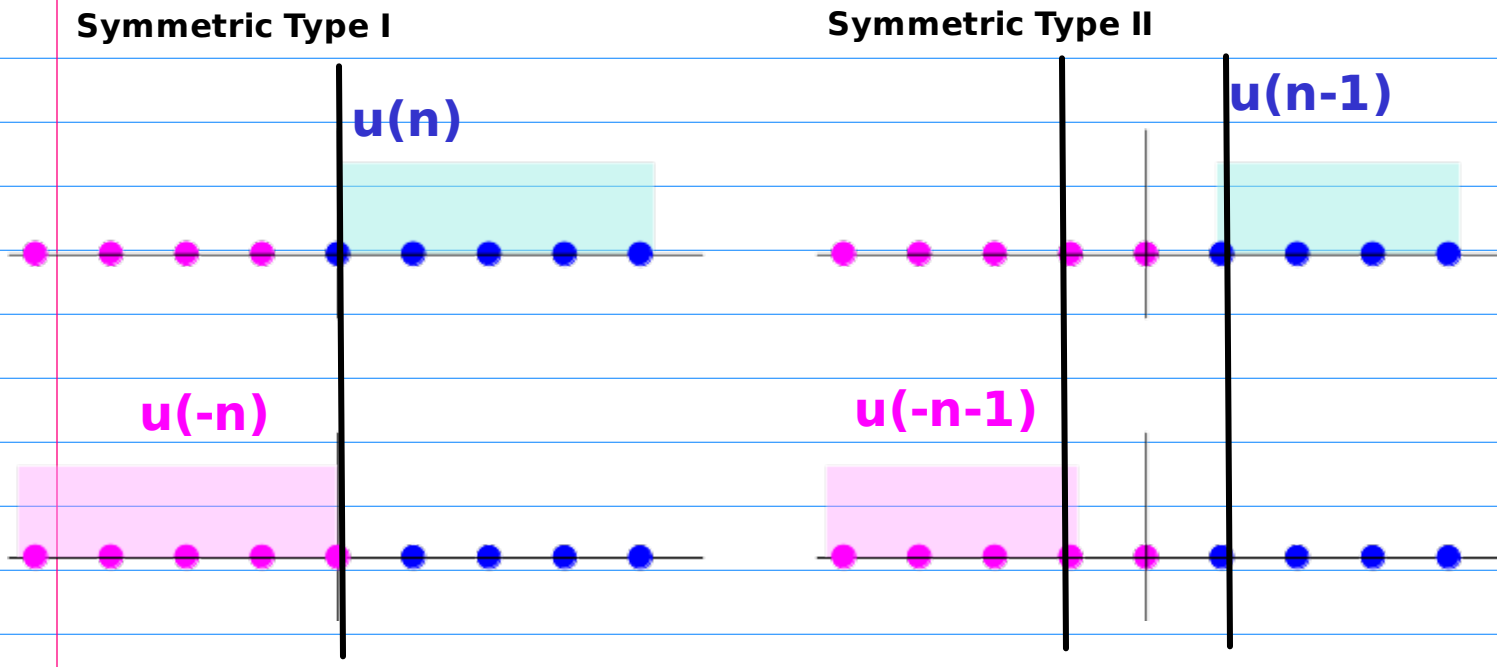


$a^{-n}\,u(n-1)$

Symmetric Range Types

Symmetric Type I	Symmetric Type II

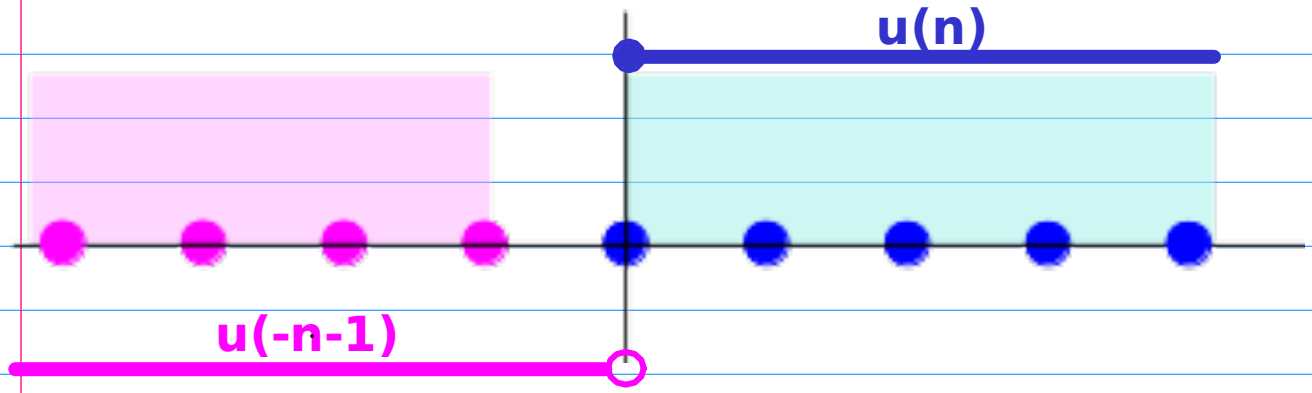




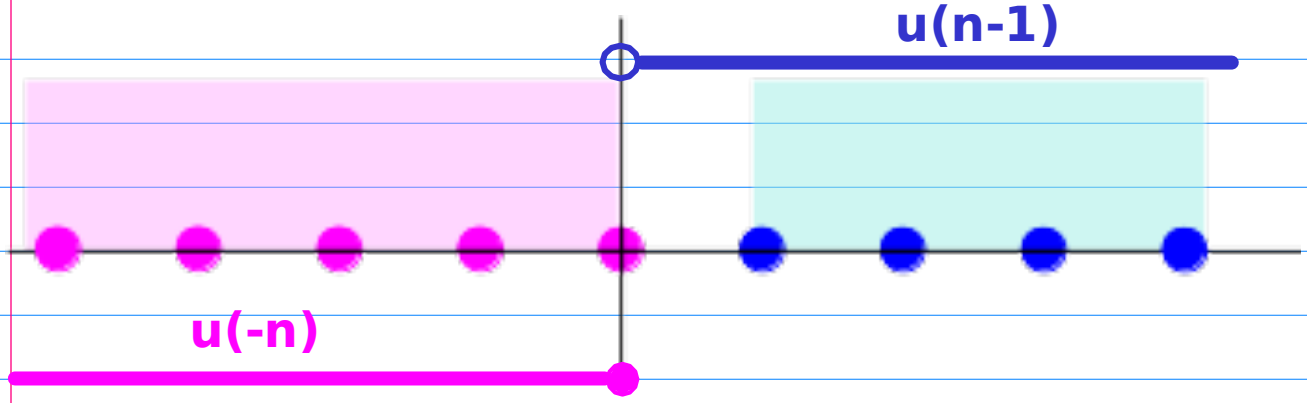
Complementary Range Types

Complement Type I	Complement Type II
u(n) (1) (2)	**u(-n)** (3) (4)
u(-n-1) (5) (6)	**u(n-1)** (7) (8)

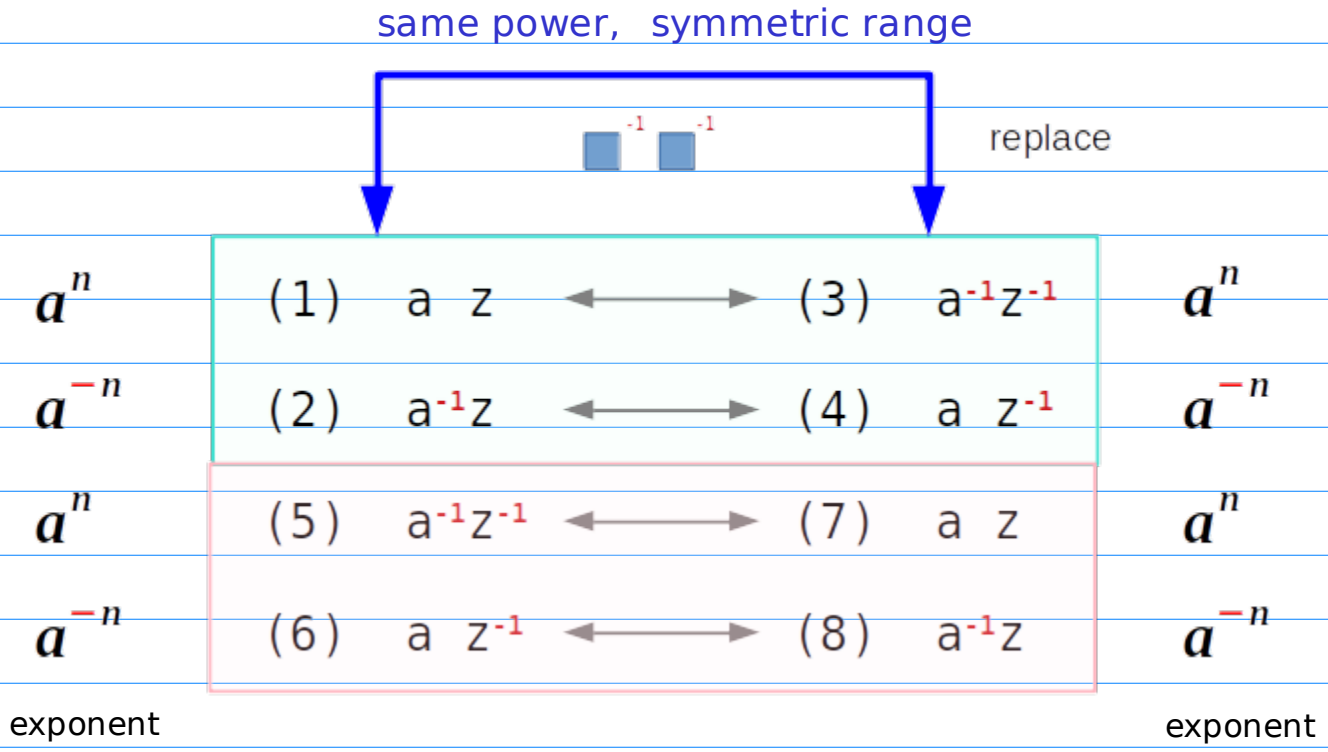
Complement Type I



Complement Type II



Symmetric Range Relations
- CR view

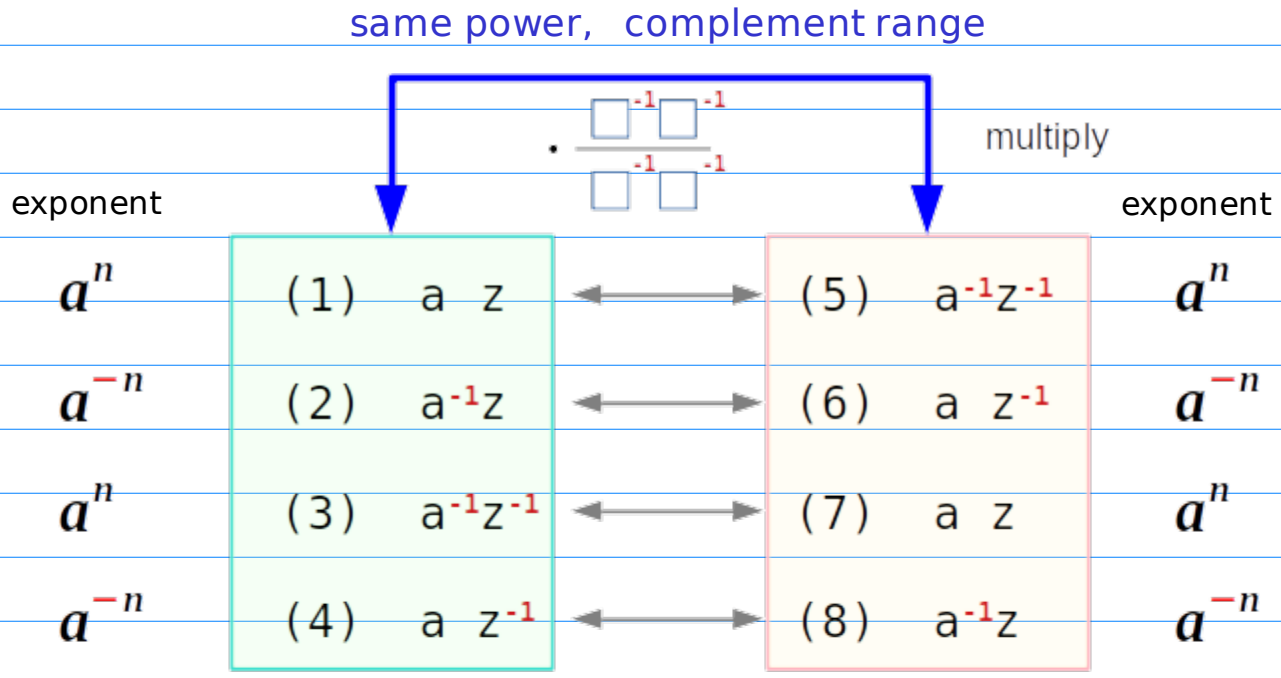


Symmetric range relations

replace

same power, symmetric range

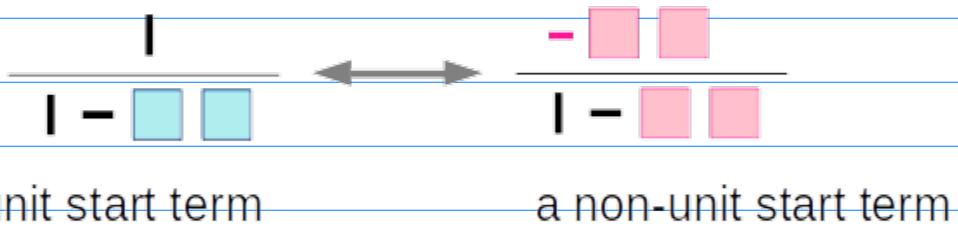
Complementary Range Relations - CR view

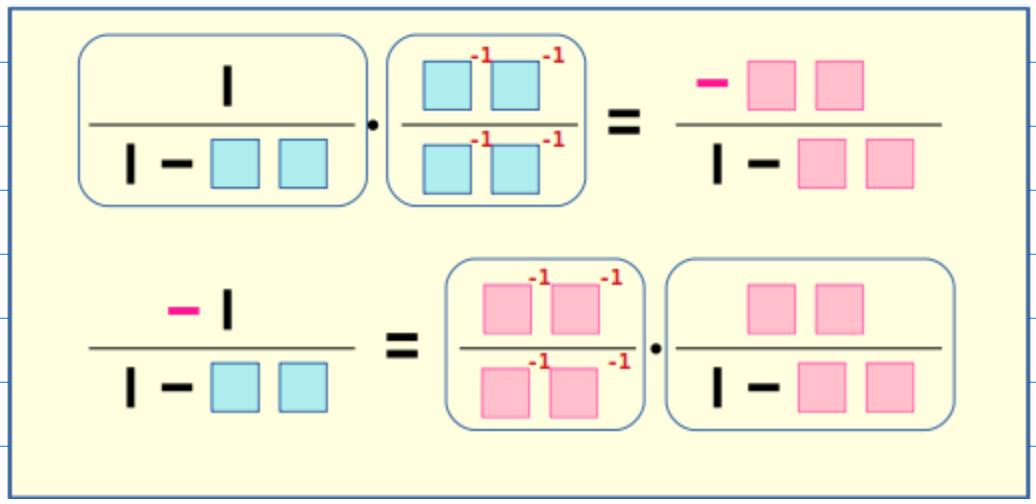


Complementary range relations

same power, complement range

multiply



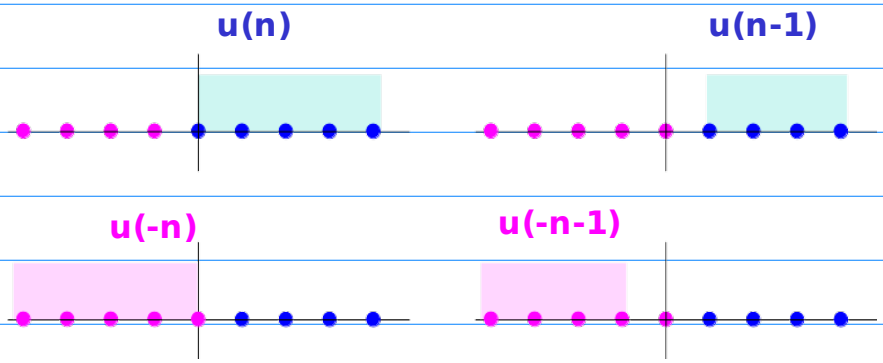


Symmetric Range Relations
- CR & Range view

same power, symmetric range

replace

(1) $a^{+1}z^{+1}$	$a^{+n}\cdot u(n)$	(3) $a^{-1}z^{-1}$	$a^{+n}\cdot u(-n)$
(2) $a^{-1}z^{+1}$	$a^{-n}\cdot u(n)$	(4) $a^{+1}z^{-1}$	$a^{-n}\cdot u(-n)$
(5) $a^{-1}z^{-1}$	$a^{+n}\cdot u(-n-1)$	(7) $a^{+1}z^{+1}$	$a^{+n}\cdot u(n-1)$
(6) $a^{+1}z^{-1}$	$a^{-n}\cdot u(-n-1)$	(8) $a^{-1}z^{+1}$	$a^{-n}\cdot u(n-1)$



