Fundamental Matrix Spaces (4A)

Young Won Lim
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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Row & Column Spaces

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

ROW Space **subspace** of R^n

$$= span\{\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_m\}$$

COLUMN Space **subspace** of R^m

$$= span\{\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_n\}$$

$$\begin{aligned} \boldsymbol{r}_1 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \\ \boldsymbol{r}_2 &= \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \\ & \quad\ \ \vdots \qquad \vdots \qquad\qquad \vdots \\ \boldsymbol{r}_m &= \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{aligned}$$

$$\longleftarrow \quad \boldsymbol{n} \quad \longrightarrow$$

$\boldsymbol{r}_i \in R^n$

$$\begin{array}{cccc} \boldsymbol{c}_1 & \boldsymbol{c}_2 & & \boldsymbol{c}_n \\ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} & \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \end{array} \quad \updownarrow \boldsymbol{m}$$

$\boldsymbol{c}_i \in R^m$

Row Space

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

ROW Space **subspace** of R^n

$$= span\{\boldsymbol{r_1}, \boldsymbol{r_2}, \cdots, \boldsymbol{r_m}\}$$

$$= \{\boldsymbol{w}\}$$

$$\boldsymbol{r_i} \in R^n$$

$$\boldsymbol{w} = k_1\boldsymbol{r_1} + k_2\boldsymbol{r_2} + \cdots + k_m\boldsymbol{r_m}$$

$$\boldsymbol{r_1} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$

$$\boldsymbol{r_2} = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$$

$$\vdots$$

$$\boldsymbol{r_m} = \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\xleftrightarrow{\quad \boldsymbol{n} \quad}$$

$$= k_1 \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$

$$+ k_2 \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$$

$$\vdots$$

$$+ k_m \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Column Spaces

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

COLUMN Space **subspace** of R^m

$$= span\{\mathbf{c_1}, \mathbf{c_2}, \cdots, \mathbf{c_n}\}$$

$$= \{\mathbf{w}\}$$

$\mathbf{c}_i \in R^m$

$$\begin{array}{cccccc} & \mathbf{c_1} & \mathbf{c_2} & & \mathbf{c_n} \\ \mathbf{m} & \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} & \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \end{array}$$

$$\mathbf{w} = k_1 \mathbf{c_1} + k_2 \mathbf{c_2} + \cdots + k_n \mathbf{c_n}$$

$$= k_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + k_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots + k_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Null Space

$$\begin{array}{c|c} & n \\ m & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} m$$

NULL Space

subspace of R^n

solution space

$$= \begin{pmatrix} a_{11}x_1 + & a_{12}x_2 + & \cdots & a_{1n}x_n \\ a_{21}x_1 + & a_{22}x_2 + & \cdots & a_{2n}x_n \\ \vdots & \vdots & & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \cdots & a_{mn}x_n \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$$\boldsymbol{A}\boldsymbol{x} = x_1 \boldsymbol{c}_1 + x_2 \boldsymbol{c}_2 + \cdots + x_n \boldsymbol{c}_n = \boldsymbol{0} \qquad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$$

$$\boldsymbol{A}\boldsymbol{x} = x_1 \boldsymbol{c}_1 + x_2 \boldsymbol{c}_2 + \cdots + x_n \boldsymbol{c}_n = \boldsymbol{b} \qquad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

Null Space

$$\begin{array}{cc} & n \\ m & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} m$$

NULL Space — **subspace** of R^n

solution space — $A x = \mathbf{0}$

Invertible A — $x = A^{-1}\mathbf{0} = \mathbf{0}$ — only trivial solution — $\{\mathbf{0}\}$

Non-invertible A $\cancel{A^{-1}}$

zero row(s) in a RREF	free variables	parameters *s, t, u, …*	
one	one	a <u>line</u> through the origin	R^1
two	two	a <u>plane</u> through the origin	R^2
three	three	a <u>3-dim</u> space through the origin	R^3

Solution Space of **Ax=b** (1)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$$

$$\cancel{0 = 1}$$

$$1 \cdot x_1 + 3 \cdot x_3 = -1$$
$$1 \cdot x_2 - 4 \cdot x_3 = 2$$

$$1 \cdot x_1 - 5 \cdot x_2 + 1 \cdot x_3 = 4$$

Solve for a leading variable

$$x_1 = -1 - 3 \cdot x_3 \qquad\qquad x_1 = 4 + 5 \cdot x_2 - 1 \cdot x_3$$
$$x_2 = 2 + 4 \cdot x_3$$

Treat a free variable as a parameter

$$x_3 = t \qquad\qquad x_2 = s \quad x_3 = t$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

Solution Space of **Ax=b** (2)

