Power Density Spectrum - Discrete Time

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

Definition

the bilateral z-transform of $R_{XX}[n]$

$$S_{XX}(z) = \sum_{n=-\infty}^{\infty} R_{XX}[n] z^{-n}$$

$$R_{XX}[n] = \frac{1}{2\pi j} \oint_C S_{XX}(z) z^{n-1} dz$$

Definition

the bilateral z-transform of $R_{XX}[n]$

$$S_{XX}(z) = \sum_{n=-\infty}^{\infty} R_{XX}[n] z^{-n}$$

substitute $z = e^{j\Omega}$

$$S_{XX}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} R_{XX}[n] e^{-jn\Omega}$$

- the magnitude $|z| = \left|e^{j\Omega}\right| = 1$
- the angle $Arg(z) = Arg(e^{j\Omega}) = \Omega$
- the discrete frequency Ω has units of radians per sample

Definition

The power density spectrum $S_{XX}(e^{j\Omega})$

$$S_{XX}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} R_{XX}[n] e^{-jn\Omega}$$

- $S_{XX}(e^{j\Omega})$ has a period 2π
- usually $-\pi \leq \Omega < \pi$

Definition

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} \boxed{S_{XX}(e^{j\Omega})} e^{jn\Omega} d\Omega = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \boxed{\sum_{m=-\infty}^{\infty} R_{XX}[m] e^{-jm\Omega}} e^{jn\Omega} d\Omega$$

$$= \sum_{n=-\infty}^{\infty} R_{XX}[n] \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{j(n-m)\Omega} d\Omega$$

$$= \sum_{n=-\infty}^{\infty} R_{XX}[n] \frac{\sin[(n-m)\pi]}{[(n-m)\pi]}$$

$$\frac{\sin[(n-m)\pi]}{[(n-m)\pi]} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

Definition

the discrete time Fourier transform (DTFT) of $R_{XX}[n]$

$$S_{XX}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} R_{XX}[n] e^{-jn\Omega}$$

the inverse DTFT of $S_{XX}(e^{j\Omega})$

$$R_{XX}[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} S_{XX}(e^{j\Omega}) e^{jn\Omega} d\Omega$$

1. $S_{XX}(e^{j\Omega}) \geq 0$
2. $S_{XX}(e^{-j\Omega}) = S_{XX}(e^{+j\Omega})$ for real $X[n]$
3. $S_{XX}(e^{+j\Omega})$ is real
4. $\frac{1}{2\pi} \int_{-\pi}^{+\pi} S_{XX}(e^{j\Omega}) d\Omega = E[X^2[n]]$

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} S_{XX}(e^{j\Omega})d\Omega = A\left[E\left[X^2[n]\right]\right]$$

$$\lim_{N\to\infty}\left\{\frac{1}{2N+1}\sum_{n=-N}^{+N} E\left[X^2[n]\right]\right\}$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega = A\left[E\left[X^2(t)\right]\right]$$

Definition

$X[n]$ is a **mean ergodic** if the **time average** of samples converges to the statistical average in the mean square sense

$$\lim_{N\to\infty}\left\{\frac{1}{2N+1}\sum_{n=-N}^{+N} X[n]\right\} = \overline{X}$$

the necessary and sufficient condition is

$$\lim_{T\to\infty}\left\{\frac{1}{2N+1}\sum_{n=-2N}^{+2N}\left(1-\frac{|n|}{2N+1}\right)C_{XX}[n]\right\} = 0$$

Definition

A stationary sequence $X[n]$
with autocorrelation function $R_N[k]$
is called **autocorrelation ergodic**
if for all k, $R_N[k]$ converges to $R_{XX}[k]$ as $N \to \infty$

$$R_N[k] = \frac{1}{2N+1} \sum_{n=-N}^{+N} x[n]x[n+k]$$

$$R_{XX}[k] = \lim_{N\to\infty} R_N[k]$$
$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} x[n]x[n+k]$$