symmetric ranges
type I

symmetric ranges
type II

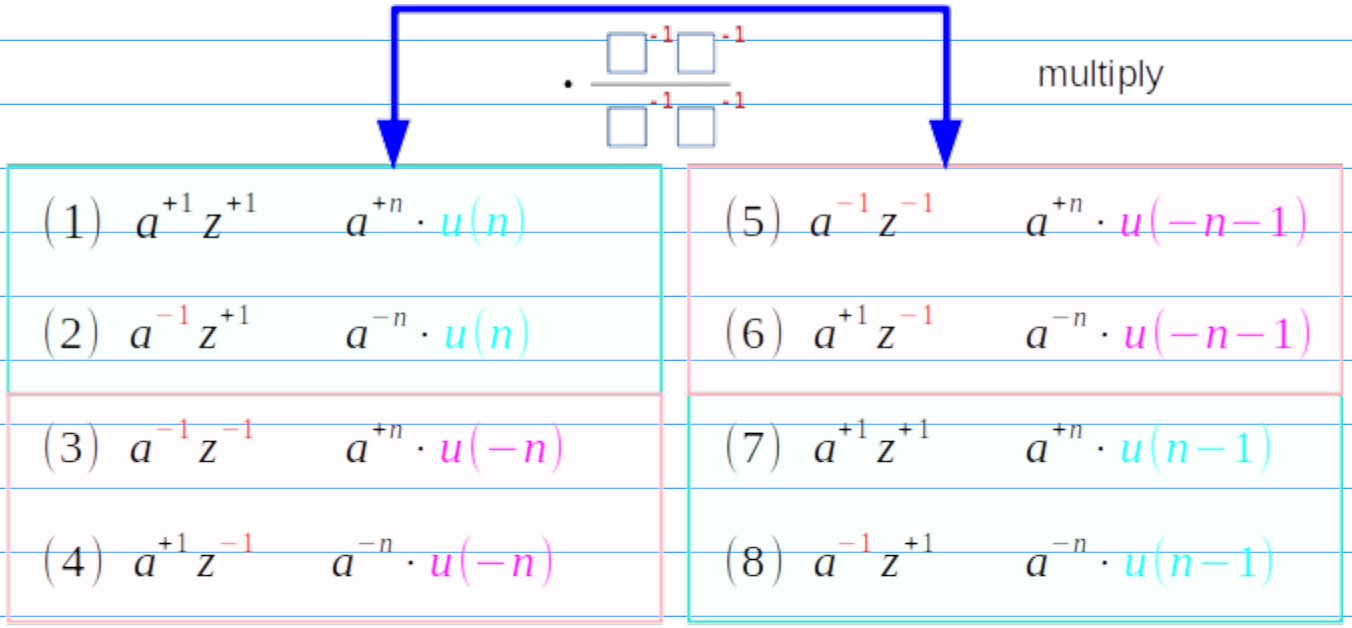
Symmetric range relations

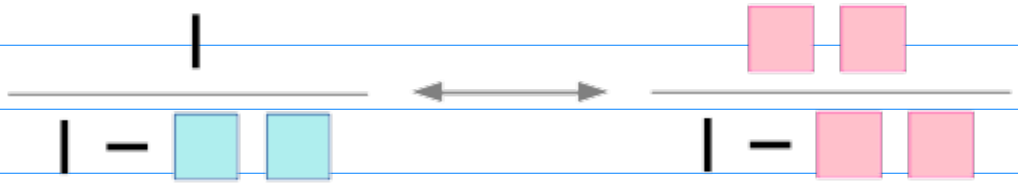
replace

same power, symmetric range

Complementary Range Relations
- CR & Range view

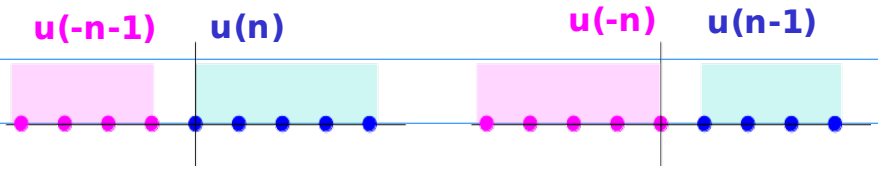
same power, complement range





a unit start term

a non-unit start term

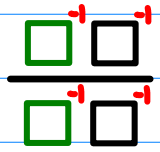


complementary ranges
type I

complementary ranges
type II

Complementary range relations

same power, complement range



multiply

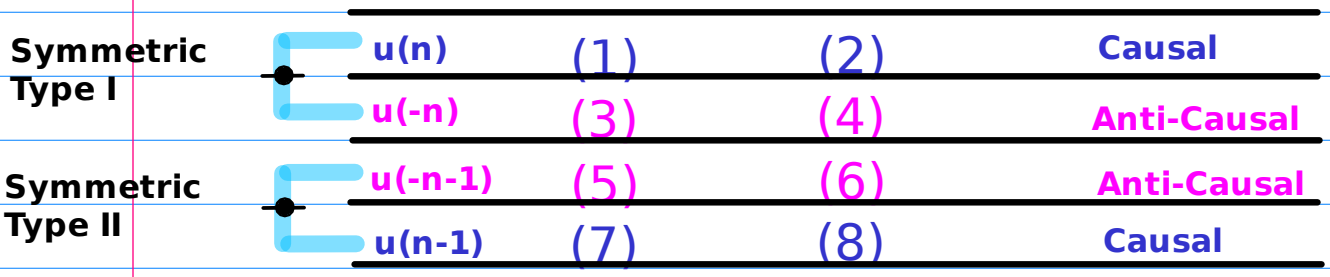
symmetric pair ordering

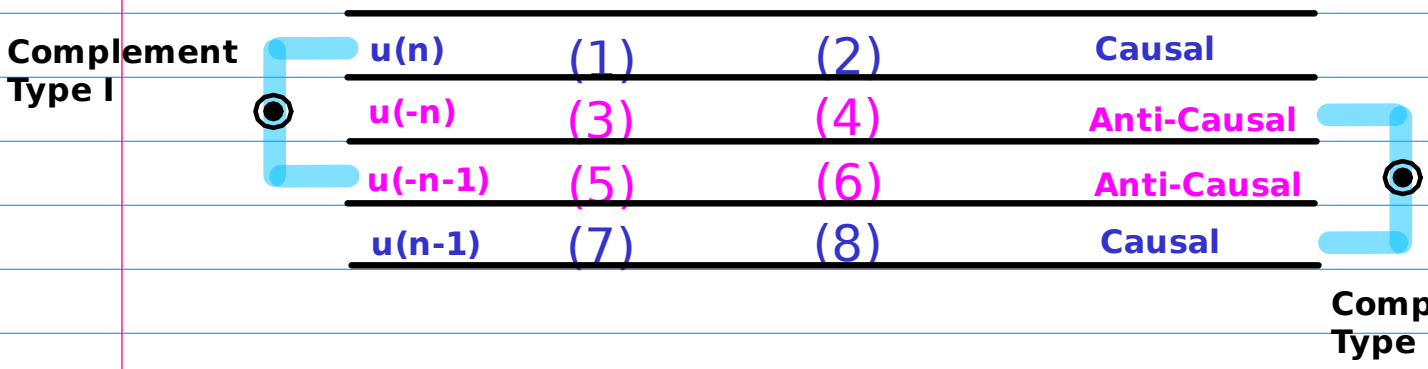
Positive Exponent	Negative Exponent
(1)	(2)
(3)	(4)
(5)	(6)
(7)	(8)

u(n)	(1)	(2)	Causal
u(-n)	(3)	(4)	Anti-Causal
u(-n-1)	(5)	(6)	Anti-Causal
u(n-1)	(7)	(8)	Causal

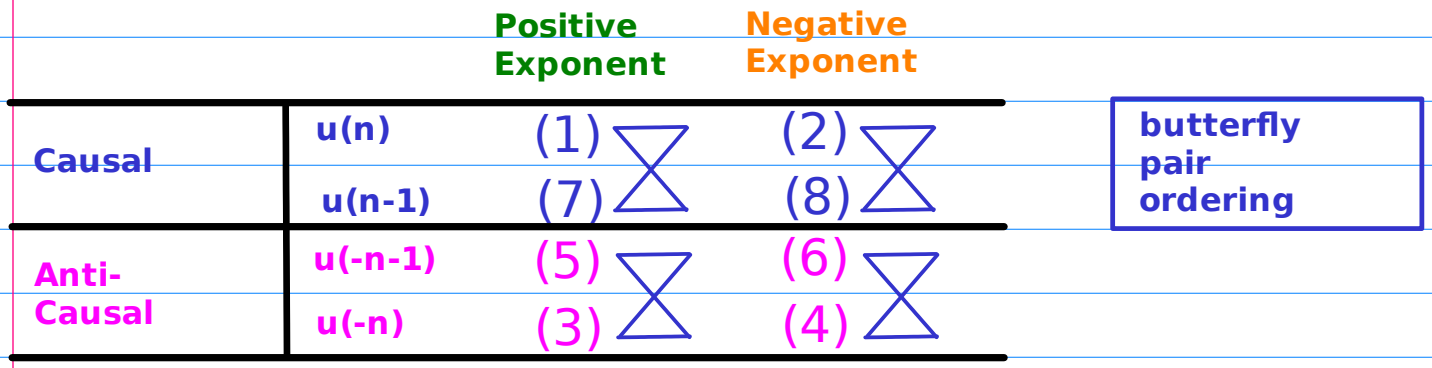
ranges include the origin	(1)	(2)	u(n)
	(3)	(4)	u(-n)
ranges exclude the origin	(5)	(6)	u(-n-1)
	(7)	(8)	u(n-1)

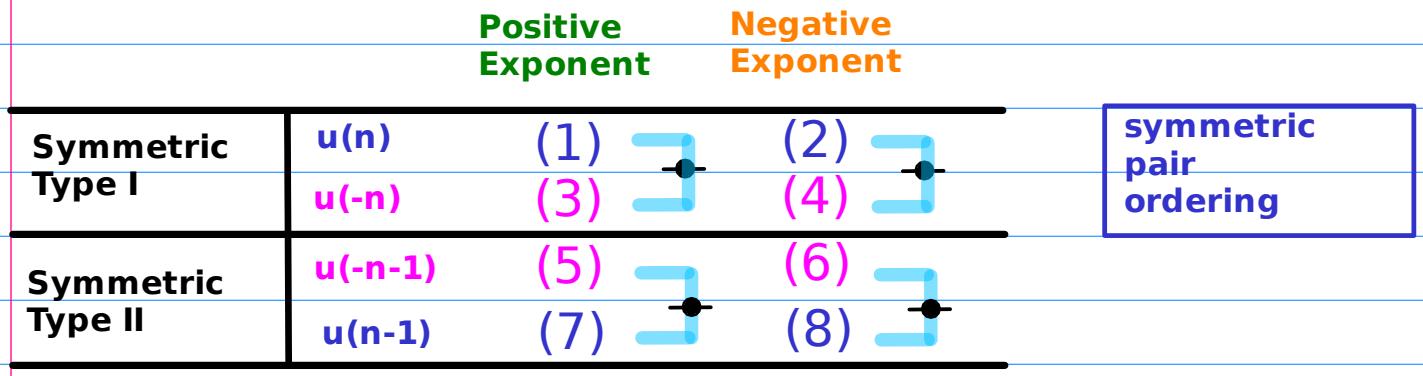
symmetric pair ordering

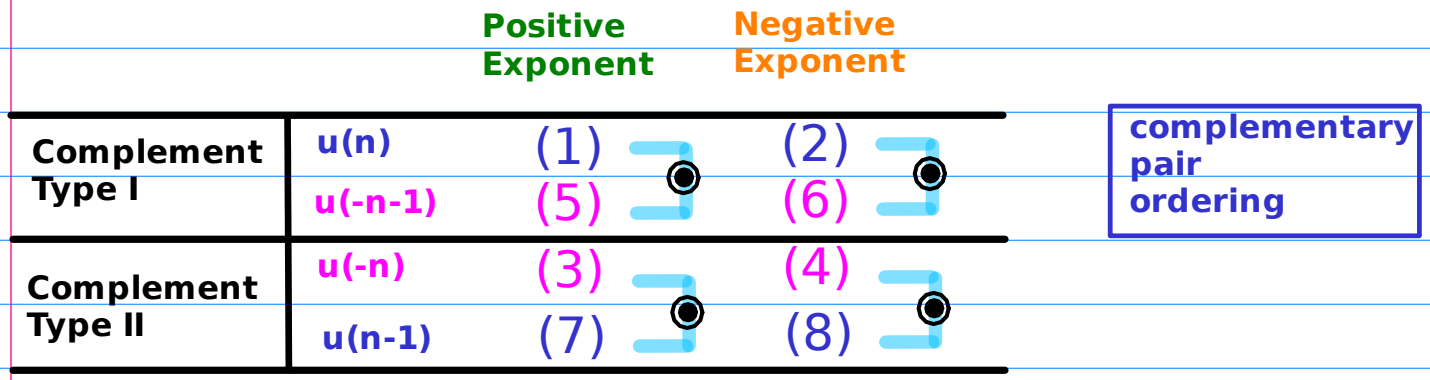




Three types of orderings







Multiplying a or a^{-1}

Multiplying z or z^{-1}

Exponent only Shifting

	$*a$	$*a^{-1}$
Positive Exponent	$a^{n+1} \leftarrow a^n$ **Left Shift** $\leftarrow$	$a^{n-1} \leftarrow a^n$ **Right Shift** $\rightarrow$
Negative Exponent	$a^{-n+1} \leftarrow a^{-n}$ **Right Shift** $\rightarrow$	$a^{-n-1} \leftarrow a^{-n}$ **Left Shift** $\leftarrow$

Exponent & Range Shifting

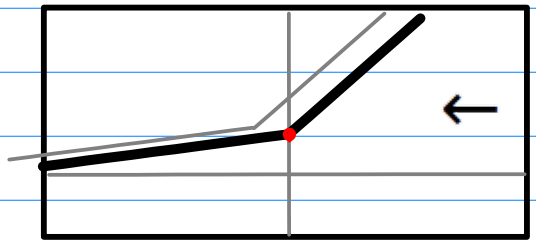
$*z$	$*z^{-1}$
$n \leftarrow n-1$ **Right Shift** $\Rightarrow$	$n \leftarrow n+1$ **Left Shift** $\Leftarrow$

$* a$

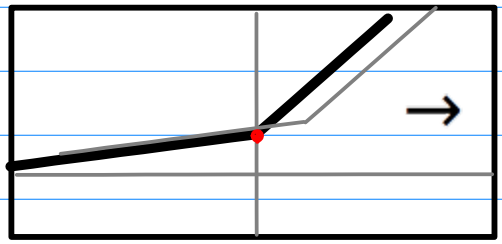
$* a^{-1}$

Positive Exponent

reach the same value earier

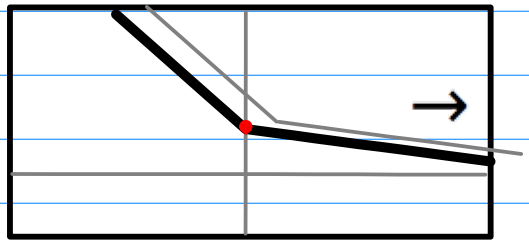


reach the same value later

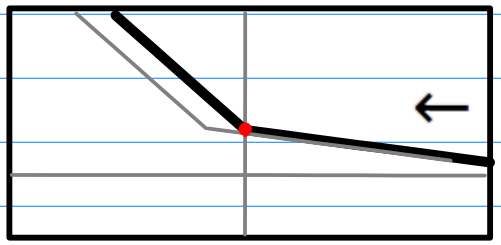


Negative Exponent

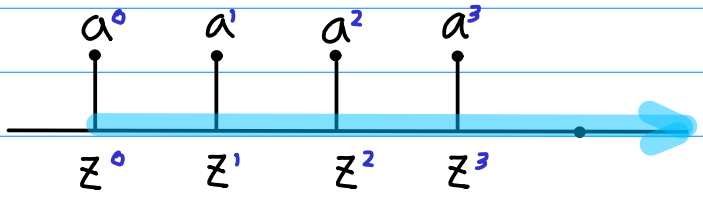
reach the same value later



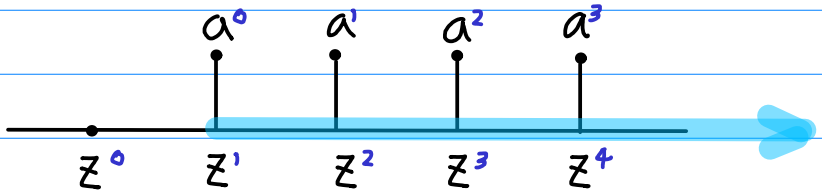
reach the same value earlier

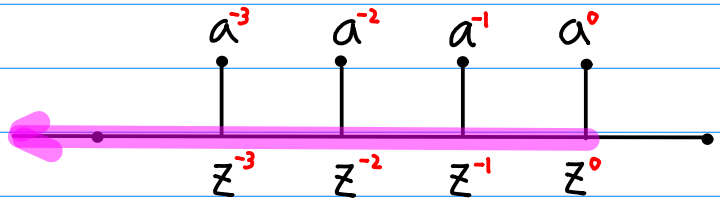


$*z$ | $*z^{-1}$

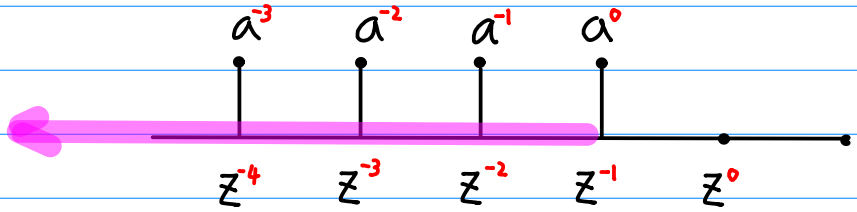


Shift Right time slots — $*z$ ⇅ $*z^{-1}$ — Shift Left time slots





Shift Left time slots — $*z^{-1}$ ⇅ $*z$ — Shift Right time slots



Shifting Geometric Sequences

Shifting Exponent only | **Shifting Exponent & Range**

	$*a$	$*a^{-1}$	$*z$	$*z^{-1}$
Positive Exponent	$a^{n+1} \leftarrow a^n$ **Left Shift** $\leftarrow$	$a^{n-1} \leftarrow a^n$ **Right Shift** $\rightarrow$	$n \leftarrow n-1$ **Right Shift** $\Rightarrow$	$n \leftarrow n+1$ **Left Shift** $\Leftarrow$
Negative Exponent	$a^{-n+1} \leftarrow a^{-n}$ **Right Shift** $\rightarrow$	$a^{-n-1} \leftarrow a^{-n}$ **Left Shift** $\leftarrow$		

	Positive Exponent		**Negative Exponent**	
Left Shifted	(1) *a, (7) /z	$\frac{a}{1-az}$ $\lvert z\rvert < a^{-1}$	$a^{n+1}\,u(n)$	(2) /a, (8) /z	$\frac{a^{-1}}{1-a^{-1}z}$ $\lvert z\rvert < a$	$a^{-n-1}\,u(n)$
RIght Shifted	(1) *z, (7) /a	$\frac{z}{1-az}$ $\lvert z\rvert < a^{-1}$	$a^{n-1}\,u(n-1)$	(2) *z, (8) *a	$\frac{z}{1-a^{-1}z}$ $\lvert z\rvert < a$	$a^{-n+1}\,u(n-1)$
Left Shifted	(3) /z, (5) *a	$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$a^{n+1}\,u(-n-1)$	(4) /z, (6) /a	$-\frac{z^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$	$a^{-n-1}\,u(-n-1)$
RIght Shifted	(3) /a, (5) *z	$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$a^{n-1}\,u(-n)$	(4) *a, (6) *z	$-\frac{a}{1-az^{-1}}$ $\lvert z\rvert > a$	$a^{-n+1}\,u(-n)$

Initial and Shifted Geometric Sequences

Positive Exponent

o-including (1)

$\frac{1}{1-az}$ $|z|<a^{-1}$ a^n u(n)

o-excluding (7)

$\frac{az}{1-az}$ $|z|<a^{-1}$ a^n u(n-1)

o-execluding (5)

$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ a^n u(-n-1)

o-including (3)

$-\frac{1}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ a^n u(-n)

Negative Exponent

o-including (2)

$\frac{1}{1-a^{-1}z}$ $|z|<a^{-1}$ a^{-n} u(n)

o-excluding (8)

$\frac{a^{-1}z}{1-a^{-1}z}$ $|z|<a^{-1}$ a^{-n} u(n-1)

o-excluding (6)

$-\frac{az^{-1}}{1-az^{-1}}$ $|z|>a^{-1}$ a^{-n} u(-n-1)

o-including (4)

$-\frac{1}{1-az^{-1}}$ $|z|>a$ a^{-n} u(-n)

Positive Exponent

Left Shifted: (1) *a ←, (7) /z ⇐

$\frac{a}{1-az}$ $|z|<a^{-1}$ a^{n+1} u(n)

RIght Shifted: (1) *z ⇒, (7) /a →

$\frac{z}{1-az}$ $|z|<a^{-1}$ a^{n-1} u(n-1)

Left Shifted: (5) *a ←, (3) /z ⇐

$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ a^{n+1} u(-n-1)

RIght Shifted: (5) *z ⇒, (3) /a →

$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ a^{n-1} u(-n)

Negative Exponent

(2) /a ←, (8) /z ⇐

$\frac{a^{-1}}{1-a^{-1}z}$ $|z|<a$ a^{-n-1} u(n)

(2) *z ⇒, (8) *a →

$\frac{z}{1-a^{-1}z}$ $|z|<a$ a^{-n+1} u(n-1)