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

$x_3 = t$ ⬅ free variable

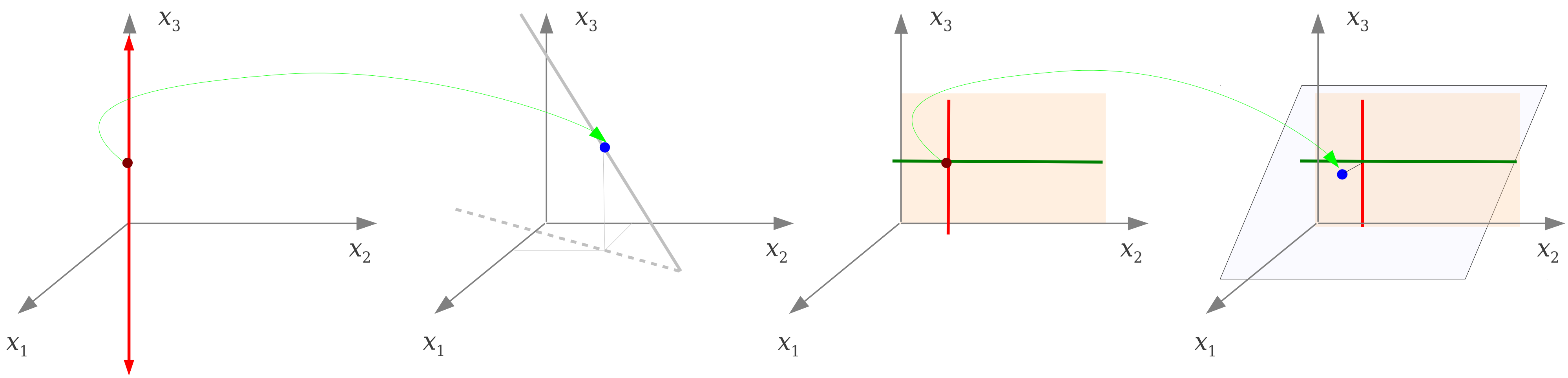
$$\begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

$x_2 = s$ ⬅ free variable

$x_3 = t$ ⬅ free variable

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$



infinitely many solutions *infinitely many solutions*

Solution Space of **Ax=b** (3)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

General Solution of $Ax = b$ ⇒

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}} + \boxed{t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}}$$

Particular Solution of $Ax = b$

General Solution of $Ax = 0$

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}} + \boxed{s \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

Particular Solution of $Ax = b$

General Solution of $Ax = 0$

Linear System & Inner Product (1)

Linear Equations

Corresponding Homogeneous Equation

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

$$\boldsymbol{a} = (a_1, a_2, \cdots, a_n)$$

$$\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$$

normal vector

$$\boldsymbol{a} \cdot \boldsymbol{x} = b$$

$$\boldsymbol{a} \cdot \boldsymbol{x} = 0$$

each **solution** vector $\boldsymbol{x}$ of a **homogeneous** equation
orthogonal to the coefficient vector $\boldsymbol{a}$

Homogeneous Linear System

$$\begin{array}{l} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = 0 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = 0 \end{array}$$

$$\begin{array}{l} \boldsymbol{r_1} \cdot \boldsymbol{x} = 0 \\ \boldsymbol{r_2} \cdot \boldsymbol{x} = 0 \\ \cdots \\ \boldsymbol{r_m} \cdot \boldsymbol{x} = 0 \end{array}$$

Linear System & Inner Product (2)

Homogeneous Linear System

$$\begin{array}{lcr} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 & & \boldsymbol{r_1}\cdot\boldsymbol{x} = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 & & \boldsymbol{r_2}\cdot\boldsymbol{x} = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots & & \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 & & \boldsymbol{r_m}\cdot\boldsymbol{x} = 0 \end{array}$$

each **solution** vector $\boldsymbol{x}$ of a **homogeneous** equation **orthogonal** to the row vector $\boldsymbol{r_i}$ of the coefficient matrix

Homogeneous Linear System $\boldsymbol{A}\cdot\boldsymbol{x} = \boldsymbol{0}$ $\boldsymbol{A} : m\times n$

solution set consists of all vectors in R^n
that are **orthogonal** to every row vector of $\boldsymbol{A}$

Linear System & Inner Product (3)

Non-Homogeneous Linear System $\boldsymbol{A}\cdot\boldsymbol{x} = \boldsymbol{b}$ $\boldsymbol{A} : m \times n$

