

# Power Density Spectrum - Discrete Time

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles,Jr. and B. Shi



# Bilateral z-Transform of $R_{XX}[n]$

$N$  Gaussian random variables

## Definition

the bilateral z-transform of  $R_{XX}[n]$

$$S_{XX}(z) = \sum_{n=-\infty}^{\infty} R_{XX}[n]z^{-n}$$

$$R_{XX}[n] = \frac{1}{2\pi j} \oint_C S_{XX}(z)z^{n-1} dz$$

# Evaluation of $R_{XX}[n]$ at $e^{j\Omega}$

$N$  Gaussian random variables

## Definition

the bilateral z-transform of  $R_{XX}[n]$

$$S_{XX}(z) = \sum_{n=-\infty}^{\infty} R_{XX}[n]z^{-n}$$

substitute  $z = e^{j\Omega}$

$$S_{XX}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} R_{XX}[n]e^{-jn\Omega}$$

- the magnitude  $|z| = |e^{j\Omega}| = 1$
- the angle  $Arg(z) = Arg(e^{j\Omega}) = \Omega$
- the discrete frequency  $\Omega$  has units of radians per sample

# Power density spectrum of a DT sequence

$N$  Gaussian random variables

## Definition

The power density spectrum  $S_{XX}(e^{j\Omega})$

$$S_{XX}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} R_{XX}[n]e^{-jn\Omega}$$

- $S_{XX}(e^{j\Omega})$  has a period  $2\pi$
- usually  $-\pi \leq \Omega < \pi$

# Discrete Time Fourier Transform of $R_{XX}[n]$

$N$  Gaussian random variables

## Definition

$$\begin{aligned}\frac{1}{2\pi} \int_{-\pi}^{+\pi} S_{XX}(e^{j\Omega}) e^{jn\Omega} d\Omega &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} \left[ \sum_{m=-\infty}^{\infty} R_{XX}[m] e^{-jm\Omega} \right] e^{jn\Omega} d\Omega \\ &= \sum_{n=-\infty}^{\infty} R_{XX}[n] \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{j(n-m)\Omega} d\Omega \\ &= \sum_{n=-\infty}^{\infty} R_{XX}[n] \frac{\sin[(n-m)\pi]}{[(n-m)\pi]}\end{aligned}$$

$$\frac{\sin[(n-m)\pi]}{[(n-m)\pi]} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

# Discrete Time Fourier Transform of $R_{XX}[n]$

$N$  Gaussian random variables

## Definition

the discrete time Fourier transform (DTFT) of  $R_{XX}[n]$

$$S_{XX}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} R_{XX}[n]e^{-jn\Omega}$$

the inverse DTFT of  $S_{XX}(e^{j\Omega})$

$$R_{XX}[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} S_{XX}(e^{j\Omega})e^{jn\Omega} d\Omega$$



# Properties of Power Density Spectrum - DT

$N$  Gaussian random variables

- 1  $S_{XX}(e^{j\Omega}) \geq 0$
- 2  $S_{XX}(e^{-j\Omega}) = S_{XX}(e^{+j\Omega})$  for real  $X[n]$
- 3  $S_{XX}(e^{+j\Omega})$  is real
- 4  $\frac{1}{2\pi} \int_{-\pi}^{+\pi} S_{XX}(e^{j\Omega}) d\Omega = E[X^2[n]]$

# Properties of Power Density Spectrum - DT

$N$  Gaussian random variables

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} S_{XX}(e^{j\Omega}) d\Omega = A [E [X^2[n]]]$$

$$\lim_{N \rightarrow \infty} \left\{ \frac{1}{2N+1} \sum_{n=-N}^{+N} E [X^2[n]] \right\}$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega = A [E [X^2(t)]]$$

# Mean Ergodic Process - discrete time

$N$  Gaussian random variables

## Definition

$X[n]$  is a **mean ergodic** if the **time average** of samples converges to the statistical average in the mean square sense

$$\lim_{N \rightarrow \infty} \left\{ \frac{1}{2N+1} \sum_{n=-N}^{+N} X[n] \right\} = \bar{X}$$

the necessary and sufficient condition is

$$\lim_{T \rightarrow \infty} \left\{ \frac{1}{2N+1} \sum_{n=-2N}^{+2N} \left( 1 - \frac{|n|}{2N+1} \right) C_{XX}[n] \right\} = 0$$

# Auto-correlation Ergodic Process - Discrete Time

$N$  Gaussian random variables

## Definition

A stationary sequence  $X[n]$   
with autocorrelation function  $R_N[k]$   
is called **autocorrelation ergodic**  
if for all  $k$ ,  $R_N[k]$  converges to  $R_{XX}[k]$  as  $N \rightarrow \infty$

$$R_N[k] = \frac{1}{2N+1} \sum_{n=-N}^{+N} x[n]x[n+k]$$

$$\begin{aligned} R_{XX}[k] &= \lim_{N \rightarrow \infty} R_N[k] \\ &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} x[n]x[n+k] \end{aligned}$$