(6) /a ←, (4) /z ⇐

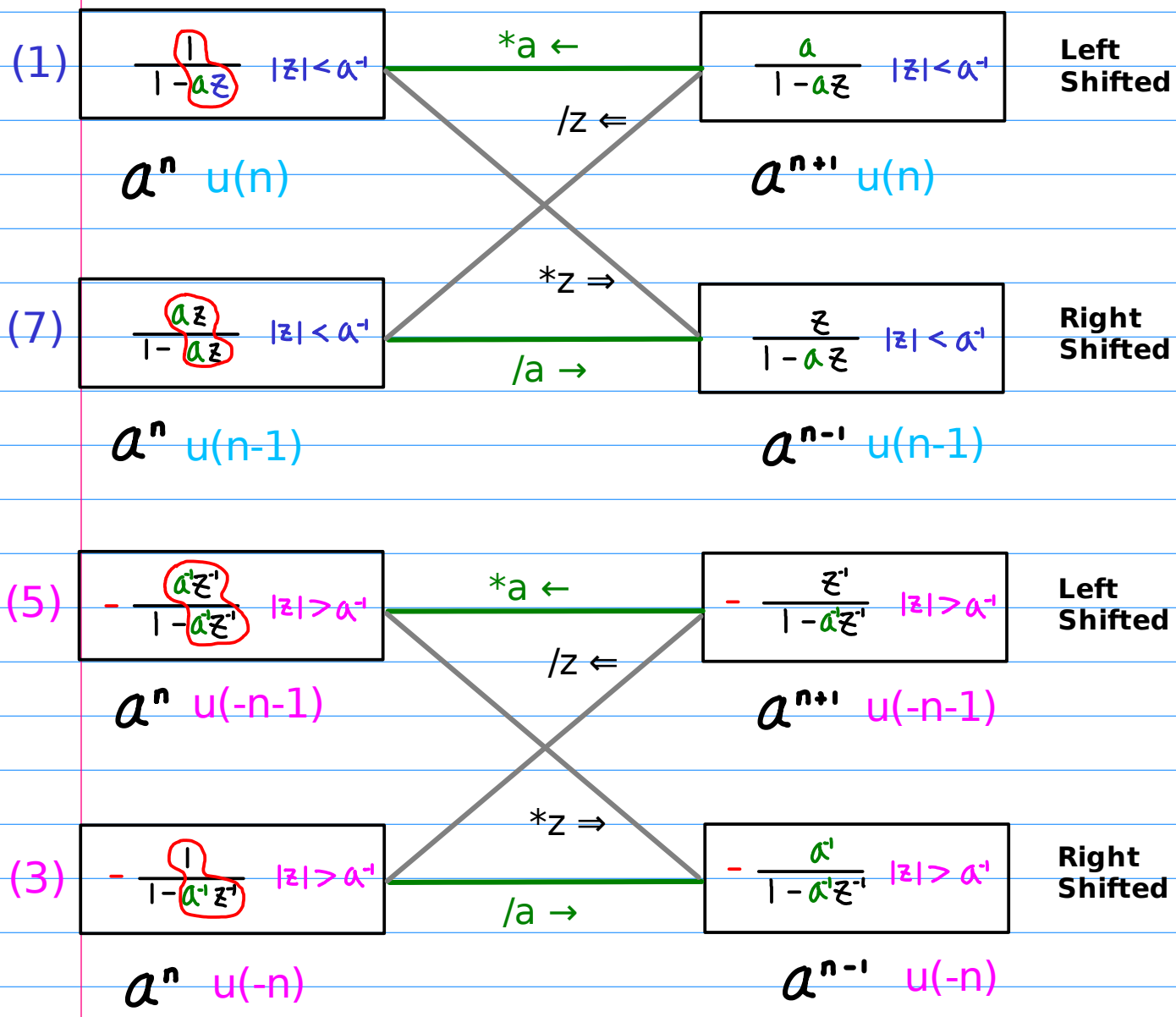
$-\frac{z^{-1}}{1-az^{-1}}$ $|z|>a$ a^{-n-1} u(-n-1)

(6) *z ⇒, (4) *a →

$-\frac{a}{1-az^{-1}}$ $|z|>a$ a^{-n+1} u(-n)

Shifting Geometric Series (1) positive exponent

Positive Exponent a /z $n \leftarrow n+1$ *z $n \leftarrow n-1$

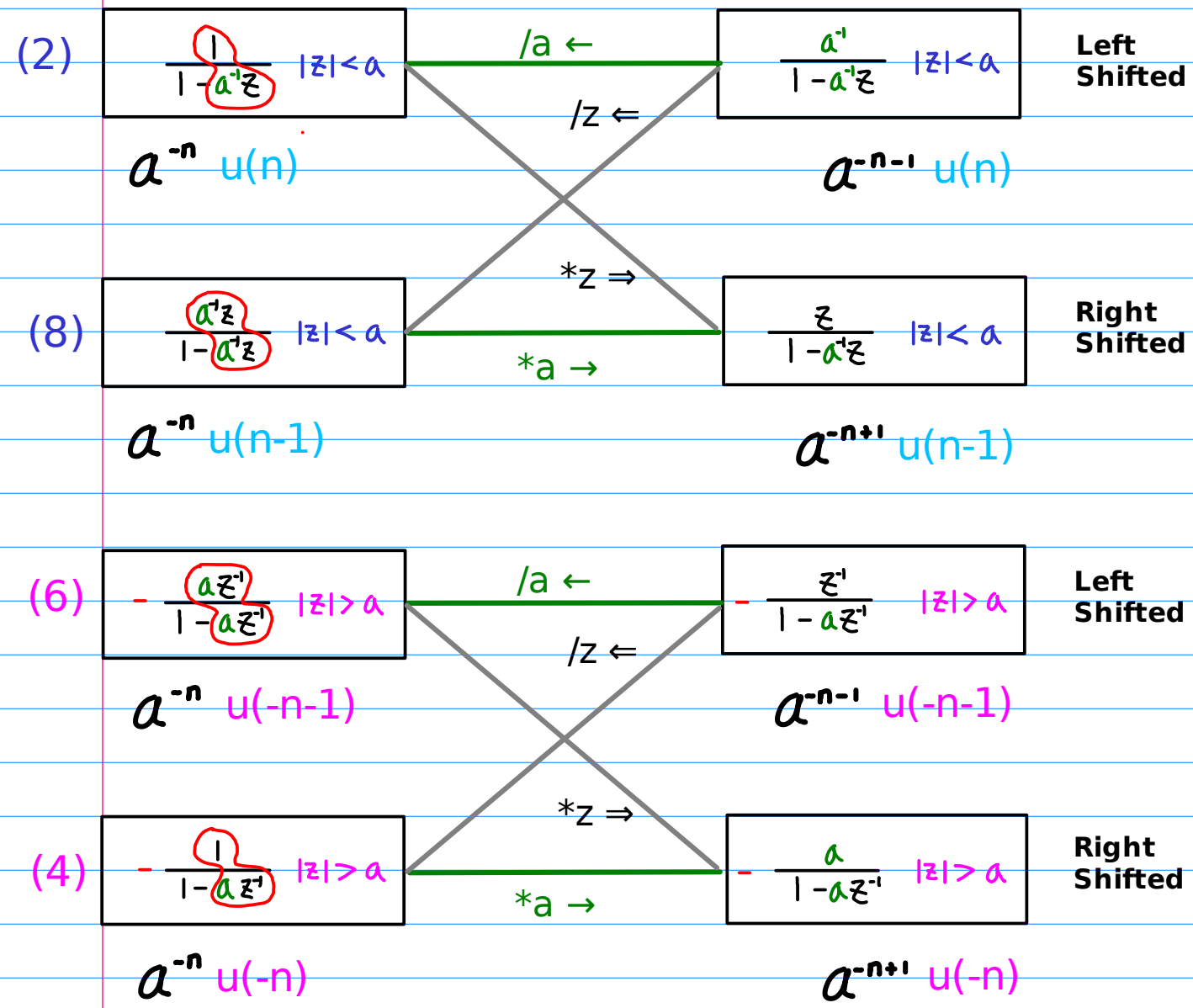


Causal	**u(n)**	(1)	(2)
	u(n-1)	(7)	(8)
Anti-Causal	**u(-n-1)**	(5)	(6)
	u(-n)	(3)	(4)

butterfly pair ordering

Shifting Geometric Series (2) negative exponent

Negative Exponent a^{-1} /z n ← n+1 *z n ← n-1

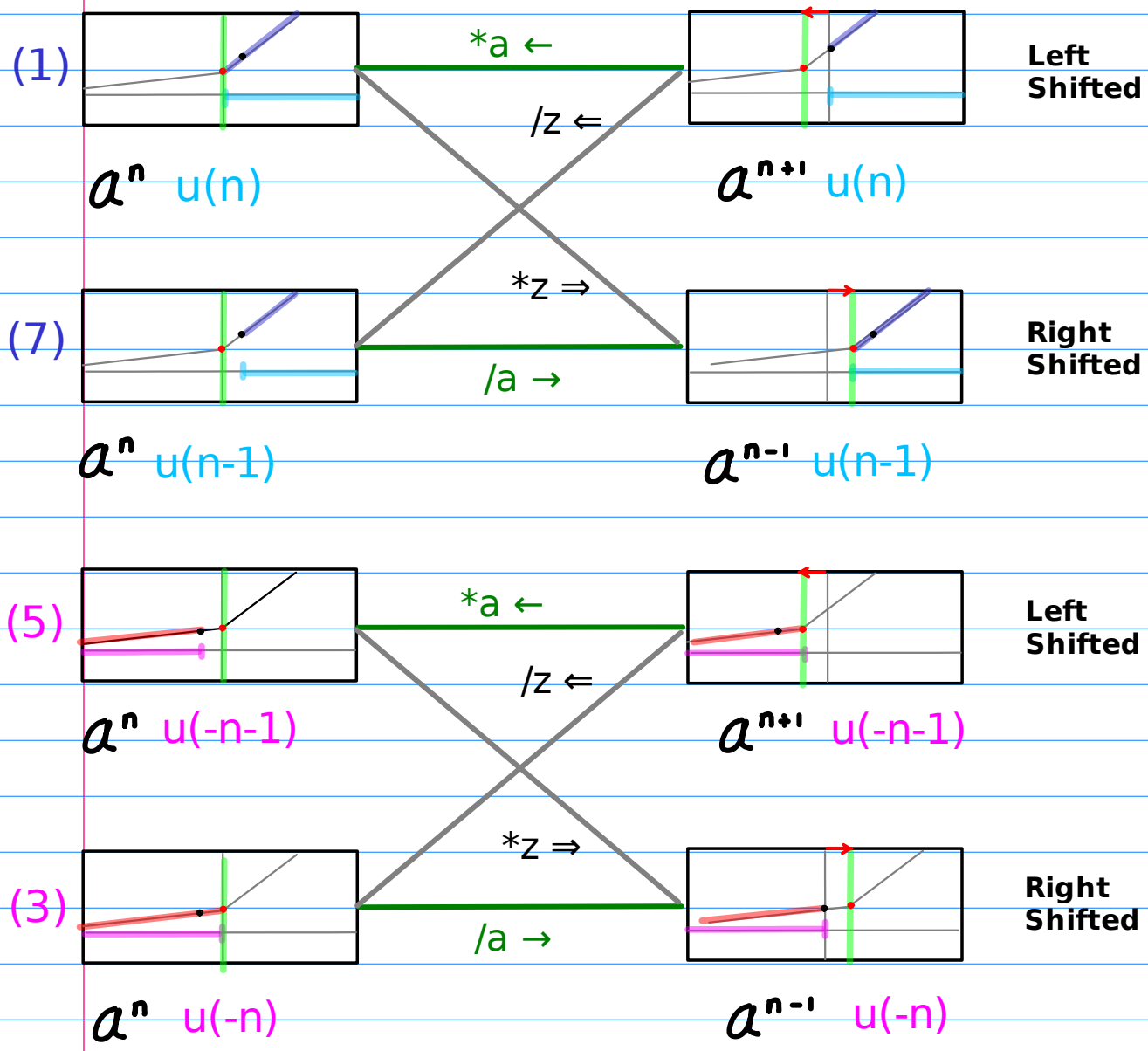


Causal	**u(n)**	(1)	(2)
	u(n-1)	(7)	(8)
Anti-Causal	**u(-n-1)**	(5)	(6)
	u(-n)	(3)	(4)

butterfly pair ordering

Shifting Geometric Series

(1) Series view - positive exponent

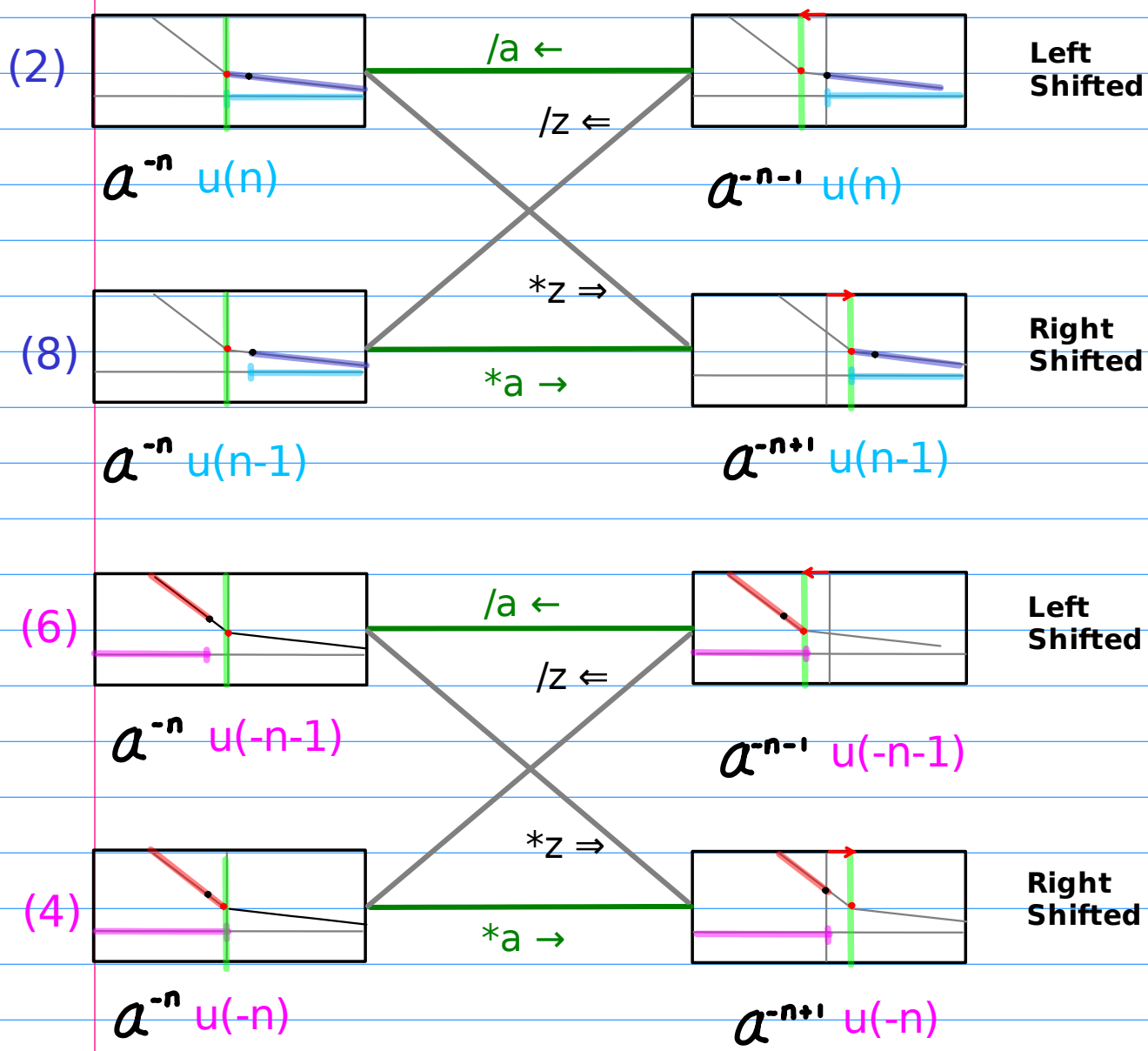


Causal	u(n)	(1)	(2)
	u(n-1)	(7)	(8)
Anti-Causal	u(-n-1)	(5)	(6)
	u(-n)	(3)	(4)

butterfly pair ordering

Shifting Geometric Series

(2) Series view - negative exponent

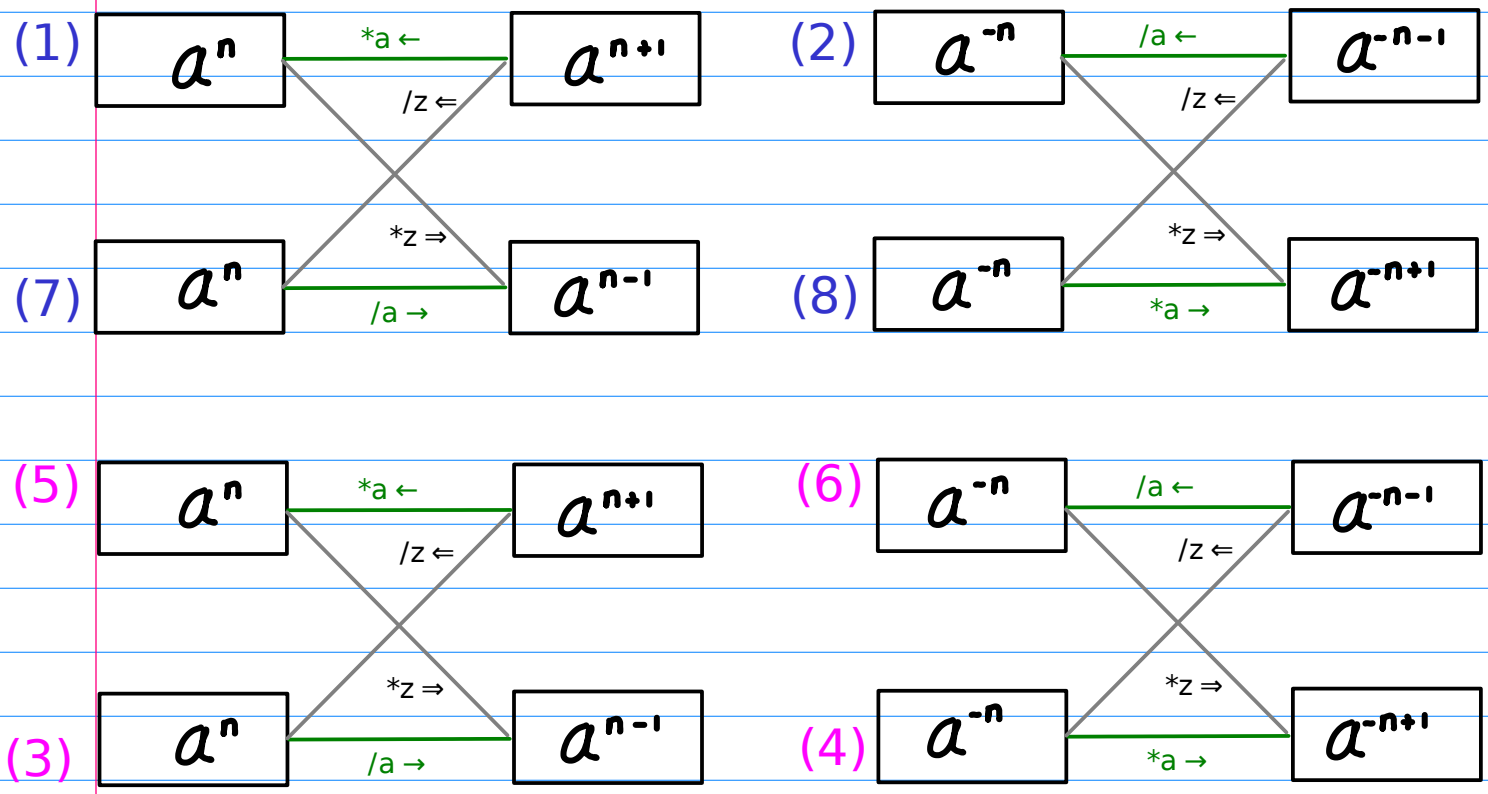


Causal	u(n)	(1)	(2)
	u(n-1)	(7)	(8)
Anti-Causal	u(-n-1)	(5)	(6)
	u(-n)	(3)	(4)

butterfly pair ordering

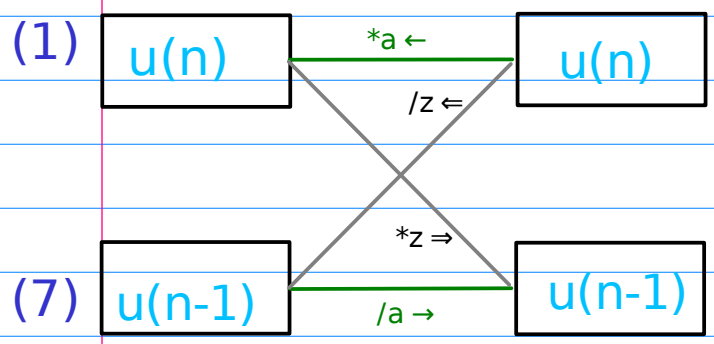
Shifting Geometric Series

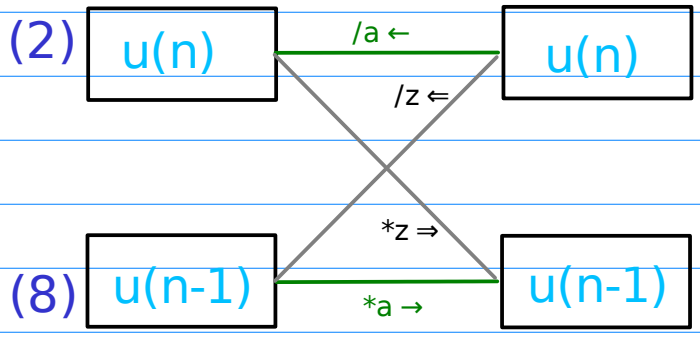
(3) Exponent view (postive, negative exponents)