Homogeneous Linear System $\boldsymbol{A}\cdot\boldsymbol{x} = \boldsymbol{0}$

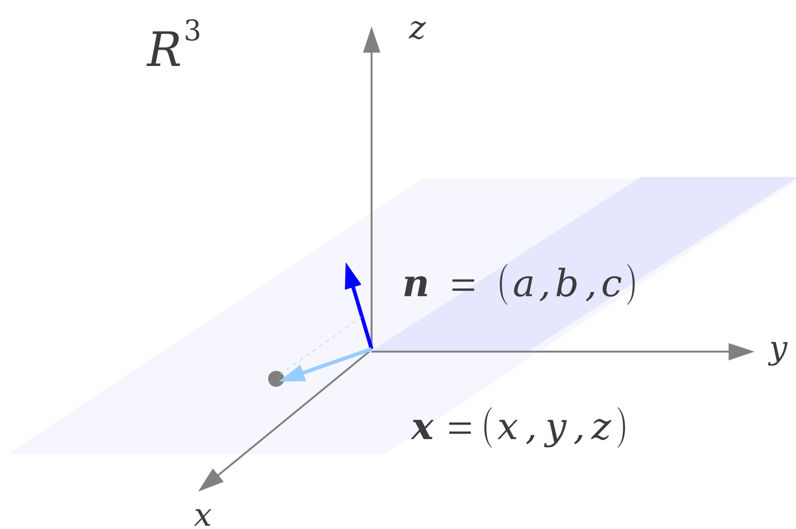
a particular solution

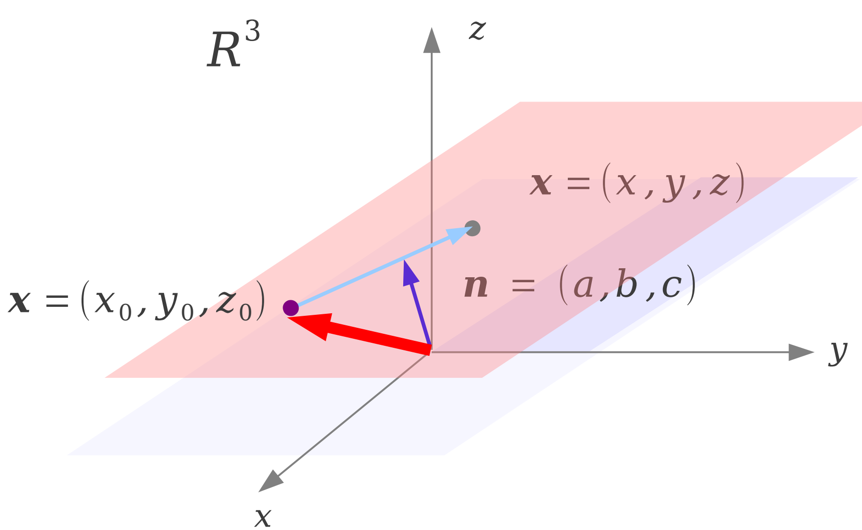
$\boldsymbol{A}\cdot\boldsymbol{x} = \boldsymbol{b}$

solution set consists of all vectors in R^n that are **orthogonal** to every row vector of $\boldsymbol{A}$

+

a particular solution $\boldsymbol{x_0}$ $\boldsymbol{A}\cdot\boldsymbol{x_0} = \boldsymbol{b}$





Linear System & Inner Product (4)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$3 \begin{cases} 2 & \begin{cases} \mathbf{r_1} \cdot \mathbf{x} = 0 \\ \mathbf{r_2} \cdot \mathbf{x} = 0 \end{cases} \\ 1 & \{ \text{a line through the origin} \quad R^1 \end{cases} \qquad 3 \begin{cases} 1 & \{ \mathbf{r_1} \cdot \mathbf{x} = 0 \\ 2 & \{ \text{a plane through the origin} \quad R^2 \end{cases}$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

Consistent Linear System **Ax=b**

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + & a_{12}x_2 + & \cdots & a_{1n}x_n \\ a_{21}x_1 + & a_{22}x_2 + & \cdots & a_{2n}x_n \\ \vdots & \vdots & & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \cdots & a_{mn}x_n \end{pmatrix}$$

$$= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ consistent ⟺

$x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots + x_n\boldsymbol{c}_n = \boldsymbol{b}$

expressed in linear combination of column vectors

⟺ $\boldsymbol{b}$ is in the column space of $\boldsymbol{A}$

$$\boldsymbol{A}\boldsymbol{x} = x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots + x_n\boldsymbol{c}_n = \boldsymbol{b}$$

Rank and Nullity

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

ROW Space **subspace** of R^n

$= span\{\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_m\}$

COLUMN Space **subspace** of R^m

$= span\{\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_n\}$

NULL Space **subspace** of R^n **solution space** $A\boldsymbol{x} = \boldsymbol{0}$

Invertible A $\boldsymbol{x} = A^{-1}\boldsymbol{0} = \boldsymbol{0}$ only trivial solution

Non-invertible A zero row(s) in a RREF free variables parameters *s, t, u, …*

dim(row space of A) = dim(column space of A) = **rank**(A)

dim(null space of A) = **nullity**(A)

Solution Space of **Ax=0**

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

the same case

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

General Solution of $Ax = \mathbf{0}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

dim(row space of A)
dim(col space of A)

rank(A) = 2 **rank**(A) = 1

dim(null space of A)

nullity(A) = 1 **nullity**(A) = 2

Elementary Row Operation (1)