# Estimating the Power Density Spectrum

$N$  Gaussian random variables

## Definition

$$\hat{R}_N[k] = \frac{1}{N} \sum_{n=0}^{N-1-|k|} X[n]X[n+|k|] \quad |k| < N$$

$$\begin{aligned} \hat{R}_N[k] &= \frac{1}{N} \sum_{n=0}^{N-1-k} X[n]X[n+k] & 0 < k < N \\ &= \frac{1}{N} \sum_{n=0}^{N-1+k} X[n]X[n-k] & -N < k < 0 \end{aligned}$$

# DTFT, FFT

$N$  Gaussian random variables

## Definition

$$X_N(\Omega_k) = \sum_{n=0}^{N-1} X[n]e^{-j\Omega_k n} \quad k = 0, 1, \dots, N-1$$

$$\Omega_k = \frac{2\pi k}{N} \quad k = 0, 1, \dots, N-1$$

# Average Power $P_{XX}$ - Continuous Time

$N$  Gaussian random variables

## Definition

the average power  $P_{XX}$  for the random process  $X_T(\omega)$

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{\lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}} d\omega$$

the power density spectrum  $S_{XX}(\omega)$

$$\boxed{S_{XX}(\omega)} = \boxed{\lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}}$$

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{S_{XX}(\omega)} d\omega$$

# Average Power $P_{XX}$ - Continuous Time

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the power density spectrum  $S_{XX}(\omega)$

$$\boxed{S_{XX}(\Omega)} = \boxed{\lim_{N \rightarrow \infty} \frac{E[|X_T[n]|^2]}{N}}$$

$$P_{XX} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \boxed{S_{XX}(\Omega)} d\Omega$$



# Periodogram

$N$  Gaussian random variables

## Definition

Periodogram : the estimate of the power density spectrum

$$\hat{S}_N(\Omega_k) = \frac{1}{N} |X_N(\Omega_k)|^2 \quad k = 0, 1, \dots, N-1$$

$$\lim_{N \rightarrow \infty} E \left[ \hat{S}_N(\Omega_k) \right] = S_{XX}(\Omega_k) \quad k = 0, 1, \dots, N-1$$

continuous time

$$S_{XX}(\omega) = \lim_{T \rightarrow \infty} \frac{E \left[ |X_T(\omega)|^2 \right]}{2T}$$

# Periodogram Proof (1)

$N$  Gaussian random variables

## Definition

$$\begin{aligned} E \left[ \widehat{S}_N(\Omega_k) \right] &= \frac{1}{N} E \left[ \left\{ \sum_{n=0}^{N-1} X[n] e^{-j\Omega_k n} \right\} \left\{ \sum_{m=0}^{N-1} X[m] e^{-j\Omega_k m} \right\} \right] \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} E [X[n] X[m]] e^{-j\Omega_k(n-m)} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} R_{XX} [n-m] e^{-j\Omega_k(n-m)} \end{aligned}$$

# Periodogram Proof (2)

$N$  Gaussian random variables

## Definition

$$\begin{aligned} E \left[ \hat{S}_N(\Omega_k) \right] &= \frac{1}{N} \sum_{n=-(N-1)}^{+(N-1)} (N - |k|) R_{XX}[k] e^{-j\Omega_k k} \\ &= \frac{N}{N} \sum_{n=-(N-1)}^{+(N-1)} R_{XX}[k] e^{-j\Omega_k k} \\ &\quad - \frac{1}{N} \sum_{n=-(N-1)}^{+(N-1)} |k| R_{XX}[k] e^{-j\Omega_k k} \end{aligned}$$

# The Variance of the Periodogram (1)

$N$  Gaussian random variables

## Definition

$$\sigma_S^2 = E \left[ \widehat{S}_N(\Omega_k)^2 \right] - E \left[ \widehat{S}_N(\Omega_k) \right]^2$$

$$E \left[ \widehat{S}_N(\Omega_k)^2 \right] = E \left[ \left( \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} E[X[n]X[m]] e^{-j\Omega_k(n-m)} \right)^2 \right]$$

$$= \frac{1}{N^2} \sum_{n,m,p,q} E[X[n]X[m]X[p]X[q]] e^{-j\Omega_k(n-m)} e^{-j\Omega_k(q-p)}$$

# The Variance of the Periodogram (2)

$N$  Gaussian random variables

## Definition

$$\begin{aligned} E [X[n]X[m]X[p]X[q]] &= R_{XX}[n-m] \cdot R_{XX}[p-q] \\ &\quad + R_{XX}[n-p] \cdot R_{XX}[m-q] \\ &\quad + R_{XX}[n-q] \cdot R_{XX}[m-p] \end{aligned}$$

$$\begin{aligned} E \left[ \widehat{S}_N(\Omega_k)^2 \right] &= 2E \left[ \widehat{S}_N(\Omega_k) \right]^2 \\ &\quad + \frac{1}{N^2} \left| \sum_{n,p} R_{XX}[n-p] e^{-j\Omega_k(n-p)} \right|^2 \end{aligned}$$

# The Smoothed Estimate

$N$  Gaussian random variables

## Definition

$$\sigma_{\hat{S}}^2 \geq E \left[ \hat{S}_N(\Omega_k) \right]^2$$

$$\hat{S}_{M,P}(\Omega_k) = \frac{1}{P} \sum_{p=1}^P \hat{S}_{M,p}(\Omega_k) \quad k = 0, 1, \dots, M-1$$