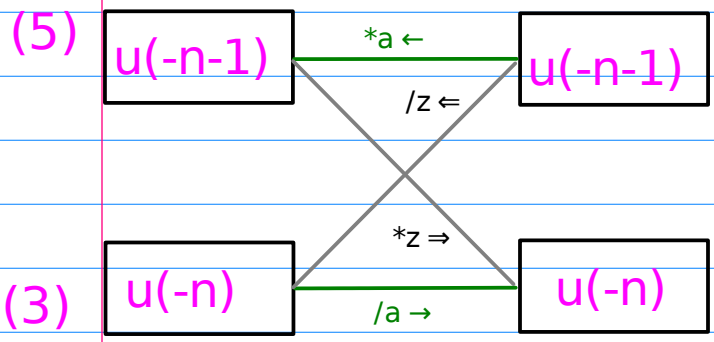


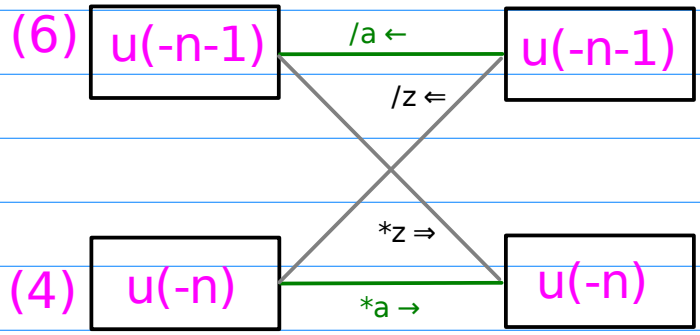
Shifting Geometric Series

(4) Range view (postive, negative exponents)

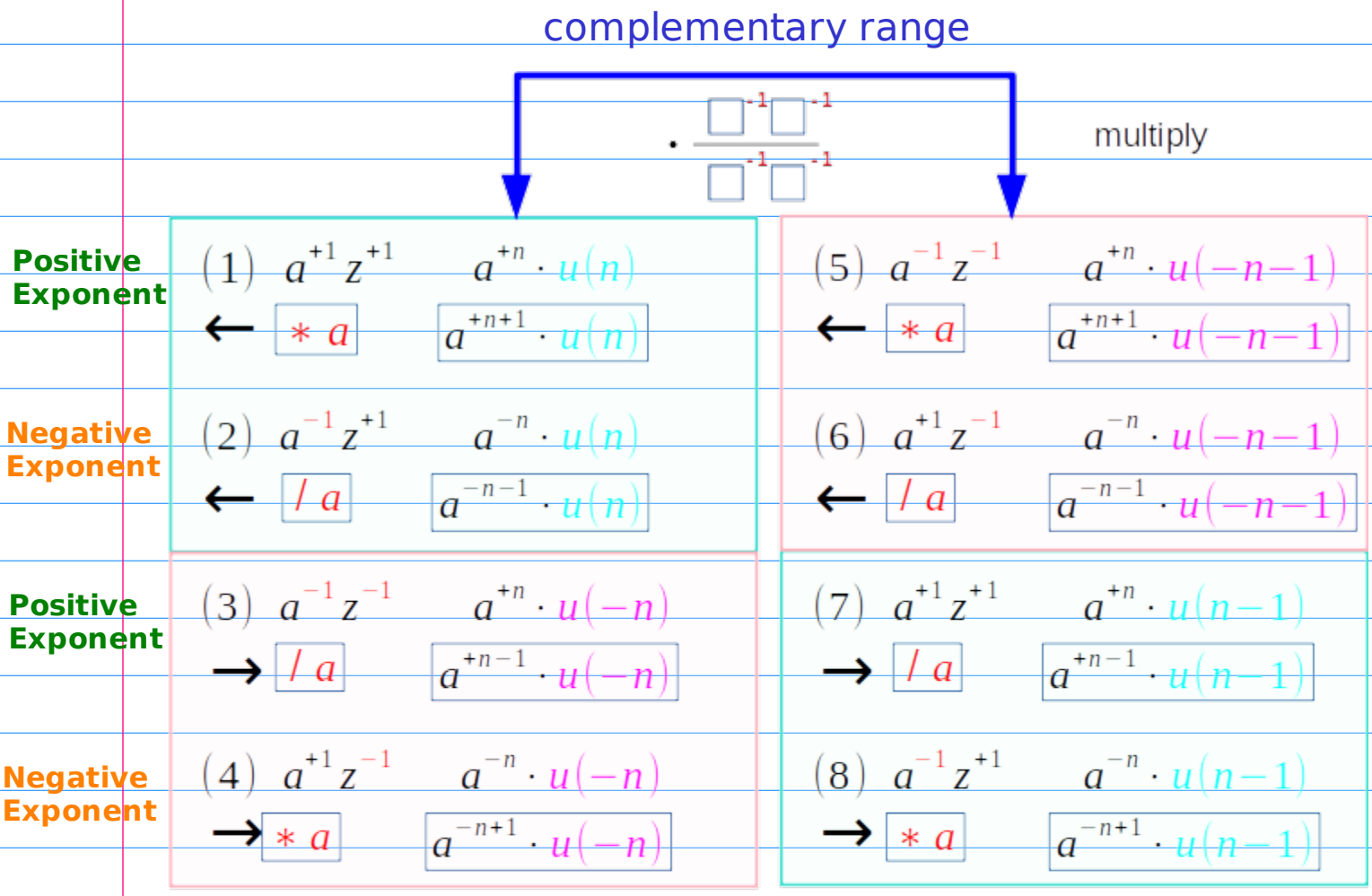


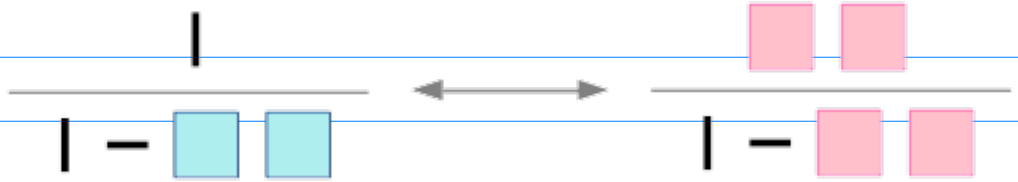






Shifting Geometric Series by *a or /a

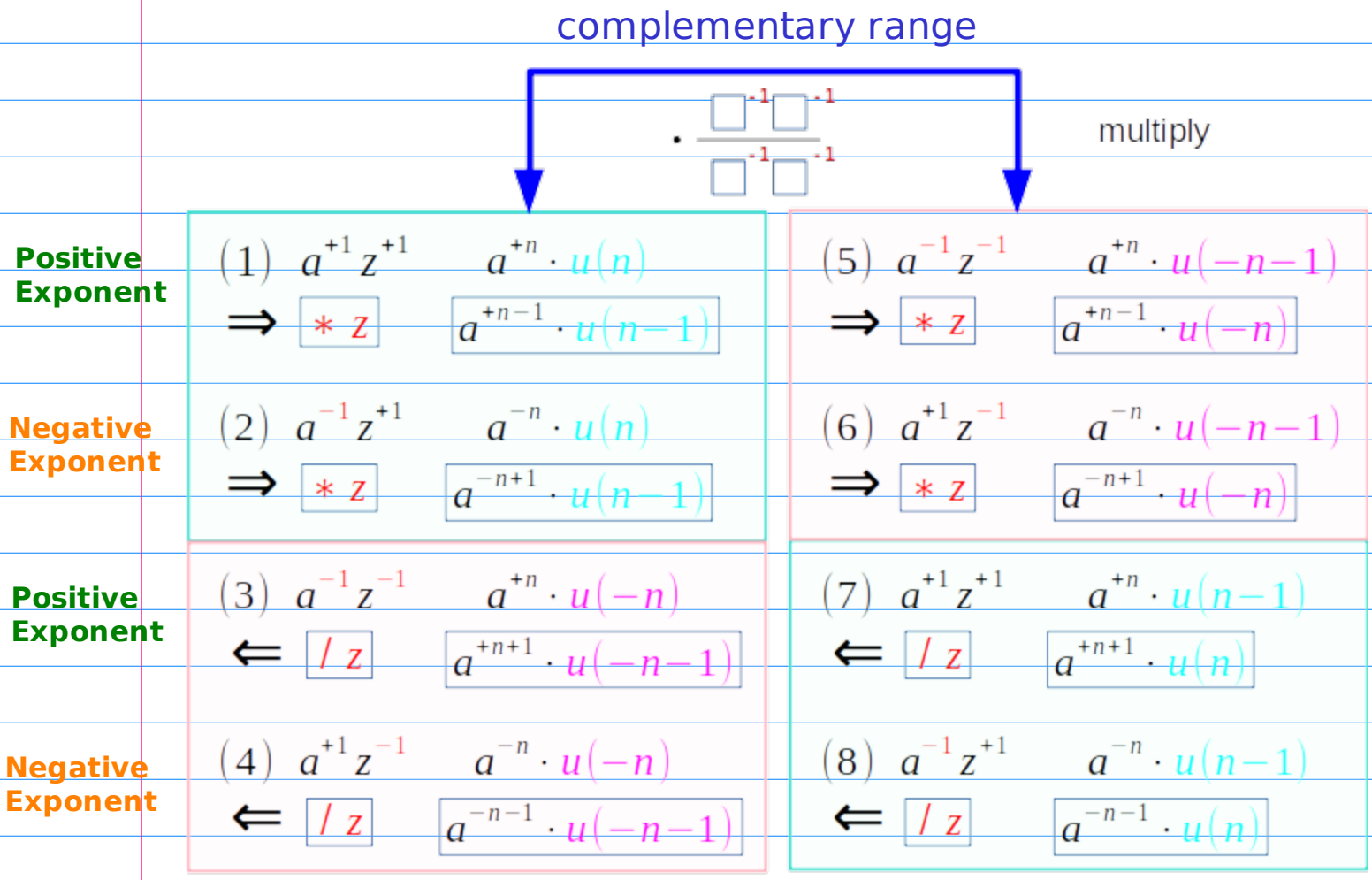


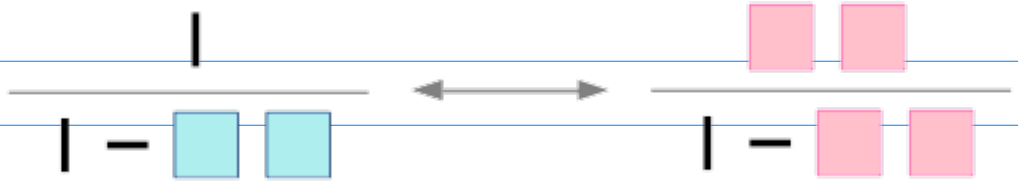


a unit start term

a non-unit start term

Shifting Geometric Series by *z or /z





a unit start term

a non-unit start term

u(n)	(1)	(2)
u(-n)	(3)	(4)
u(-n-1)	(5)	(6)
u(n-1)	(7)	(8)

symmetric pair ordering .

Positive Exponent **Negative Exponent**

(1) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n)$ ← $*\,a$ $a^{+n+1}\cdot u(n)$

(2) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n)$ ← $/\,a$ $a^{-n-1}\cdot u(n)$

(3) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n)$ → $/\,a$ $a^{+n-1}\cdot u(-n)$

(4) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n)$ → $*\,a$ $a^{-n+1}\cdot u(-n)$

(5) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n-1)$ ← $*\,a$ $a^{+n+1}\cdot u(-n-1)$

(6) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n-1)$ ← $/\,a$ $a^{-n-1}\cdot u(-n-1)$

(7) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n-1)$ → $/\,a$ $a^{+n-1}\cdot u(n-1)$

(8) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n-1)$ → $*\,a$ $a^{-n+1}\cdot u(n-1)$

Positive Exponent **Negative Exponent**

(1) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n)$ ⇒ $*\,z$ $a^{+n-1}\cdot u(n-1)$

(2) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n)$ ⇒ $*\,z$ $a^{-n+1}\cdot u(n-1)$

(3) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n)$ ⇐ $/\,z$ $a^{+n+1}\cdot u(-n-1)$

(4) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n)$ ⇐ $/\,z$ $a^{-n-1}\cdot u(-n-1)$

(5) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n-1)$ ⇒ $*\,z$ $a^{+n-1}\cdot u(-n)$

(6) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n-1)$ ⇒ $*\,z$ $a^{-n+1}\cdot u(-n)$

(7) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n-1)$ ⇐ $/\,z$ $a^{+n+1}\cdot u(n)$

(8) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n-1)$ ⇐ $/\,z$ $a^{-n-1}\cdot u(n)$

Causal	**u(n)**	(1)	(2)
	u(n-1)	(7)	(8)
Anti-Causal	**u(-n-1)**	(5)	(6)
	u(-n)	(3)	(4)

butterfly pair ordering

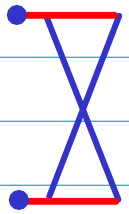
Positive Exponent **Negative Exponent**

(1) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n)$ ← $*\,a$ $a^{+n+1}\cdot u(n)$

(2) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n)$ ← $/\,a$ $a^{-n-1}\cdot u(n)$

(7) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n-1)$ → $/\,a$ $a^{+n-1}\cdot u(n-1)$

(8) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n-1)$ → $*\,a$ $a^{-n+1}\cdot u(n-1)$



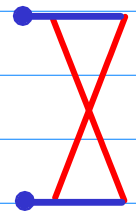
(5) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n-1)$ ← $*\,a$ $a^{+n+1}\cdot u(-n-1)$

(6) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n-1)$ ← $/\,a$ $a^{-n-1}\cdot u(-n-1)$

(3) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n)$ → $/\,a$ $a^{+n-1}\cdot u(-n)$

(4) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n)$ → $*\,a$ $a^{-n+1}\cdot u(-n)$

Positive Exponent **Negative Exponent**

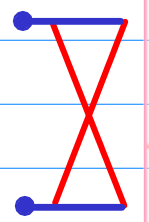


(1) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n)$ ⇒ $*\,z$ $a^{+n-1}\cdot u(n-1)$

(2) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n)$ ⇒ $*\,z$ $a^{-n+1}\cdot u(n-1)$

(7) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n-1)$ ⇐ $/\,z$ $a^{+n+1}\cdot u(n)$

(8) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n-1)$ ⇐ $/\,z$ $a^{-n-1}\cdot u(n)$



(5) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n-1)$ ⇒ $*\,z$ $a^{+n-1}\cdot u(-n)$

(6) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n-1)$ ⇒ $*\,z$ $a^{-n+1}\cdot u(-n)$

(3) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n)$ ⇐ $/\,z$ $a^{+n+1}\cdot u(-n-1)$

(4) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n)$ ⇐ $/\,z$ $a^{-n-1}\cdot u(-n-1)$

Complement Type I	**u(n)**	(1)	(2)
	u(-n-1)	(5)	(6)
Complement Type II	**u(-n)**	(3)	(4)
	u(n-1)	(7)	(8)

complementary pair ordering

Positive Exponent **Negative Exponent**

(1) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n)$ (2) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n)$

← $*\,a$ $a^{+n+1}\cdot u(n)$ ← $/\,a$ $a^{-n-1}\cdot u(n)$

(5) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n-1)$ (6) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n-1)$

← $*\,a$ $a^{+n+1}\cdot u(-n-1)$ ← $/\,a$ $a^{-n-1}\cdot u(-n-1)$

(3) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n)$ (4) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n)$

→ $/\,a$ $a^{+n-1}\cdot u(-n)$ → $*\,a$ $a^{-n+1}\cdot u(-n)$

(7) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n-1)$ (8) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n-1)$

→ $/\,a$ $a^{+n-1}\cdot u(n-1)$ → $*\,a$ $a^{-n+1}\cdot u(n-1)$

Positive Exponent **Negative Exponent**

(1) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n)$ (2) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n)$

⇒ $*\,z$ $a^{+n-1}\cdot u(n-1)$ ⇒ $*\,z$ $a^{-n+1}\cdot u(n-1)$

(5) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n-1)$ (6) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n-1)$

⇒ $*\,z$ $a^{+n-1}\cdot u(-n)$ ⇒ $*\,z$ $a^{-n+1}\cdot u(-n)$

(3) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n)$ (4) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n)$

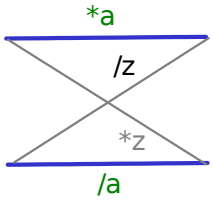
⇐ $/\,z$ $a^{+n+1}\cdot u(-n-1)$ ⇐ $/\,z$ $a^{-n-1}\cdot u(-n-1)$

(7) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n-1)$ (8) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n-1)$

⇐ $/\,z$ $a^{+n+1}\cdot u(n)$ ⇐ $/\,z$ $a^{-n-1}\cdot u(n)$

Shifted Geometric Series (1)

by multiplying a or a^{-1}



Positive Exponent

(1) ←

$\frac{1}{1-az^{-1}}$ $|z|<a^{-1}$ $a^n u(n)$ $\times a$ $\frac{a}{1-az^{-1}}$ $|z|<a^{-1}$ $a^{n+1} u(n)$

(7) →

$\frac{az^{-1}}{1-az^{-1}}$ $|z|<a^{-1}$ $a^n u(n-1)$ $\times a^{-1}$ $\frac{z^{-1}}{1-az^{-1}}$ $|z|<a^{-1}$ $a^{n-1} u(n-1)$

(5) ←

$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ $a^n u(-n-1)$ $\times a$ $-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ $a^{n+1} u(-n-1)$

(3) →

$-\frac{1}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ $a^n u(-n)$ $\times a^{-1}$ $-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ $a^{n-1} u(-n)$

Negative Exponent

(2) ←

$\frac{1}{1-a^{-1}z}$ $|z|<a$ $a^{-n} u(n)$ $\times a^{-1}$ $\frac{a^{-1}}{1-a^{-1}z}$ $|z|<a$ $a^{-n-1} u(n)$

(8) →

$\frac{a^{-1}z}{1-a^{-1}z}$ $|z|<a$ $a^{-n} u(n-1)$ $\times a$ $\frac{z}{1-a^{-1}z}$ $|z|<a$ $a^{-n+1} u(n-1)$

(6) ←

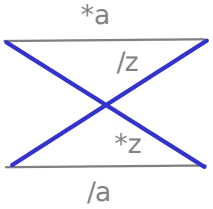
$-\frac{az^{-1}}{1-az^{-1}}$ $|z|>a$ $a^{-n} u(-n-1)$ $\times a^{-1}$ $-\frac{z^{-1}}{1-az^{-1}}$ $|z|>a$ $a^{-n-1} u(-n-1)$

(4) →

$-\frac{1}{1-az^{-1}}$ $|z|>a$ $a^{-n} u(-n)$ $\times a$ $-\frac{a}{1-az^{-1}}$ $|z|>a$ $a^{-n+1} u(-n)$

Shifted Geometric Series (2)

by multiplying z or z^{-1}



Positive Exponent

(1) ⇐ $\frac{az}{1-az}$ $|z|<a^{-1}$ | $n \leftarrow n+1$ | $a^n u(n-1)$ z^{-1} | $\frac{a}{1-az}$ $|z|<a^{-1}$ | $a^{n+1} u(n)$

(7) ⇒ $\frac{1}{1-az}$ $|z|<a^{-1}$ | $n \leftarrow n-1$ | $a^n u(n)$ $\times z$ | $\frac{z}{1-az}$ $|z|<a^{-1}$ | $a^{n-1} u(n-1)$

(5) ⇐ $-\frac{1}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ | $n \leftarrow n+1$ | $a^n u(-n)$ $\times z^{-1}$ | $-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ | $a^{n+1} u(-n-1)$

(3) ⇒ $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ | $n \leftarrow n-1$ | $a^n u(-n-1) \times z$ | $-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ | $a^{n-1} u(-n)$