ROW Space **subspace** of R^n

$= span\{\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_m\}$

COLUMN Space **subspace** of R^m

$= span\{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}$

NULL Space **subspace** of R^n

solution space $A\mathbf{x} = \mathbf{0}$

free variables parameters *s, t, u, …*

Elementary row operations do not change the **null space** of a matrix

Elementary row operations do not change the **row space** of a matrix

Elementary row operations ***do*** **change** the **col space** of a matrix

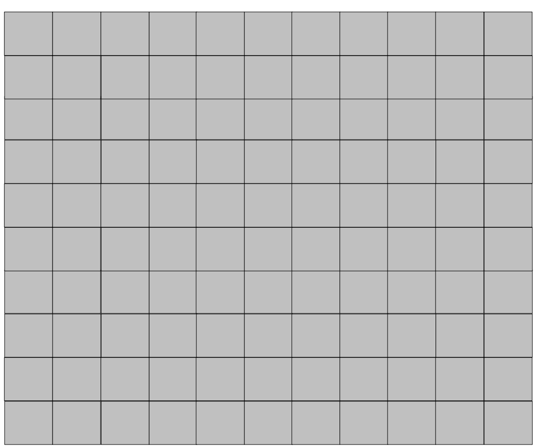
Elementary row operations do not change
the **linear dependence** and **linear independence** relationship among column vectors

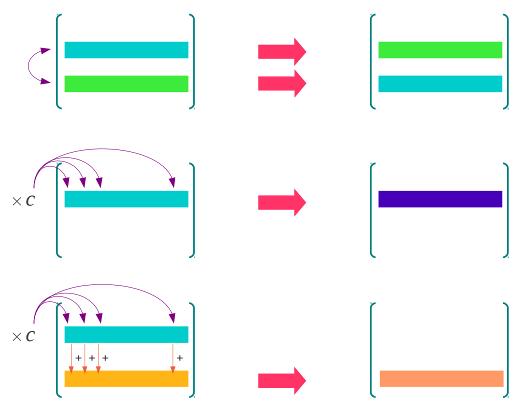
Elementary Row Operation (2)

Elementary row operations do not change the **null space** of a matrix
Elementary row operations do not change the **row space** of a matrix
Elementary row operations do not change the **linear dependence** and **linear independence** relationship among column vectors

Elementary row operations ***do* change** the **col space** of a matrix

A





R

1	0		0		0	0	0		0	
0	1		0		0	0	0		0	
0	0	0	1		0	0	0			
0	0	0	0	0	1	0	0		0	
0	0	0	0	0	0	1	0			
0	0	0	0	0	0	0	1		0	
0	0	0	0	0	0	0	0	0	1	
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

Elementary Row Operation (3)

Elementary row operations

- do not change the **null space** of a matrix
- do not change the **row space** of a matrix
- do not change the **linear dependence** and **linear independence** relationship among column vectors

- ***do* change** the **col space** of a matrix

A

row space of **A** = **row** space of **R**
null space of A = **null** space of **R**
col space of **A** ≠ **col** space of **R**

R

a given set of column vectors

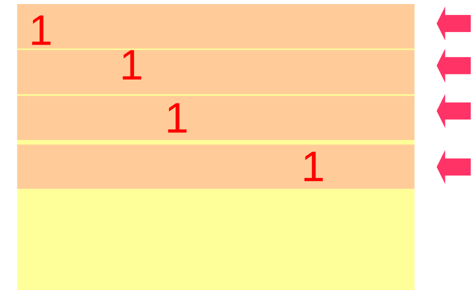
linearly independent ⟺ linearly independent

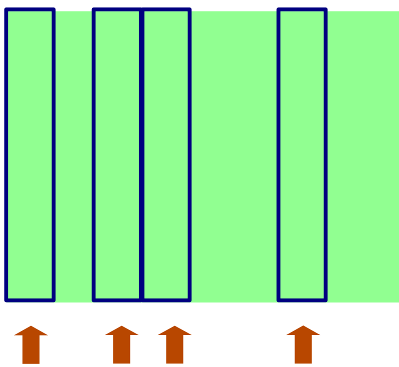
basis ⟺ basis

the corresponding set of column vectors

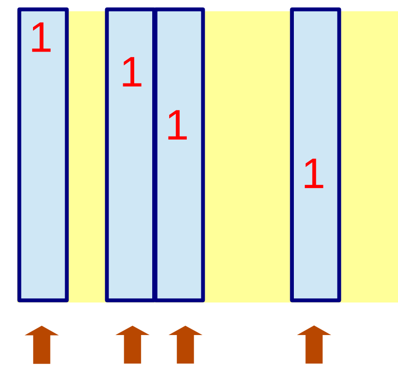
Bases of Row & Column Spaces (1)

basis of row space of **A** $=$ basis of row space of **R**





basis of col space of **A** $\neq$ basis of col space of **R**

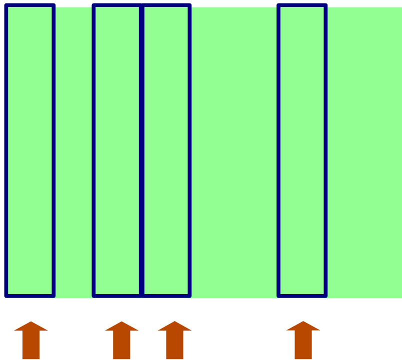


the corresponding set of column vectors

a given set of column vectors

dim(row space of A) = dim(column space of A) = **rank**(A)