Definition

$$\hat{R}_N[k] = \frac{1}{N} \sum_{n=0}^{N-1-|k|} X[n]X[n+|k|] \qquad |k| < N$$

$$\hat{R}_N[k] = \frac{1}{N} \sum_{n=0}^{N-1-k} X[n]X[n+k] \qquad 0 < k < N$$

$$= \frac{1}{N} \sum_{n=0}^{N-1+k} X[n]X[n-k] \qquad -N < k < 0$$

Definition

$$X_N(\Omega_k) = \sum_{n=0}^{N-1} X[n] e^{-j\Omega_k n} \qquad k = 0, 1, ..., N-1$$

$$\Omega_k = \frac{2\pi k}{N} \qquad k = 0, 1, ..., N-1$$

Definition

the average power P_{XX} for the **random process** $X_T(\omega)$

$$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \boxed{\lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}} d\omega$$

the power density spectrum $S_{XX}(\omega)$

$$\boxed{S_{XX}(\omega)} = \boxed{\lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}}$$

$$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \boxed{S_{XX}(\omega)} d\omega$$

Definition

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the power density spectrum $S_{XX}(\omega)$

$$\boxed{S_{XX}(\Omega)} = \boxed{\lim_{N\to\infty} \frac{E\left[|X_T[n]|^2\right]}{N}}$$

$$P_{XX} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \boxed{S_{XX}(\Omega)} d\Omega$$

Definition

Periodogram : the estimate of the power density spectrum

$$\widehat{S}_N(\Omega_k) = \frac{1}{N}|X_N(\Omega_k)|^2 \qquad k = 0, 1, ..., N-1$$

$$\lim_{N\to\infty} E\left[\widehat{S}_N(\Omega_k)\right] = S_{XX}(\Omega_k) \qquad k = 0, 1, ..., N-1$$

continuous time

$$\boxed{S_{XX}(\omega)} = \boxed{\lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}}$$

Periodogram Proof (1)

N Gaussian random variables

Definition

$$E\left[\widehat{S}_N(\Omega_k)\right] = \frac{1}{N} E\left[\left\{\sum_{n=0}^{N-1} X[n]e^{-j\Omega_k n}\right\}\left\{\sum_{m=0}^{N-1} X[m]e^{-j\Omega_k m}\right\}\right]$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} E\left[X[n]X[m]\right] e^{-j\Omega_k(n-m)}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} R_{XX}\left[n-m\right] e^{-j\Omega_k(n-m)}$$

Periodogram Proof (2)

N Gaussian random variables

Definition

$$E\left[\widehat{S}_N(\Omega_k)\right] = \frac{1}{N} \sum_{n=-(N-1)}^{+(N-1)} (N - |k|) R_{XX}[k] e^{-j\Omega_k k}$$

$$= \frac{N}{N} \sum_{n=-(N-1)}^{+(N-1)} R_{XX}[k] e^{-j\Omega_k k}$$

$$- \frac{1}{N} \sum_{n=-(N-1)}^{+(N-1)} |k| R_{XX}[k] e^{-j\Omega_k k}$$

The Variance of the Periodogram (1)

N Gaussian random variables

Definition

$$\sigma_S^2 = E\left[\widehat{S}_N(\Omega_k)^2\right] - E\left[\widehat{S}_N(\Omega_k)\right]^2$$

$$E\left[\widehat{S}_N(\Omega_k)^2\right] = E\left[\left(\frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} E\left[X[n]X[m]\right] e^{-j\Omega_k(n-m)}\right)^2\right]$$

$$= \frac{1}{N^2}\sum_{n,m,p,q} E\left[X[n]X[m]X[p]X[q]\right] e^{-j\Omega_k(n-m)} e^{-j\Omega_k(q-p)}$$

Definition

$$
\begin{aligned}
E\left[X[n]X[m]X[p]X[q]\right] = {} & R_{XX}[n-m]\cdot R_{XX}[p-q] \\
& + R_{XX}[n-p]\cdot R_{XX}[m-q] \\
& + R_{XX}[n-q]\cdot R_{XX}[m-p]
\end{aligned}
$$

$$
\begin{aligned}
E\left[\widehat{S}_N(\Omega_k)^2\right] = {} & 2E\left[\widehat{S}_N(\Omega_k)\right]^2 \\
& + \frac{1}{N^2}\left|\sum_{n,p} R_{XX}[n-p]e^{-j\Omega_k(n-p)}\right|^2
\end{aligned}
$$

Definition

$$\sigma_{\widehat{S}}^2 \geq E\left[\widehat{S}_N(\Omega_k)\right]^2$$

$$\widehat{S}_{M,P}(\Omega_k) = \frac{1}{P}\sum_{p=1}^{P}\widehat{S}_{M,p}(\Omega_k) \qquad k = 0, 1, \ldots, M-1$$