Negative Exponent

(2) ⇐ $\frac{a^{-1}z}{1-a^{-1}z}$ $|z|<a$ | $n \leftarrow n+1$ | $a^{-n} u(n-1) \times z^{-1}$ | $\frac{a^{-1}}{1-a^{-1}z}$ $|z|<a$ | $a^{-n-1} u(n)$

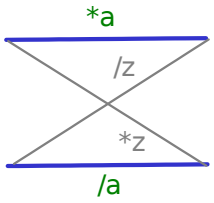
(8) ⇒ $\frac{1}{1-a^{-1}z}$ $|z|<a$ | $n \leftarrow n-1$ | $a^{-n} u(n)$ $\times z$ | $\frac{z}{1-a^{-1}z}$ $|z|<a$ | $a^{-n+1} u(n-1)$

(6) ⇐ $-\frac{1}{1-az^{-1}}$ $|z|>a$ | $n \leftarrow n+1$ | $a^{-n} u(-n)$ $\times z^{-1}$ | $-\frac{z^{-1}}{1-az^{-1}}$ $|z|>a$ | $a^{-n-1} u(-n-1)$

(4) ⇒ $-\frac{az^{-1}}{1-az^{-1}}$ $|z|>a$ | $n \leftarrow n-1$ | $a^{-n} u(-n-1) \times z$ | $-\frac{a}{1-az^{-1}}$ $|z|>a$ | $a^{-n+1} u(-n)$

Shifted Geometric Series (3)

by multiplying a or a^{-1} Assume a > 1



(1)	a^{n} u(n)	*a	←	a^{n+1}	u(n)
(2)	a^{-n} u(n)	/a	←	a^{-n-1}	u(n)
(3)	a^{n} u(-n)	/a	→	a^{n-1}	u(-n)
(4)	a^{-n} u(-n)	*a	→	a^{-n+1}	u(-n)
(5)	a^{n} u(-n-1)	*a	←	a^{n+1}	u(-n-1)
(6)	a^{-n} u(-n-1)	/a	←	a^{-n-1}	u(-n-1)
(7)	a^{n} u(n-1)	/a	→	a^{n-1}	u(n-1)
(8)	a^{-n} u(n-1)	*a	→	a^{-n+1}	u(n-1)

row major ordering

(1)	(2)	*a	/a	←	←
(3)	(4)	/a	*a	→	→
(5)	(6)	*a	/a	←	←
(7)	(8)	/a	*a	→	→

complementary pair ordering

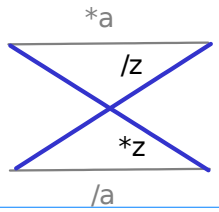
(1)	(2)	*a	/a	←	←
(5)	(6)	*a	/a	←	←
(3)	(4)	/a	*a	→	→
(7)	(8)	/a	*a	→	→

butterfly pair ordering

(1)	(2)	*a	/a	←	←
(7)	(8)	/a	*a	→	→
(5)	(6)	*a	/a	←	←
(3)	(4)	/a	*a	→	→

Shifted Geometric Series (4)

by multiplying z or z^{-1} Assume $a > 1$



(1)	a^n u(n)	*z	$\Rightarrow$	a^{n-1}	u(n-1)
(2)	a^{-n} u(n)	*z	$\Rightarrow$	a^{-n+1}	u(n-1)
(3)	a^n u(-n)	/z	$\Leftarrow$	a^{n+1}	u(-n-1)
(4)	a^{-n} u(-n)	/z	$\Leftarrow$	a^{-n-1}	u(-n-1)
(5)	a^n u(-n-1)	*z	$\Rightarrow$	a^{n-1}	u(-n)
(6)	a^{-n} u(-n-1)	*z	$\Rightarrow$	a^{-n+1}	u(-n)
(7)	a^n u(n-1)	/z	$\Leftarrow$	a^{n+1}	u(n)
(8)	a^{-n} u(n-1)	/z	$\Leftarrow$	a^{-n-1}	u(n)

row major ordering

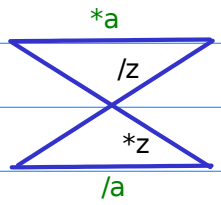
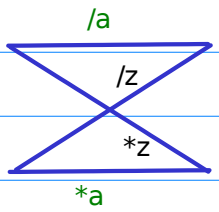
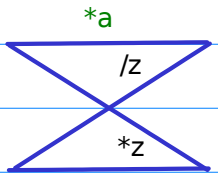
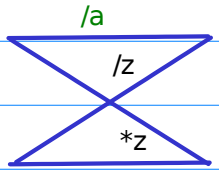
(1)	(2)	*z	*z	$\Rightarrow$	$\Rightarrow$		
(3)	(4)	/z	/z	$\Leftarrow$	$\Leftarrow$		
(5)	(6)	*z	*z	$\Rightarrow$	$\Rightarrow$		
(7)	(8)	/z	/z	$\Leftarrow$	$\Leftarrow$		

complementary pair ordering

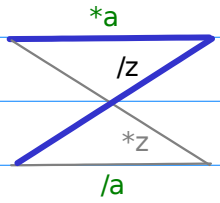
(1)	(2)	*z	*z	$\Rightarrow$	$\Rightarrow$		
(5)	(6)	*z	*z	$\Rightarrow$	$\Rightarrow$		
(3)	(4)	/z	/z	$\Leftarrow$	$\Leftarrow$		
(7)	(8)	/z	/z	$\Leftarrow$	$\Leftarrow$		

butterfly pair ordering

(1)	(2)	*z	*z	$\Rightarrow$	$\Rightarrow$		
(7)	(8)	/z	/z	$\Leftarrow$	$\Leftarrow$		
(5)	(6)	*z	*z	$\Rightarrow$	$\Rightarrow$		
(3)	(4)	/z	/z	$\Leftarrow$	$\Leftarrow$		

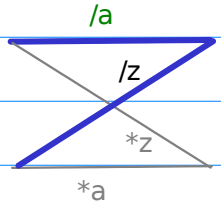
(1) $a^n u(n)$  $a^{n+1} u(n)$	(2) $a^{-n} u(n)$  $a^{-n-1} u(n)$
(7) $a^n u(n-1)$ $a^{n-1} u(n-1)$	(8) $a^{-n} u(n-1)$ $a^{-n+1} u(n-1)$
(5) $a^n u(-n-1)$  $a^{n+1} u(-n-1)$	(6) $a^{-n} u(-n-1)$  $a^{-n-1} u(-n-1)$
(3) $a^n u(-n)$ $a^{n-1} u(-n)$	(4) $a^{-n} u(-n)$ $a^{-n+1} u(-n)$

← (1) *a = (7) /z ⇐

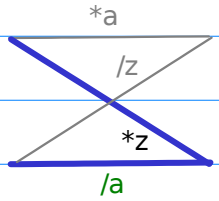


⇒ (1) *z = (7) /a →

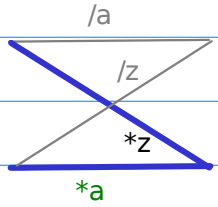
← (2) /a = (8) /z ⇐



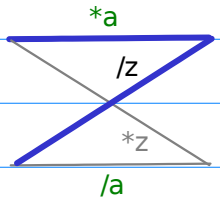
⇒ (2) *z = (8) *a →



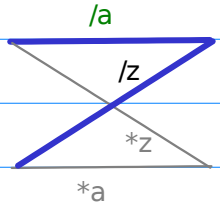
← (5) *a = (3) /z ⇐



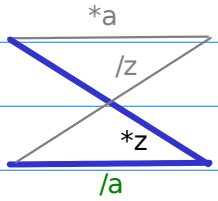
← (6) /a = (4) /z ⇐

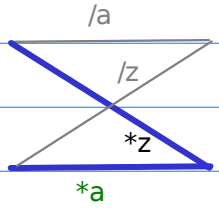


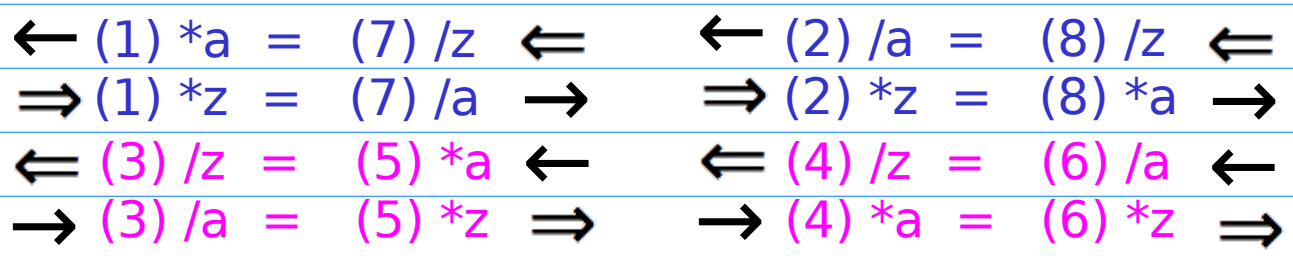
⇒ (5) *z = (3) /a →



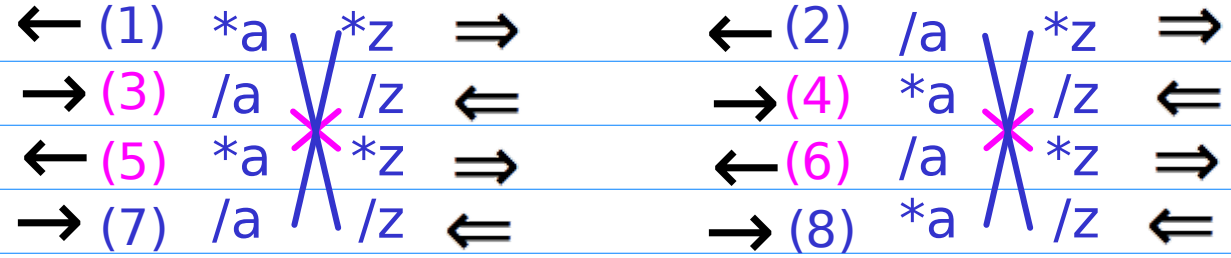
⇒ (6) *z = (4) *a →



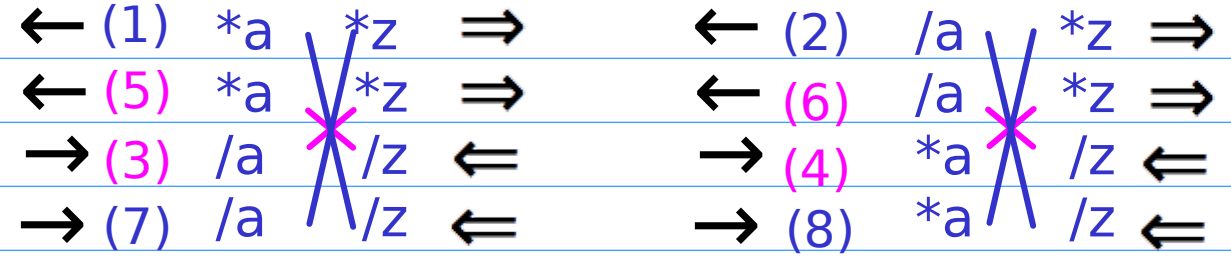




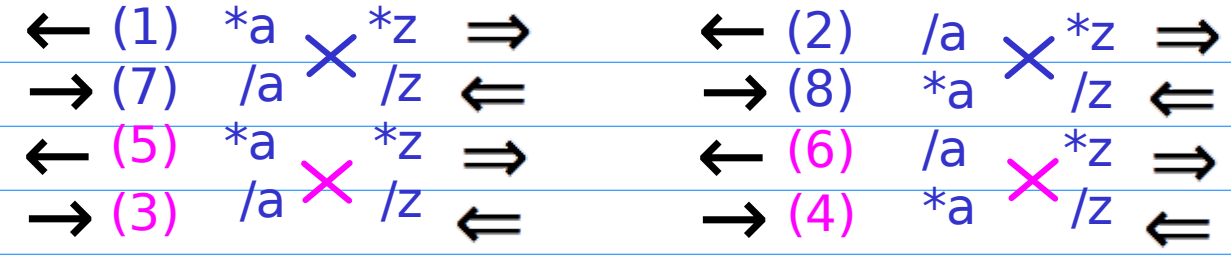
row major ordering



complementary pair ordering



butterfly pair ordering



Assume $a > 1$

(1)	a^n	u(n)	*a ← a^{n+1}	u(n)
(7)	a^n	u(n-1)	/a → a^{n-1}	u(n-1)
(5)	a^n	u(-n-1)	*a ← a^{n+1}	u(-n-1)
(3)	a^n	u(-n)	/a → a^{n-1}	u(-n)
(2)	a^{-n}	u(n)	/a ← a^{-n-1}	u(n)
(8)	a^{-n}	u(n-1)	*a → a^{-n+1}	u(n-1)
(6)	a^{-n}	u(-n-1)	/a ← a^{-n-1}	u(-n-1)
(4)	a^{-n}	u(-n)	*a → a^{-n+1}	u(-n)

2^n u(n)	2^{n+1} u(n)
2^n u(n-1)	2^{n-1} u(n-1)
2^n u(-n-1)	2^{n+1} u(-n-1)
2^n u(-n)	2^{n-1} u(-n)

$(\frac{1}{2})^{-n}$ u(n)	$(\frac{1}{2})^{-n-1}$ u(n)
$(\frac{1}{2})^{-n}$ u(n-1)	$(\frac{1}{2})^{-n+1}$ u(n-1)
$(\frac{1}{2})^{-n}$ u(-n-1)	$(\frac{1}{2})^{-n-1}$ u(-n-1)
$(\frac{1}{2})^{-n}$ u(-n)	$(\frac{1}{2})^{-n+1}$ u(-n)

$(\frac{1}{2})^n$ u(n)	$(\frac{1}{2})^{n+1}$ u(n)
$(\frac{1}{2})^n$ u(n-1)	$(\frac{1}{2})^{n-1}$ u(n-1)
$(\frac{1}{2})^n$ u(-n-1)	$(\frac{1}{2})^{n+1}$ u(-n-1)
$(\frac{1}{2})^n$ u(-n)	$(\frac{1}{2})^{n-1}$ u(-n)

2^{-n} u(n)	2^{-n-1} u(n)
2^{-n} u(n-1)	2^{-n+1} u(n-1)
2^{-n} u(-n-1)	2^{-n-1} u(-n-1)
2^{-n} u(-n)	2^{-n+1} u(-n)

Complement Type I	**u(n)**	(1)	(2)
	u(-n-1)	(5)	(6)
Complement Type II	**u(-n)**	(3)	(4)
	u(n-1)	(7)	(8)

complementary pair ordering

(1) a z	a^{n}	(2) $a^{-1}z$	a^{-n}
(5) $a^{-1}z^{-1}$	a^{n}	(6) $a\ z^{-1}$	a^{-n}
(3) $a^{-1}z^{-1}$	a^{n}	(4) $a\ z^{-1}$	a^{-n}
(7) a z	a^{n}	(8) $a^{-1}z$	a^{-n}

(1) a z	$*a$	(2) $a^{-1}z$	$/a$
(5) $a^{-1}z^{-1}$	$*a$	(6) $a\ z^{-1}$	$/a$
(3) $a^{-1}z^{-1}$	$/a$	(4) $a\ z^{-1}$	$*a$
(7) a z	$/a$	(8) $a^{-1}z$	$*a$

(1) a z	$*z$	(2) $a^{-1}z$	$*z$
(5) $a^{-1}z^{-1}$	$*z$	(6) $a\ z^{-1}$	$*z$
(3) $a^{-1}z^{-1}$	$/z$	(4) $a\ z^{-1}$	$/z$
(7) a z	$/z$	(8) $a^{-1}z$	$/z$

Geometric Series Combinations

complementary pair ordering

(1)

unit

$\frac{1}{1-az}$	$\lvert z\rvert < a^{-1}$	$a^n\ u(n)$
$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$-a^n\ u(-n-1)$

non-unit

(2)

$\frac{1}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\left(\frac{1}{a}\right)^n\ u(n)$
$-\frac{az^{-1}}{1-az^{-1}}$	$\lvert z\rvert > a$	$-\left(\frac{1}{a}\right)^n\ u(-n-1)$

(5)

(6)

(3)

unit

$-\frac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$-a^n\ u(-n)$
$\frac{az}{1-az}$	$\lvert z\rvert < a^{-1}$	$a^n\ u(n-1)$

non-unit

(4)

$-\frac{1}{1-az^{-1}}$	$\lvert z\rvert > a$	$-\left(\frac{1}{a}\right)^n\ u(-n)$
$\frac{a^{-1}z}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\left(\frac{1}{a}\right)^n\ u(n-1)$

(7)

(8)

		Positive Exponent	Negative Exponent
Complement Type I	**u(n)**	(1)	(2)
	u(-n-1)	(5)	(6)
Complement Type II	**u(-n)**	(3)	(4)
	u(n-1)	(7)	(8)

Shifted Combinations
(I) by scaling *a /a

complementary pair ordering

(1) *a

$\frac{a}{1-az}$	$\lvert z\rvert < a^{-1}$	$a^{n+1}\ u(n)$
$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$-a^{n+1}\ u(-n-1)$

(2) /a

$\frac{a^{-1}}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\left(\frac{1}{a}\right)^{n+1}\ u(n)$
$-\frac{z^{-1}}{1-az^{-1}}$	$\lvert z\rvert > a$	$-\left(\frac{1}{a}\right)^{n+1}\ u(-n-1)$

(5) *a
(3) /a

$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$-a^{n-1}\ u(-n)$
$\frac{z}{1-az}$	$\lvert z\rvert < a^{-1}$	$a^{n-1}\ u(n-1)$

(7) /a

(6) /a
(4) *a

$-\frac{a}{1-az^{-1}}$	$\lvert z\rvert > a$	$-\left(\frac{1}{a}\right)^{n-1}\ u(-n)$
$\frac{z}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\left(\frac{1}{a}\right)^{n-1}\ u(n-1)$

(8) *a

	Positive Exponent	Negative Exponent		
Complement Type I	(1) u(n) (5) u(-n-1)	(2) u(n) (6) u(-n-1)	*a u(n) *a u(-n-1)	/a u(n) /a u(-n-1)
Complement Type II	(3) u(-n) (7) u(n-1)	(4) u(-n) (8) u(n-1)	/a u(-n) /a u(n-1)	*a u(-n) *a u(n-1)

Shifted Combinations
(II) by scaling *z /z

complementary pair ordering

(1) *z

$\frac{z}{1-az}$	$\lvert z\rvert < a^{-1}$	$a^{n-1}\,u(n-1)$
$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$-a^{n-1}\,u(-n)$

(2) *z

$\frac{z}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\left(\frac{1}{a}\right)^{n-1}\,u(n-1)$
$-\frac{a}{1-az^{-1}}$	$\lvert z\rvert > a$	$-\left(\frac{1}{a}\right)^{n-1}\,u(-n)$

(5) *z

(3) /z

$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$-a^{n+1}\,u(-n-1)$
$\frac{a}{1-az}$	$\lvert z\rvert < a^{-1}$	$a^{n+1}\,u(n)$

(7) /z

(6) *z

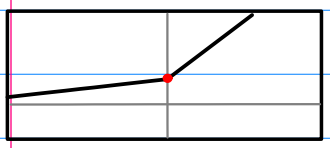
(4) /z

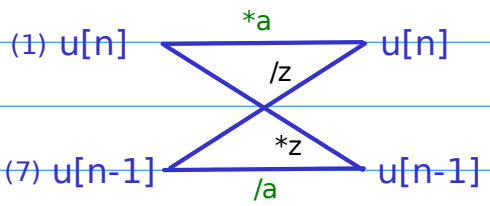
$-\frac{z^{-1}}{1-az^{-1}}$	$\lvert z\rvert > a$	$-\left(\frac{1}{a}\right)^{n+1}u(-n-1)$
$\frac{a^{-1}}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\left(\frac{1}{a}\right)^{n+1}u(n)$