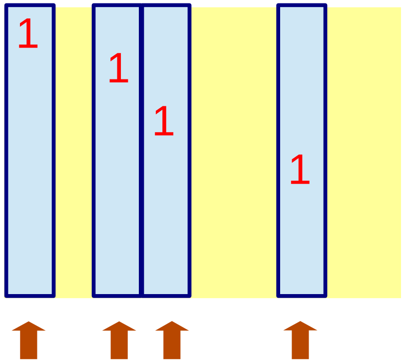
Bases of Row & Column Spaces (2)



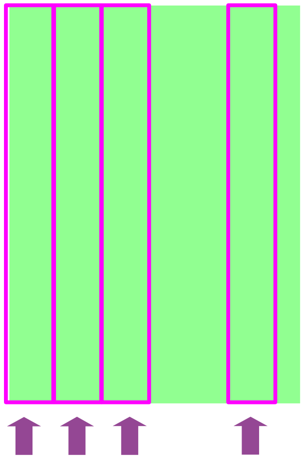
basis of
col space
of **A**

≠

basis of
col space
of **R**



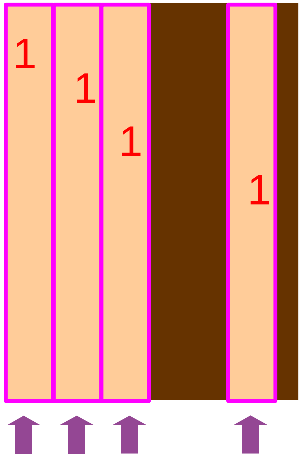
the basis consisting of columns of **A**



basis of
col space
of **A**

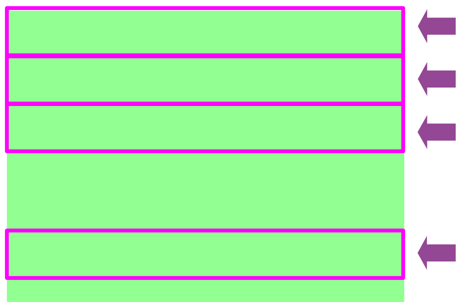
≠

basis of
col space
of **R**

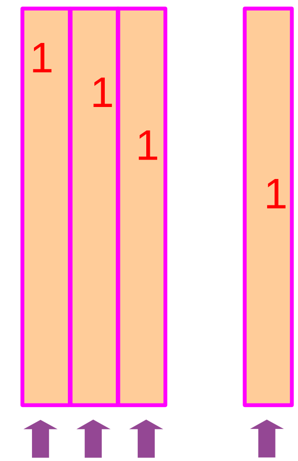


the basis consisting of rows of **A**

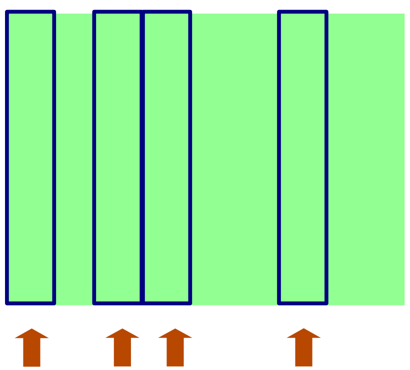
Bases of Row & Column Spaces (3)



basis of
col space
of **R**



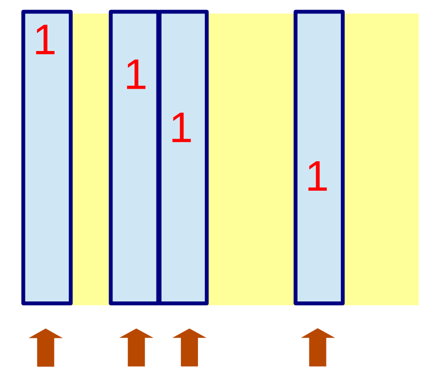
the basis consisting of rows of **A**



basis of
col space
of **A**

$\neq$

basis of
col space
of **R**



the basis consisting of columns of **A**

General Solution of **Ax=b** (1)

Non-Homogeneous Linear System $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$

Homogeneous Linear System $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{0}$

$\boldsymbol{A} : m \times n$

a particular solution

$\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$

solution set consists of all vectors in R^n
that are **orthogonal** to every row vector of $\boldsymbol{A}$

+

a particular solution $\boldsymbol{x}_0$ $\boldsymbol{A} \cdot \boldsymbol{x}_0 = \boldsymbol{b}$

The general solution of a consistent linear system can be written as

the sum of a particular solution of **Ax=b** and the general solution of **Ax=0**

General Solution of **Ax=b** (2)

Any solution of a **consistent** linear system $\boldsymbol{A}\cdot\boldsymbol{x} = \boldsymbol{b}$ | $\boldsymbol{x}_0$

A **basis** for the **null space** (solution space $\boldsymbol{A}\cdot\boldsymbol{x} = \boldsymbol{0}$) | $S = \{\boldsymbol{v}_1, \ \boldsymbol{v}_2, \ \cdots \ , \ \boldsymbol{v}_k\}$

Every solution of $\boldsymbol{A}\cdot\boldsymbol{x} = \boldsymbol{b}$

⇒ in the form $\boldsymbol{x} = \boldsymbol{x}_0 + \ c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ \cdots \ , c_1\boldsymbol{v}_k$

$\boldsymbol{x}$ is a solution of $\boldsymbol{A}\cdot\boldsymbol{x} = \boldsymbol{b}$

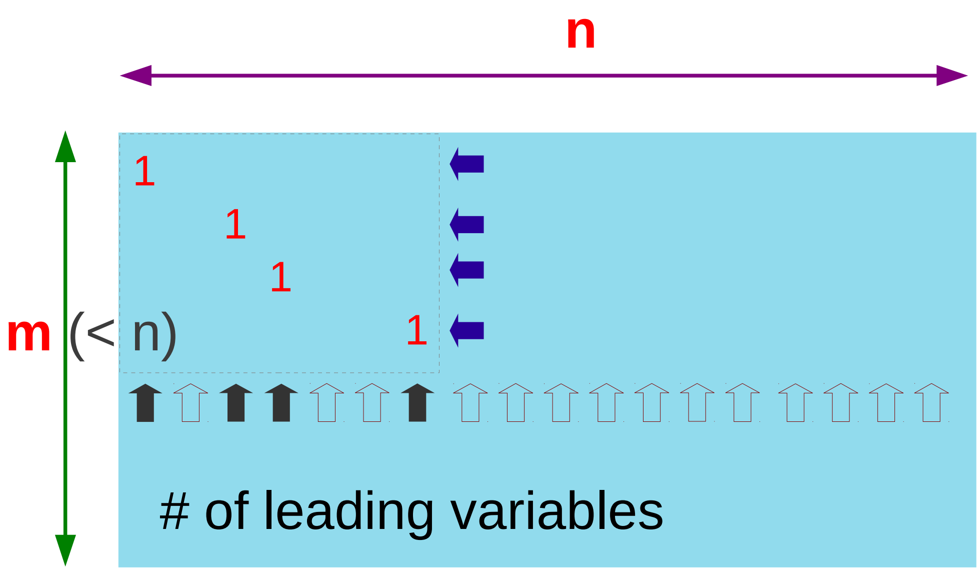
⇐ $\boldsymbol{x} = \boldsymbol{x}_0 + \ c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ \cdots \ + c_k\boldsymbol{v}_k$

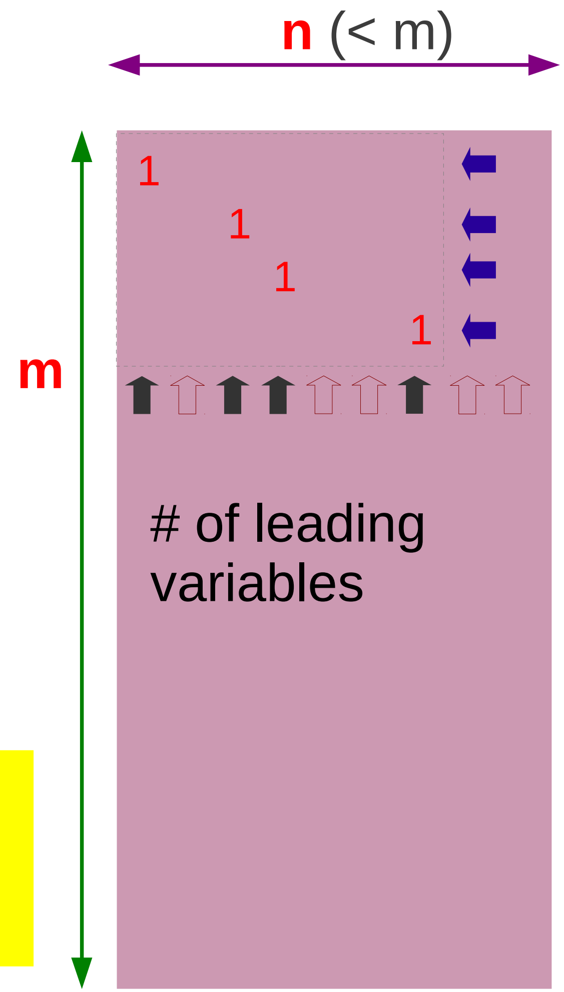
for all choices of scalars $c_1, \ c_2, \ \cdots \ c_k$

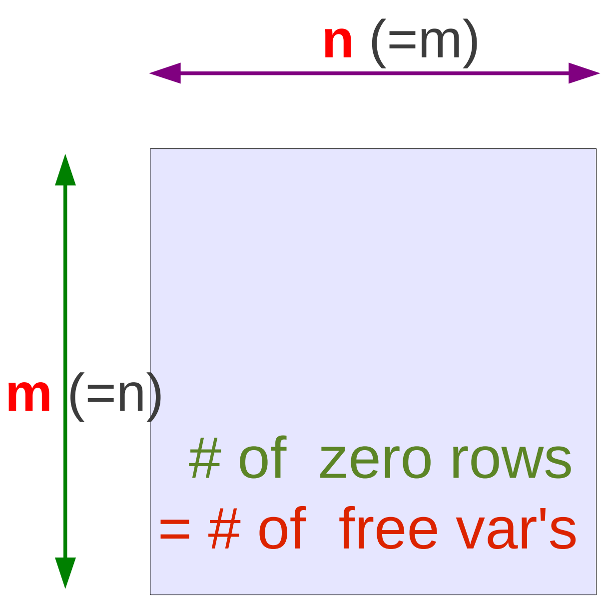
The general solution of a consistent linear system can be written as

the sum of <u>a particular solution</u> of **Ax=b** and <u>the general solution</u> of **Ax=0**