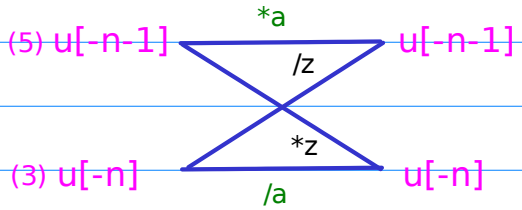
(8) /z

	Positive Exponent	Negative Exponent		
Complement Type I	(1) u(n) (5) u(-n-1)	(2) u(n) (6) u(-n-1)	*z u(n-1) *z u(-n)	*z u(n-1) *z u(-n)
Complement Type II	(3) u(-n) (7) u(n-1)	(4) u(-n) (8) u(n-1)	/z u(-n-1) /z u(n)	/z u(-n-1) /z u(n)

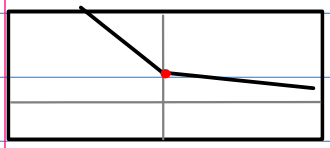
a^n

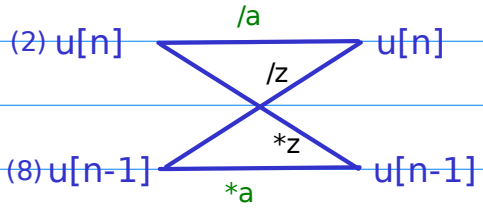


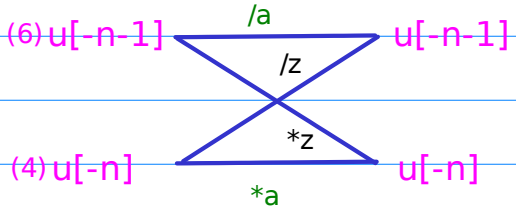




a^{-n}



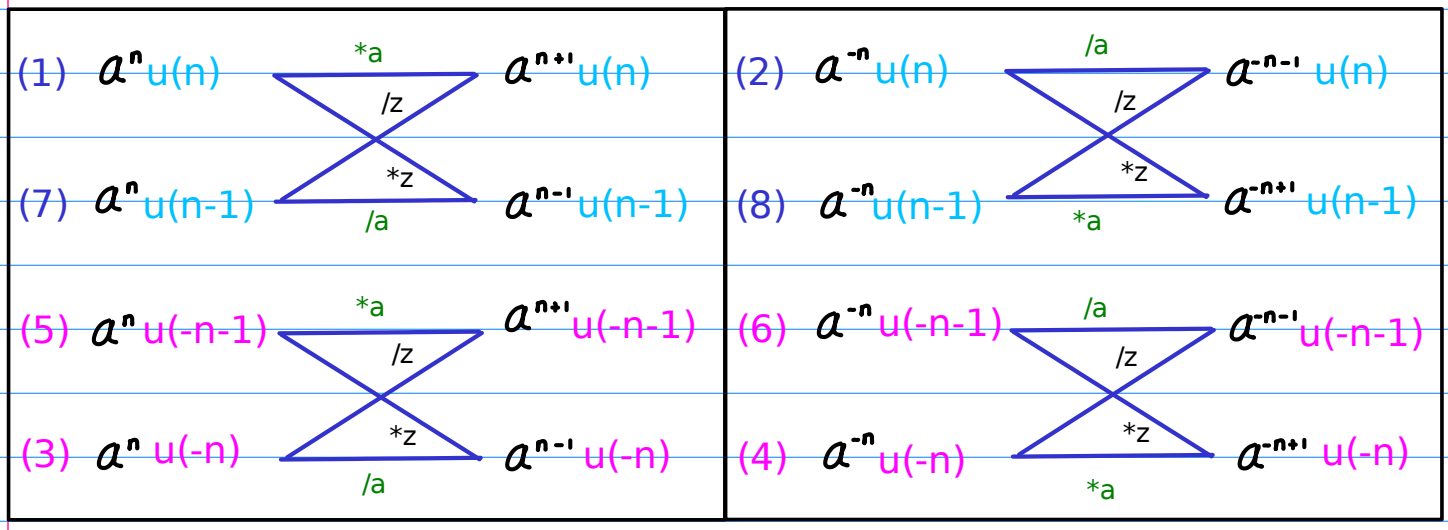




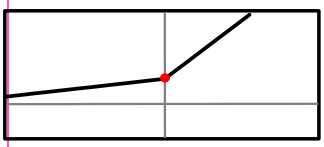
Causal	**u(n)**	(1)	(2)	**butterfly pair ordering**
	u(n-1)	(7)	(8)	
Anti-Causal	**u(-n-1)**	(5)	(6)	
	u(-n)	(3)	(4)	

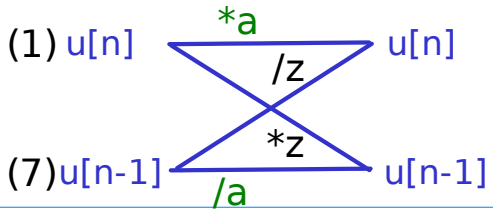
Positive Exponent	**Negative Exponent**
(1)	(2)
(3)	(4)
(5)	(6)
(7)	(8)

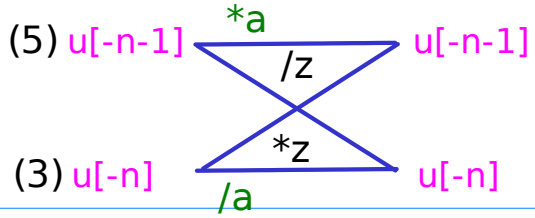
symmetric pair ordering .


(1) $a^n u(n)$
*a
/z
$a^{n+1} u(n)$
(7) $a^n u(n-1)$
*z
/a
$a^{n-1} u(n-1)$
(2) $a^{-n} u(n)$
/a
/z
$a^{-n-1} u(n)$
(8) $a^{-n} u(n-1)$
*z
*a
$a^{-n+1} u(n-1)$
(5) $a^n u(-n-1)$
*a
/z
$a^{n+1} u(-n-1)$
(3) $a^n u(-n)$
*z
/a
$a^{n-1} u(-n)$
(6) $a^{-n} u(-n-1)$
/a
/z
$a^{-n-1} u(-n-1)$
(4) $a^{-n} u(-n)$
*z
*a
$a^{-n+1} u(-n)$

a^n







(1) ***a**

$\frac{1}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{a}{1-az}$ $\lvert z\rvert < a^{-1}$
$\frac{az}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{z}{1-az}$ $\lvert z\rvert < a^{-1}$

(7) **/a**

(1) ***a**

a^n u(n) $(a^0, a^1, a^2, \cdots)$	a^{n+1} u(n) $(a^1, a^2, a^3, \cdots)$
a^n u(n-1) $(a^1, a^2, a^3, \cdots)$	a^{n-1} u(n-1) $(a^0, a^1, a^2, \cdots)$

(7) **/a**

(1) ***z**

$\frac{1}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{z}{1-az}$ $\lvert z\rvert < a^{-1}$
$\frac{az}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{a}{1-az}$ $\lvert z\rvert < a^{-1}$

(7) **/z**

(1) ***z**

a^n u(n) $(a^0, a^1, a^2, \cdots)$	a^{n-1} u(n-1) $(a^0, a^1, a^2, \cdots)$
a^n u(n-1) $(a^1, a^2, a^3, \cdots)$	a^{n+1} u(n) $(a^1, a^2, a^3, \cdots)$

(7) **/z**

(5) ***a**

$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$
$-\frac{1}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$

(3) **/a**

(5) ***a**

$-a^n$ u(-n-1) $-(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$	$-a^{n+1}$ u(-n-1) $-(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$
$-a^n$ u(-n) $-(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$	$-a^{n-1}$ u(-n) $-(\cdots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1})$

(3) **/a**

(5) ***z**

$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$
$-\frac{1}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$

(3) **/z**

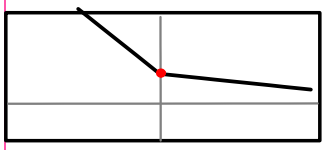
(5) ***z**

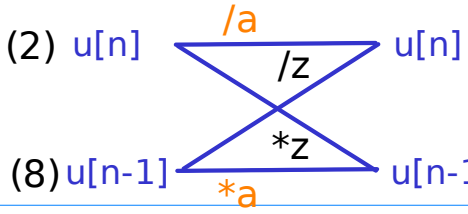
$-a^n$ u(-n-1) $-(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$	$-a^{n-1}$ u(-n) $-(\cdots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1})$
$-a^n$ u(-n) $-(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$	$-a^{n+1}$ u(-n-1) $-(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$

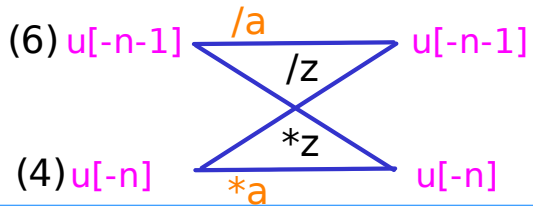
(3) **/z**

butterfly pair ordering

a^{-n}







(2) **/a**

$\frac{1}{1-a^{-1}z}$ $\lvert z\rvert < a$	$\frac{a^{-1}}{1-a^{-1}z}$ $\lvert z\rvert < a$
$\frac{a^{-1}z}{1-a^{-1}z}$ $\lvert z\rvert < a$	$\frac{z}{1-a^{-1}z}$ $\lvert z\rvert < a$

(8) ***a**

(2) **/a**

$\left(\frac{1}{a}\right)^n$ u(n) $\left(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots\right)$	$\left(\frac{1}{a}\right)^{n+1}$ u(n) $\left(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots\right)$
$\left(\frac{1}{a}\right)^n$ u(n-1) $\left(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots\right)$	$\left(\frac{1}{a}\right)^{n-1}$ u(n-1) $\left(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots\right)$

(8) ***a**

(2) ***z**

$\frac{1}{1-a^{-1}z}$ $\lvert z\rvert < a$	$\frac{z}{1-a^{-1}z}$ $\lvert z\rvert < a$
$\frac{a^{-1}z}{1-a^{-1}z}$ $\lvert z\rvert < a$	$\frac{a^{-1}}{1-a^{-1}z}$ $\lvert z\rvert < a$

(8) **/z**

(2) ***z**

$\left(\frac{1}{a}\right)^n$ u(n) $\left(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots\right)$	$\left(\frac{1}{a}\right)^{n-1}$ u(n-1) $\left(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots\right)$
$\left(\frac{1}{a}\right)^n$ u(n-1) $\left(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots\right)$	$\left(\frac{1}{a}\right)^{n+1}$ u(n) $\left(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots\right)$

(8) **/z**

(6) **/a**

$-\frac{az^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$	$-\frac{z^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$
$-\frac{1}{1-az^{-1}}$ $\lvert z\rvert > a$	$-\frac{a}{1-az^{-1}}$ $\lvert z\rvert > a$

(4) ***a**

(6) **/a**

$-\left(\frac{1}{a}\right)^n$ u(-n-1) $-\left(\cdots, a^3, a^2, a^1\right)$	$-\left(\frac{1}{a}\right)^{n+1}$ u(-n-1) $-\left(\cdots, a^2, a^1, a^0\right)$
$-\left(\frac{1}{a}\right)^n$ u(-n) $-\left(\cdots, a^3, a^1, a^0\right)$	$-\left(\frac{1}{a}\right)^{n-1}$ u(-n) $-\left(\cdots, a^3, a^2, a^1\right)$

(4) ***a**

(6) ***z**

$-\frac{az^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$	$-\frac{a}{1-az^{-1}}$ $\lvert z\rvert > a$
$-\frac{1}{1-az^{-1}}$ $\lvert z\rvert > a$	$-\frac{z^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$

(4) **/z**

(6) ***z**

$-\left(\frac{1}{a}\right)^n$ u(-n-1) $-\left(\cdots, a^3, a^2, a^1\right)$	$-\left(\frac{1}{a}\right)^{n-1}$ u(-n) $-\left(\cdots, a^3, a^2, a^1\right)$
$-\left(\frac{1}{a}\right)^n$ u(-n) $-\left(\cdots, a^2, a^1, a^0\right)$	$-\left(\frac{1}{a}\right)^{n+1}$ u(-n-1) $-\left(\cdots, a^2, a^1, a^0\right)$

(4) **/z**

butterfly pair ordering

Scale by **a**
1. Geometric Series

complementary pair ordering

	(1) *a		(2) /a	
	$\frac{1}{1-az}$ $\lvert z\rvert<a^{-1}$	$\frac{a}{1-az}$ $\lvert z\rvert<a^{-1}$	$\frac{1}{1-a^{-1}z}$ $\lvert z\rvert<a$	$\frac{a^{-1}}{1-a^{-1}z}$ $\lvert z\rvert<a$
Comp.ROC	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert>a^{-1}$	$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert>a^{-1}$	$-\frac{az^{-1}}{1-az^{-1}}$ $\lvert z\rvert>a$	$-\frac{z^{-1}}{1-az^{-1}}$ $\lvert z\rvert>a$

	(5) *a		(6) /a	
	$-\frac{1}{1-a^{-1}z^{-1}}$ $\lvert z\rvert>a^{-1}$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert>a^{-1}$	$-\frac{1}{1-az^{-1}}$ $\lvert z\rvert>a$	$-\frac{a}{1-az^{-1}}$ $\lvert z\rvert>a$
Comp.ROC	$\frac{az}{1-az}$ $\lvert z\rvert<a^{-1}$	$\frac{z}{1-az}$ $\lvert z\rvert<a^{-1}$	$\frac{a^{-1}z}{1-a^{-1}z}$ $\lvert z\rvert<a$	$\frac{z}{1-a^{-1}z}$ $\lvert z\rvert<a$

(3) /a (4) *a

(7) /a (8) *a

(1) $a^n u(n)$	*a	$a^{n+1}u(n)$	(2) $a^{-n}u(n)$	/a	$a^{-n-1}u(n)$
(7) $a^n u(n-1)$	/a	$a^{n-1}u(n-1)$	(8) $a^{-n}u(n-1)$	*a	$a^{-n+1}u(n-1)$
(5) $a^n u(-n-1)$	*a	$a^{n+1}u(-n-1)$	(6) $a^{-n}u(-n-1)$	/a	$a^{-n-1}u(-n-1)$
(3) $a^n u(-n)$	/a	$a^{n-1}u(-n)$	(4) $a^{-n}u(-n)$	*a	$a^{-n+1}u(-n)$

*z /z

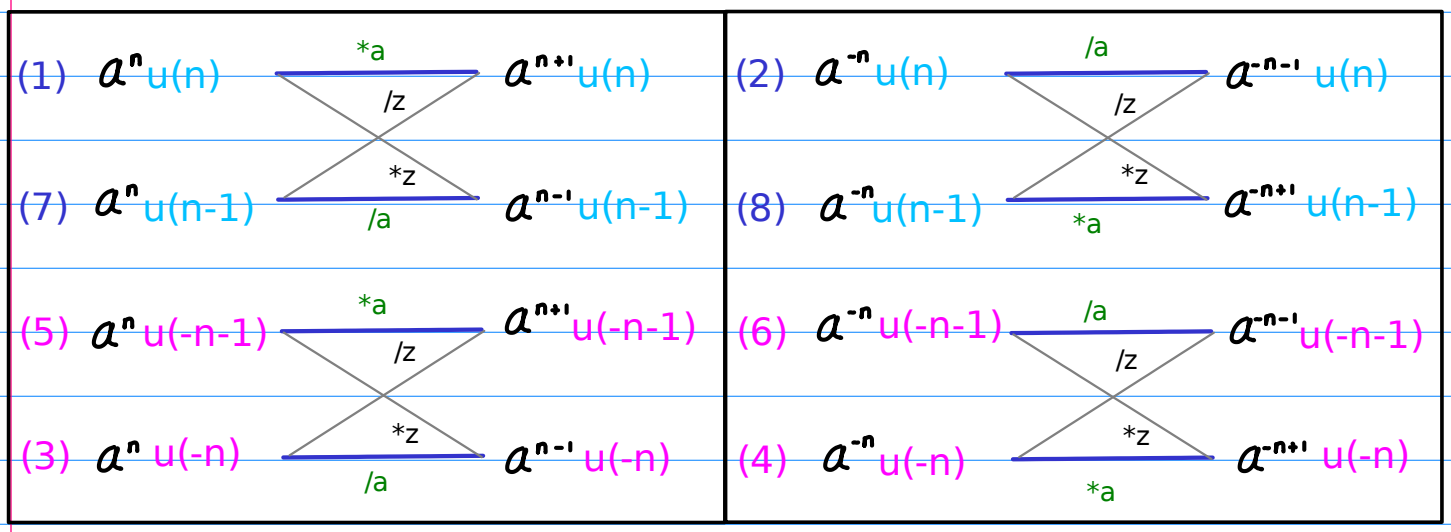
Scale by **a**
2. Sequences

complementary pair ordering

	(1)	***a**	(2)	**/a**
	$a^n\ u(n)$	$a^{n+1}\ u(n)$	$(\frac{1}{a})^n\ u(n)$	$(\frac{1}{a})^{n+1}\ u(n)$
Comp.ROC	$-a^n\ u(-n-1)$	$-a^{n+1}\ u(-n-1)$	$-(\frac{1}{a})^n\ u(-n-1)$	$-(\frac{1}{a})^{n+1}u(-n-1)$

	(5)	***a**	(6)	**/a**
	(3)	**/a**	(4)	***a**
	$-a^n\ u(-n)$	$-a^{n-1}\ u(-n)$	$-(\frac{1}{a})^n\ u(-n)$	$-(\frac{1}{a})^{n-1}\ u(-n)$
Comp.ROC	$a^n\ u(n-1)$	$a^{n-1}\ u(n-1)$	$(\frac{1}{a})^n\ u(n-1)$	$(\frac{1}{a})^{n-1}\ u(n-1)$

	(7)	**/a**	(8)	***a**



Scale by **a**
3. Sequence values

complementary pair ordering

(1)	*a	(2)	/a
$(a^0, a^1, a^2, \cdots)$	$(a^1, a^2, a^3, \cdots)$	$\left(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots\right)$	$\left(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots\right)$
Comp.ROC $-(\cdots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1})$	$-(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$	$-(\cdots, a^3, a^2, a^1)$	$-(\cdots, a^2, a^1, a^0)$

(5)	*a	(6)	/a
(3)	/a	(4)	*a
$-(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$	$-(\cdots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1})$	$-(\cdots, a^2, a^1, a^0)$	$-(\cdots, a^3, a^2, a^1)$
Comp.ROC $(a^1, a^2, a^3, \cdots)$	$(a^0, a^1, a^2, \cdots)$	$\left(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots\right)$	$\left(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots\right)$

(7)	/a	(8)	*a

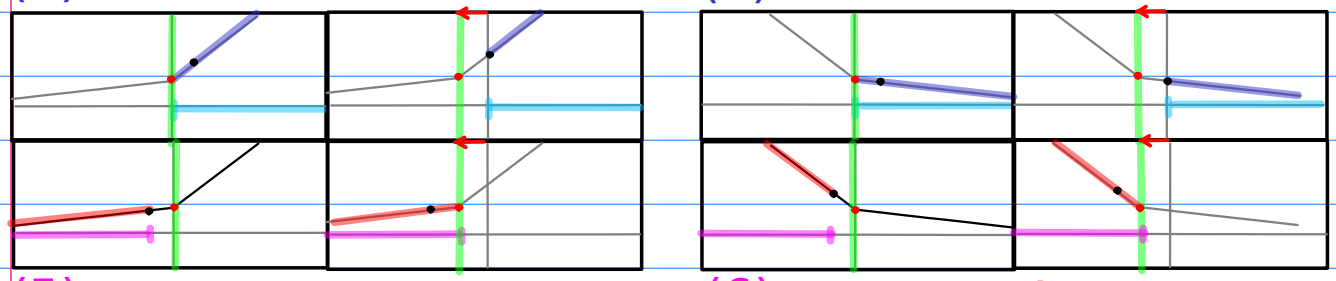
(1) $a^n u(n)$	*a, /z → $a^{n+1} u(n)$	(2) $a^{-n} u(n)$	/a, /z → $a^{-n-1} u(n)$
(7) $a^n u(n-1)$	*z, /a → $a^{n-1} u(n-1)$	(8) $a^{-n} u(n-1)$	*z, *a → $a^{-n+1} u(n-1)$
(5) $a^n u(-n-1)$	*a, /z → $a^{n+1} u(-n-1)$	(6) $a^{-n} u(-n-1)$	/a, /z → $a^{-n-1} u(-n-1)$
(3) $a^n u(-n)$	*z, /a → $a^{n-1} u(-n)$	(4) $a^{-n} u(-n)$	*z, *a → $a^{-n+1} u(-n)$

Scale by **a**
4. Graphs

complementary pair ordering

(1) ***a** (2) **/a**

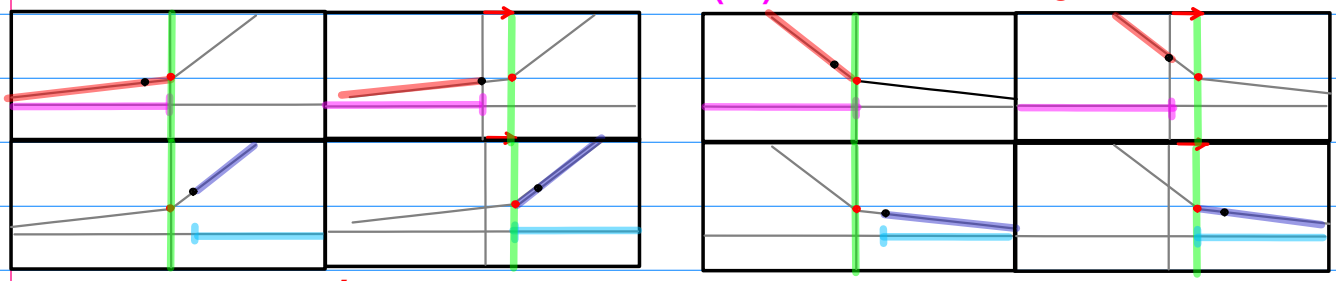
Comp.ROC



(5) ***a** (6) **/a**

(3) **/a** (4) ***a**

Comp.ROC

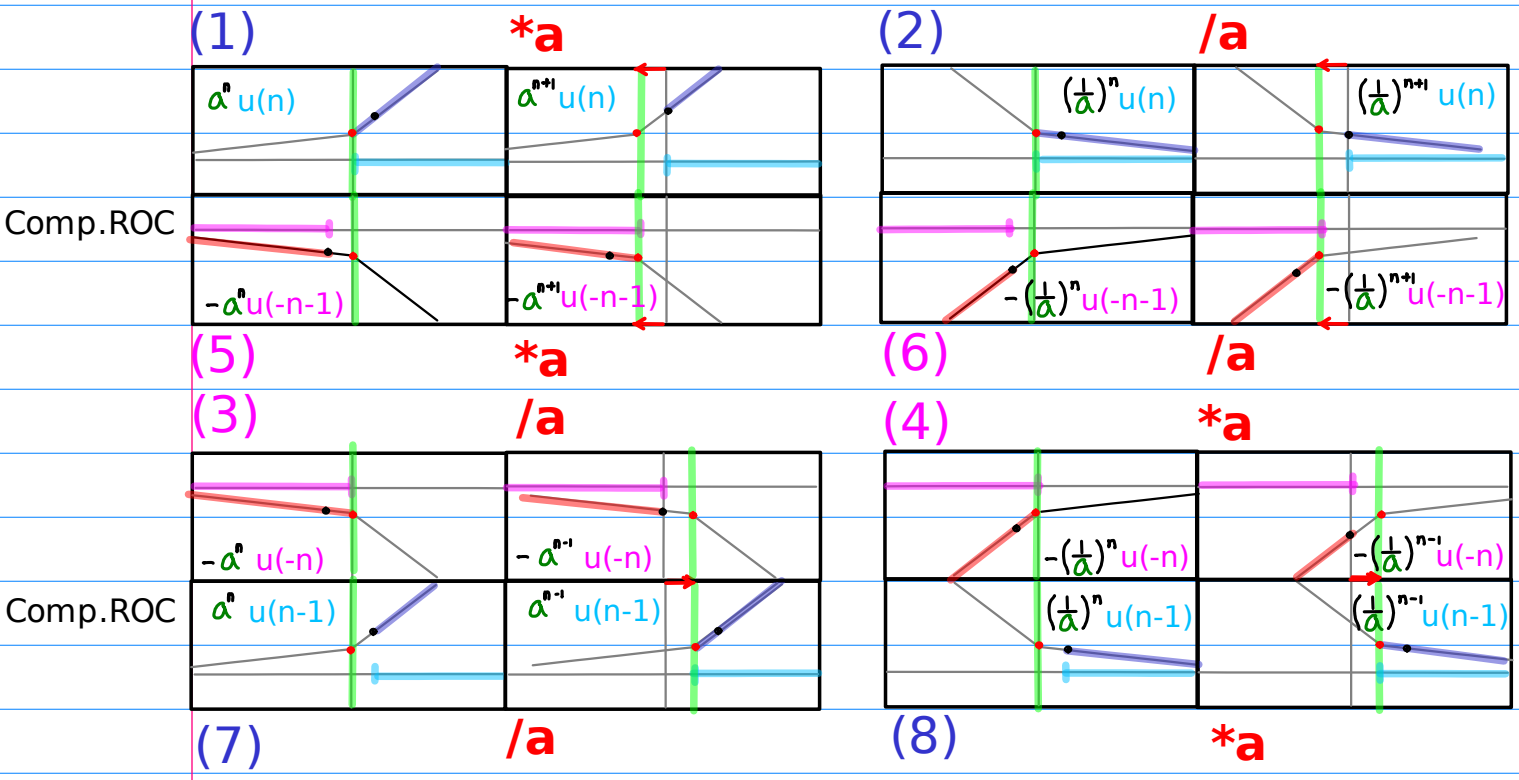


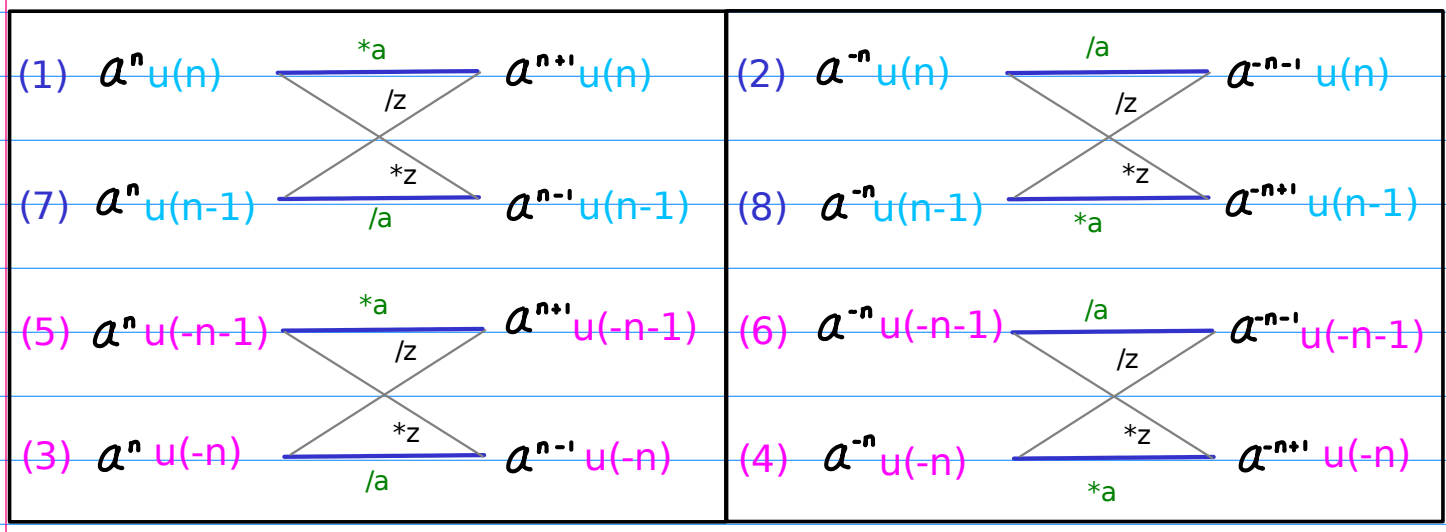
(7) **/a** (8) ***a**

(1) $a^n u(n)$	*a	$a^{n+1} u(n)$	(2) $a^{-n} u(n)$	/a	$a^{-n-1} u(n)$
(7) $a^n u(n-1)$	/a	$a^{n-1} u(n-1)$	(8) $a^{-n} u(n-1)$	*a	$a^{-n+1} u(n-1)$
(5) $a^n u(-n-1)$	*a	$a^{n+1} u(-n-1)$	(6) $a^{-n} u(-n-1)$	/a	$a^{-n-1} u(-n-1)$
(3) $a^n u(-n)$	/a	$a^{n-1} u(-n)$	(4) $a^{-n} u(-n)$	*a	$a^{-n+1} u(-n)$

Scale by **a**
5. Graphs - signs

complementary pair ordering

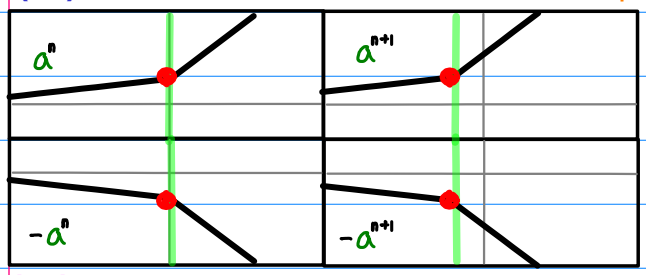




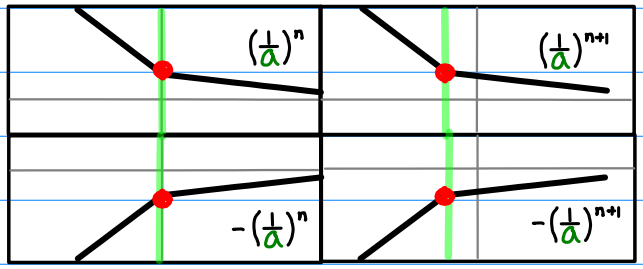
complementary pair ordering

Scale by **a**
6. Graphs - Exponents

(1) ***a** ← SHL.Exp



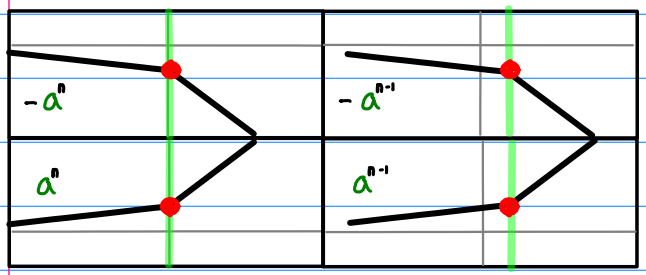
(2) **/a** ← SHL.Exp



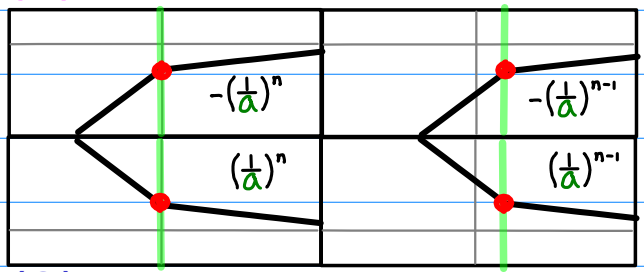
(5) ***a** ← SHL.Exp

(6) **/a** ← SHL.Exp

(3) **/a** → SHR.Exp

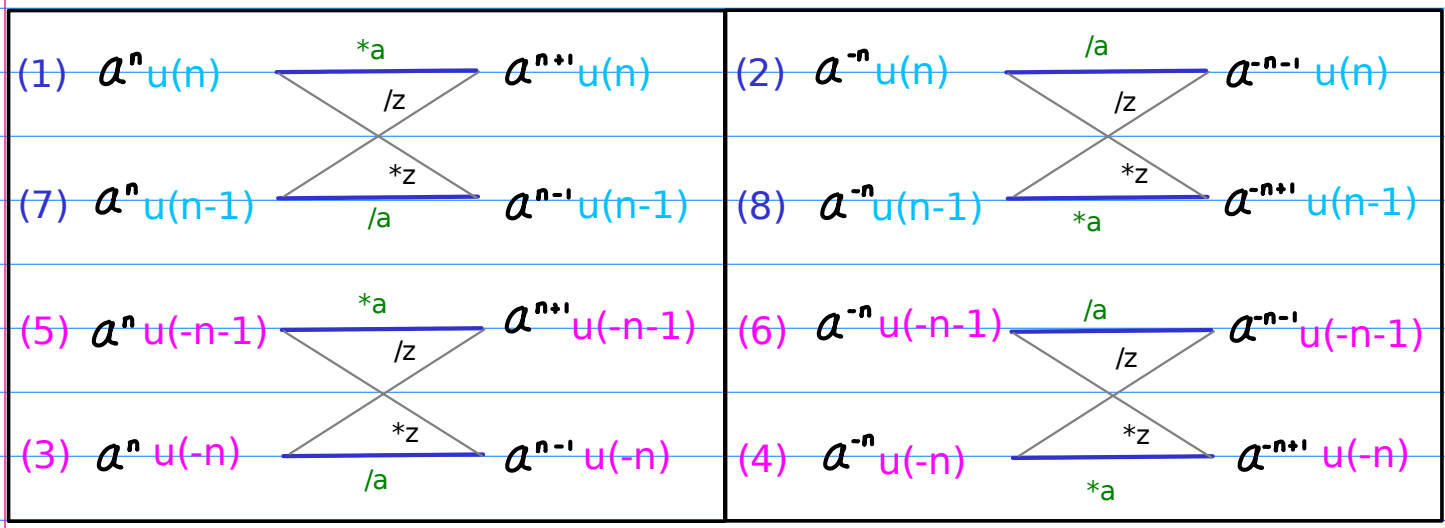


(4) ***a** → SHR.Exp



(7) **/a** → SHR.Exp

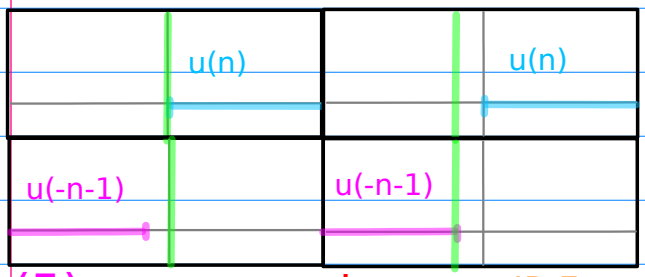
(8) ***a** → SHR.Exp



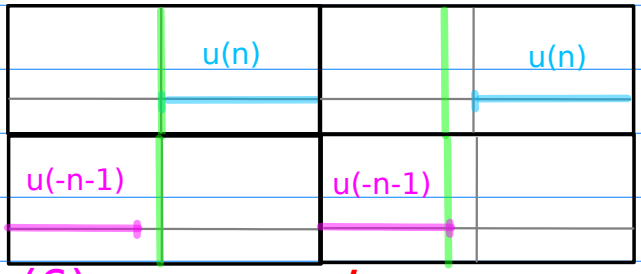
Scale by **a**
7. Graphs - Ranges

complementary pair ordering

(1) ***a** ID.Exp



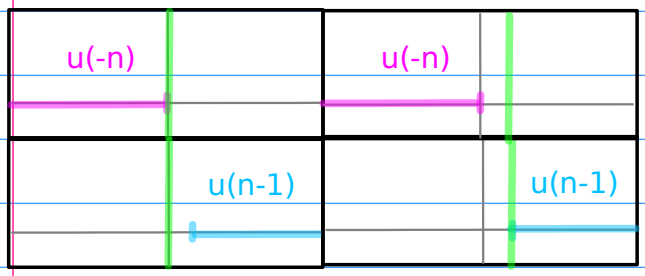
(2) **/a** ID.Exp



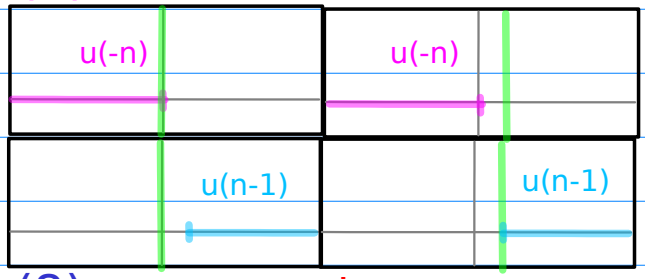
(5) ***a** ID.Exp

(6) **/a** ID.Exp

(3) **/a** ID.Exp

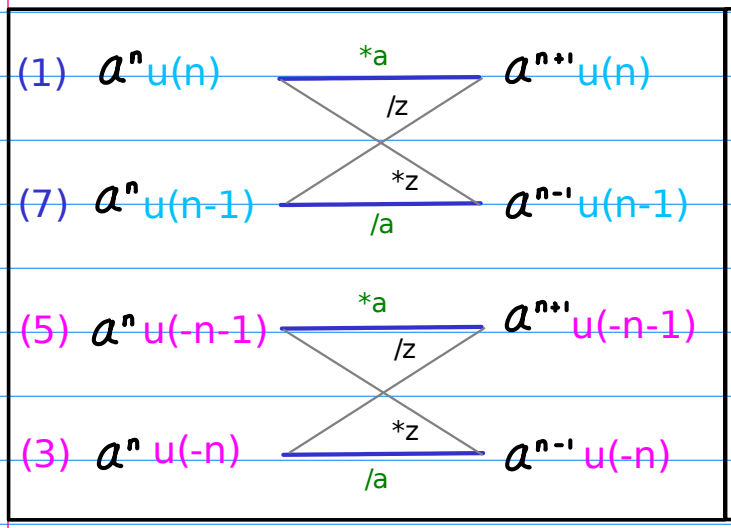


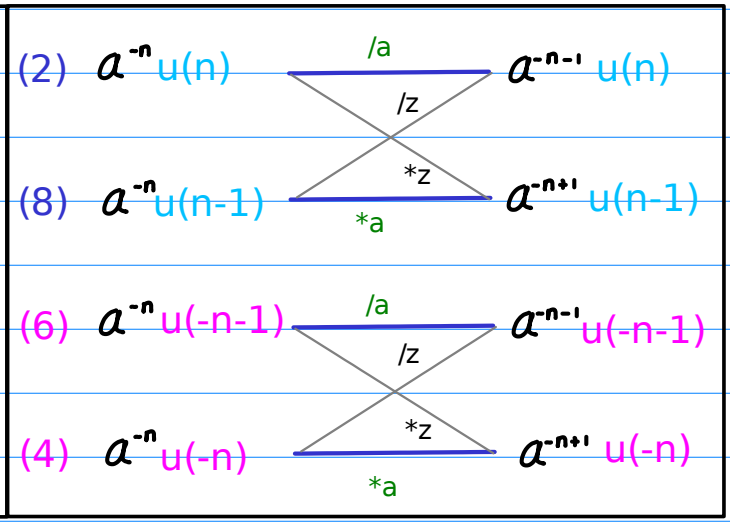
(4) ***a** ID.Exp



(7) **/a** ID.Exp

(8) ***a** ID.Exp





Scale by z
1. Geometric Series

complementary pair ordering

(1)	*z	(2)	*z
$\frac{1}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{z}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{1}{1-a^{-1}z}$ $\lvert z\rvert < a$	$\frac{z}{1-a^{-1}z}$ $\lvert z\rvert < a$
Comp.ROC $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{az^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$	$-\frac{a}{1-az^{-1}}$ $\lvert z\rvert > a$

(5)	*z	(6)	*z
(3)	/z	(4)	/z
$-\frac{1}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{1}{1-az^{-1}}$ $\lvert z\rvert > a$	$-\frac{z^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$
Comp.ROC $\frac{az}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{a}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{a^{-1}z}{1-a^{-1}z}$ $\lvert z\rvert < a$	$\frac{a^{-1}}{1-a^{-1}z}$ $\lvert z\rvert < a$
(7)	/z	(8)	/z