Rank and Nullity (1)







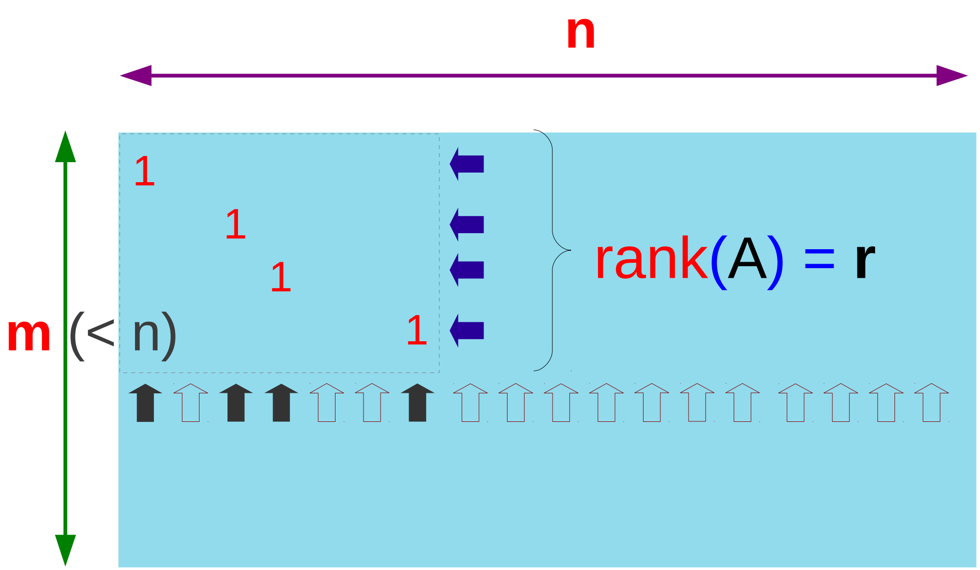
rank(A) ≤ min(**m**, n)

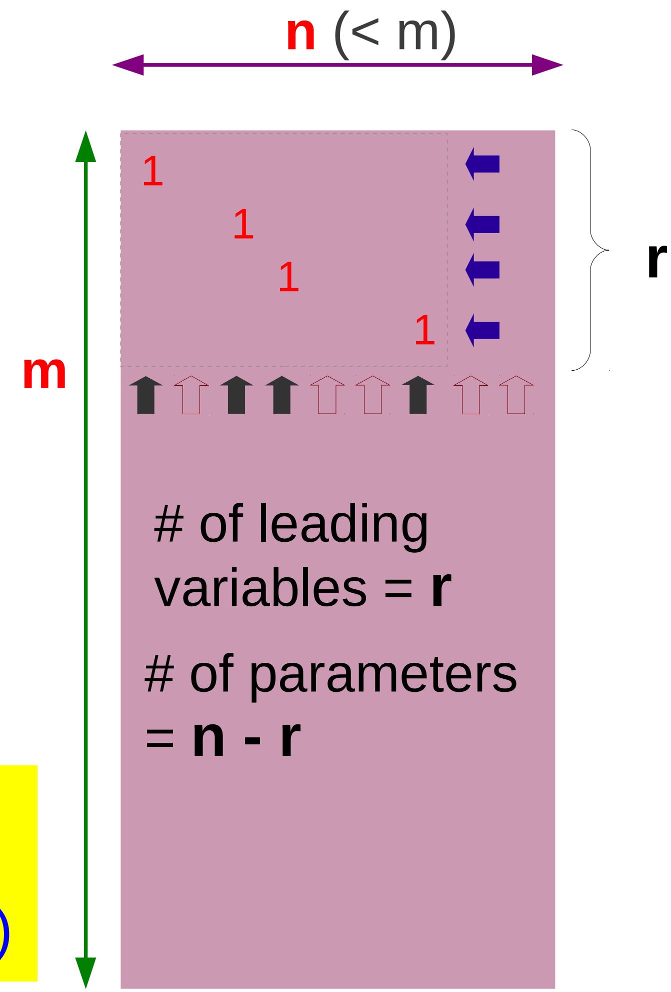
n = **rank**(A) + **nullity**(A)