(1) $a^n u(n)$	*a, /z	$a^{n+1} u(n)$	(2) $a^{-n} u(n)$	/a, /z	$a^{-n-1} u(n)$
(7) $a^n u(n-1)$	*z, /a	$a^{n-1} u(n-1)$	(8) $a^{-n} u(n-1)$	/z, *a	$a^{-n+1} u(n-1)$
(5) $a^n u(-n-1)$	*a, /z	$a^{n+1} u(-n-1)$	(6) $a^{-n} u(-n-1)$	/a, /z	$a^{-n-1} u(-n-1)$
(3) $a^n u(-n)$	*z, /a	$a^{n-1} u(-n)$	(4) $a^{-n} u(-n)$	/z, *a	$a^{-n+1} u(-n)$

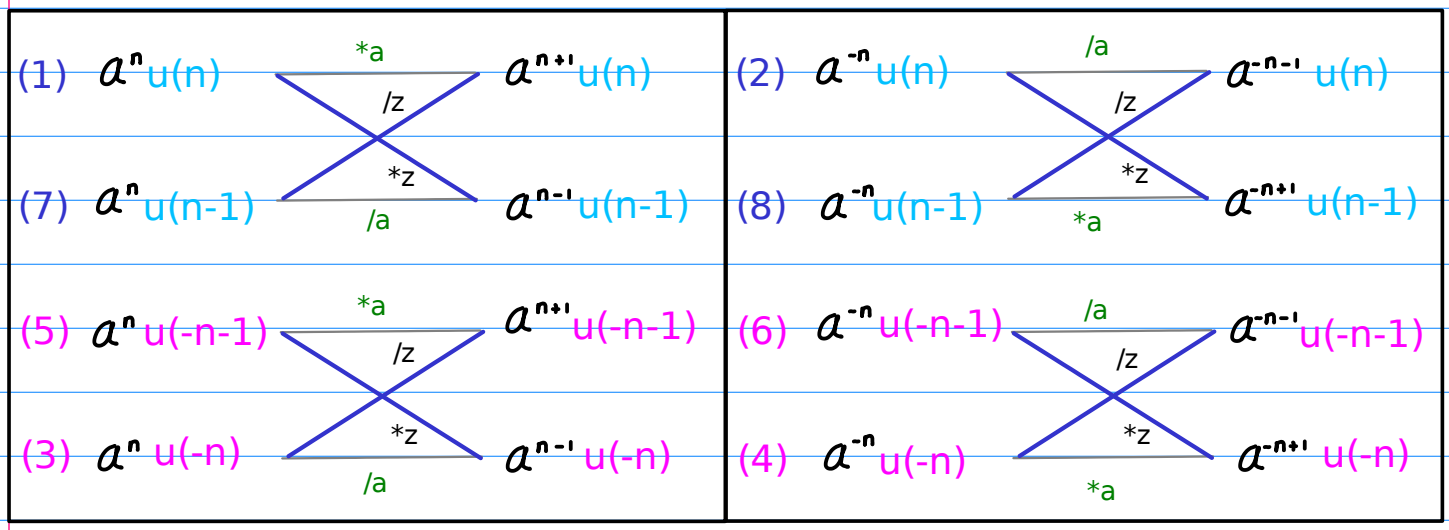
Scale by z
2. Sequences

complementary pair ordering

	(1)	*z	(2)	*z
	$a^n\ u(n)$	$a^{n-1}\ u(n-1)$	$(\frac{1}{a})^n\ u(n)$	$(\frac{1}{a})^{n-1}\ u(n-1)$
Comp.ROC	$-a^n\ u(-n-1)$	$-a^{n-1}\ u(-n)$	$-(\frac{1}{a})^n\ u(-n-1)$	$-(\frac{1}{a})^{n-1}\ u(-n)$

	(5)	*z	(6)	*z
	(3)	/z	(4)	/z
	$-a^n\ u(-n)$	$-a^{n+1}\ u(-n-1)$	$-(\frac{1}{a})^n\ u(-n)$	$-(\frac{1}{a})^{n+1}u(-n-1)$
Comp.ROC	$a^n\ u(n-1)$	$a^{n+1}\ u(n)$	$(\frac{1}{a})^n\ u(n-1)$	$(\frac{1}{a})^{n+1}\ u(n)$

	(7)	/z	(8)	/z



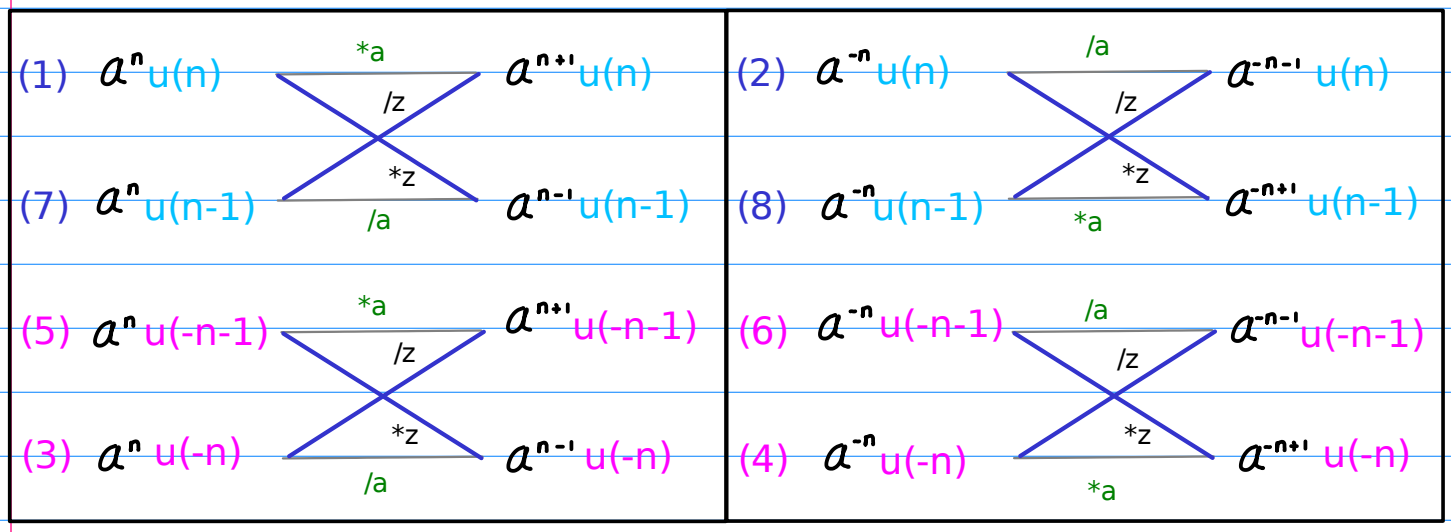
Scale by z
3. Sequence values

complementary pair ordering

	(1)	*z	(2)	*z
	$(a^0, a^1, a^2, \cdots)$	$(a^0, a^1, a^2, \cdots)$	$(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots)$	$(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots)$
Comp.ROC	$-(\cdots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1})$	$-(\cdots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1})$	$-(\cdots, a^3, a^2, a^1)$	$-(\cdots, a^3, a^2, a^1)$

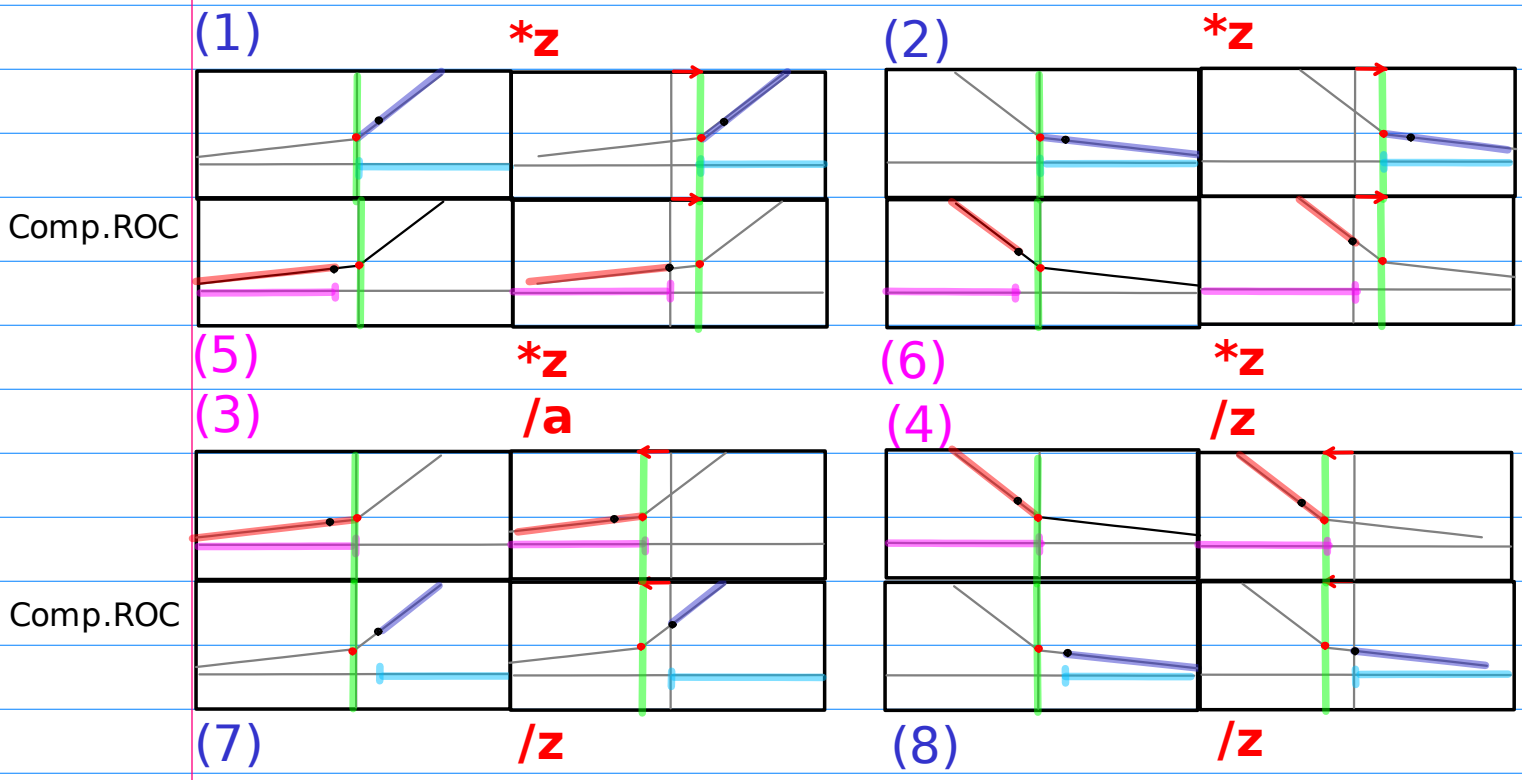
	(5)	*z	(6)	*z
	(3)	/z	(4)	/z
	$-(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$	$-(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$	$-(\cdots, a^2, a^1, a^0)$	$-(\cdots, a^2, a^1, a^0)$
Comp.ROC	$(a^1, a^2, a^3, \cdots)$	$(a^1, a^2, a^3, \cdots)$	$(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3} \cdots)$	$(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3} \cdots)$

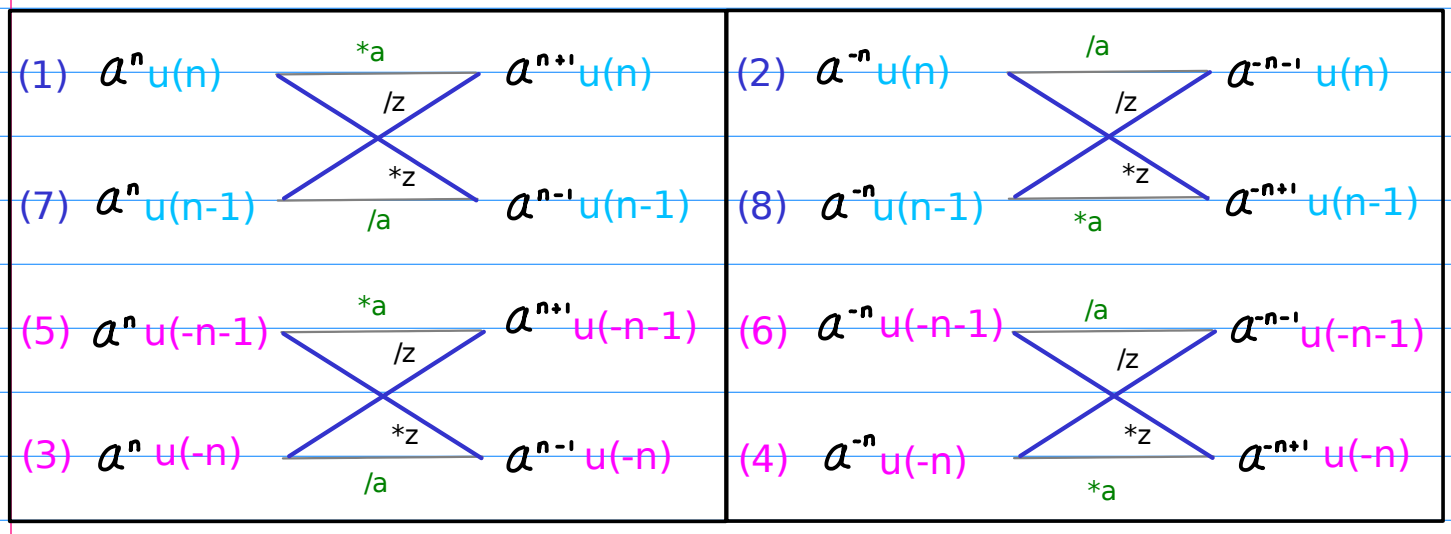
	(7)	/z	(8)	/z



Scale by z
4. Graphs

complementary pair ordering

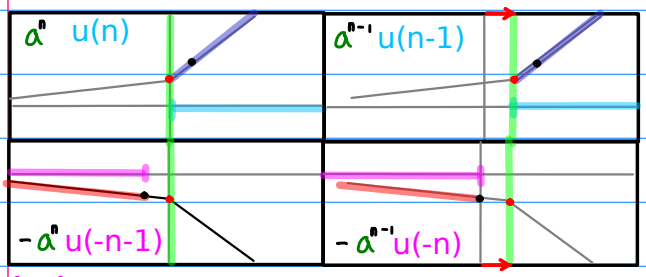




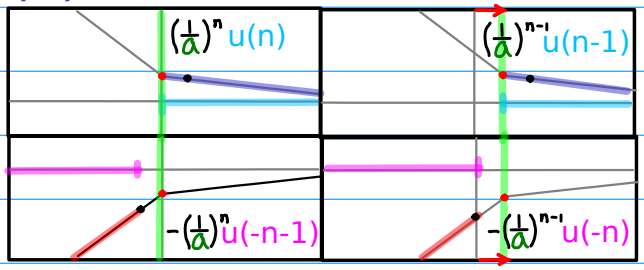
Scale by z
5. Graphs - signs

complementary pair ordering

(1) *z

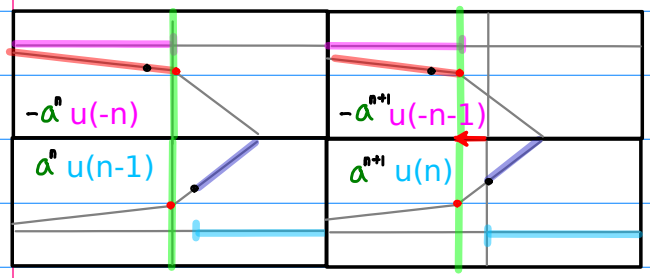


(2) *z



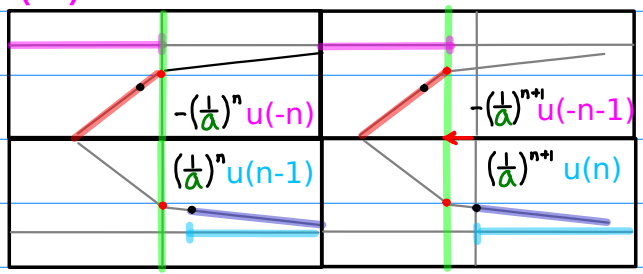
(5) *z

(3) /z



(6) *z

(4) /z



(7) /z

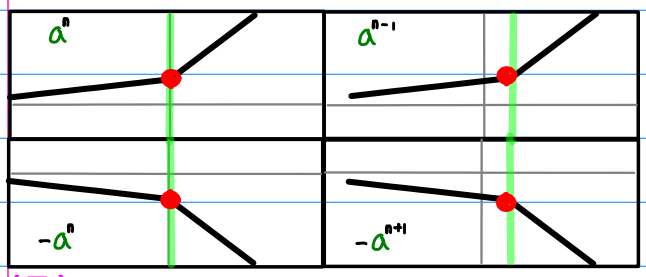
(8) /z

(1) $a^n u(n)$ —*a→ $a^{n+1}u(n)$	(2) $a^{-n}u(n)$ —/a→ $a^{-n-1}u(n)$
(7) $a^n u(n-1)$ —/a→ $a^{n-1}u(n-1)$	(8) $a^{-n}u(n-1)$ —*a→ $a^{-n+1}u(n-1)$
(5) $a^n u(-n-1)$ —*a→ $a^{n+1}u(-n-1)$	(6) $a^{-n}u(-n-1)$ —/a→ $a^{-n-1}u(-n-1)$
(3) $a^n u(-n)$ —/a→ $a^{n-1}u(-n)$	(4) $a^{-n}u(-n)$ —*a→ $a^{-n+1}u(-n)$

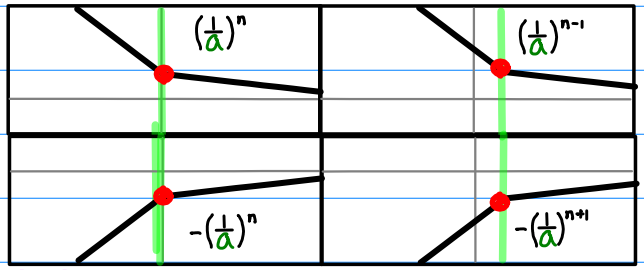
Scale by z
6. Graphs - Exponents

complementary pair ordering

(1) *z → SHR.Exp



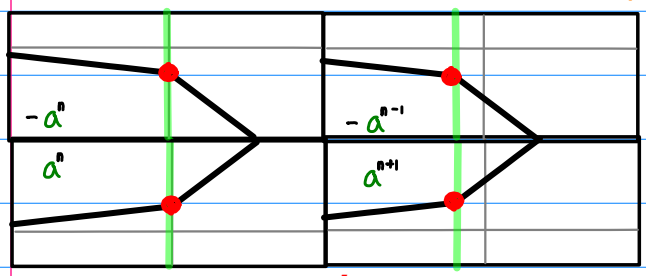
(2) *z → SHR.Exp



Comp.ROC

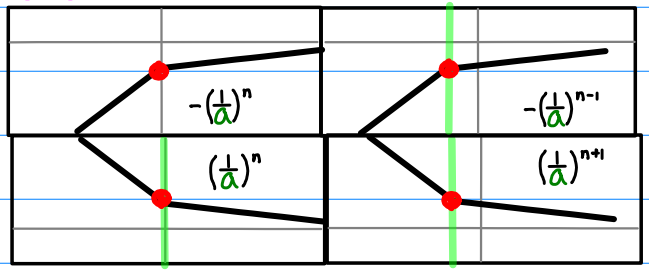
(5) *z → SHR.Exp

(3) /z ← SHL.Exp



(6) *z → SHR.Exp

(4) /z ← SHL.Exp



Comp.ROC

(7) /z ← SHL.Exp

(8) /z ← SHL.Exp

(1) $a^n u(n)$	*a	/z	$a^{n+1} u(n)$	(2) $a^{-n} u(n)$	/a	/z	$a^{-n-1} u(n)$
(7) $a^n u(n-1)$	/a	*z	$a^{n-1} u(n-1)$	(8) $a^{-n} u(n-1)$	*a	*z	$a^{-n+1} u(n-1)$
(5) $a^n u(-n-1)$	*a	/z	$a^{n+1} u(-n-1)$	(6) $a^{-n} u(-n-1)$	/a	/z	$a^{-n-1} u(-n-1)$
(3) $a^n u(-n)$	/a	*z	$a^{n-1} u(-n)$	(4) $a^{-n} u(-n)$	*a	*z	$a^{-n+1} u(-n)$

Scale by z
7. Graphs - Ranges

complementary pair ordering