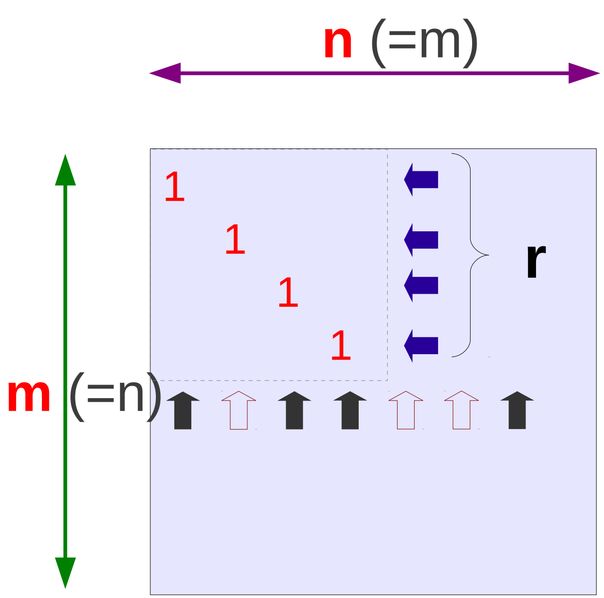
m = (# of leading variables) + (# of zero rows)

n = (# of leading variables) + (# of free variables)

Rank and Nullity (2)







rank(A) ≤ min(**m**, n)

n = **rank**(A) + **nullity**(A)

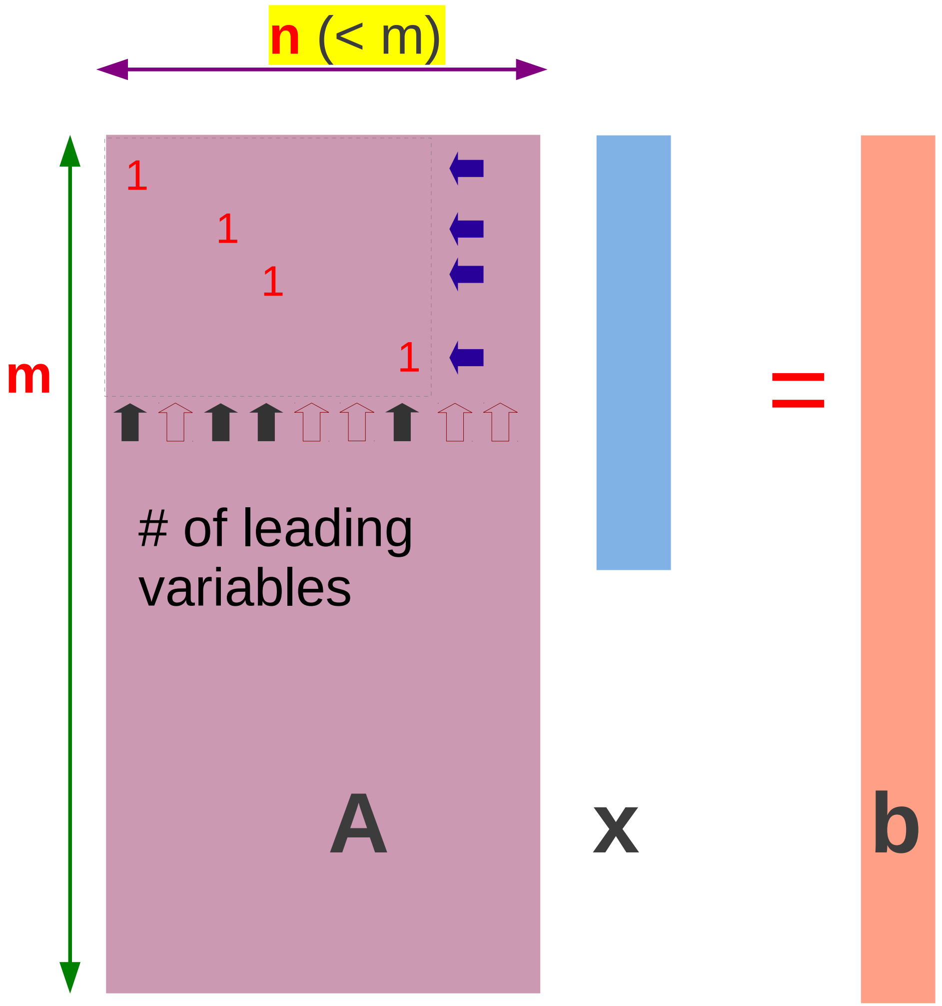
# of leading variables = **r**

# of parameters = **n - r**

rank(A)

nullity(A)

Overdetermined System



n column vectors
can span at most R^n

b is in R^m

$R^m \supset R^n$

At least one vector **b** in R^m
does not lie in column space

For such **b** in R^m

Ab = b inconsistent

$$A\mathbf{x} = x_1 \mathbf{c}_1 + x_2 \mathbf{c}_2 + \cdots + x_n \mathbf{c}_n \cancel{=} \mathbf{b}$$

n column vectors can span at most R^n **b** is in R^m

Overdetermined System Example

$$\left(\begin{array}{cc|c} 1 & 0 & b1 \\ 0 & 1 & b2 \\ 0 & 0 & b3 \end{array}\right)$$

Overdetermined **Ab = b**

may be consistent or inconsistent
depending on b1, b2, b3

b3 = 0 ➡ consistent unique solution

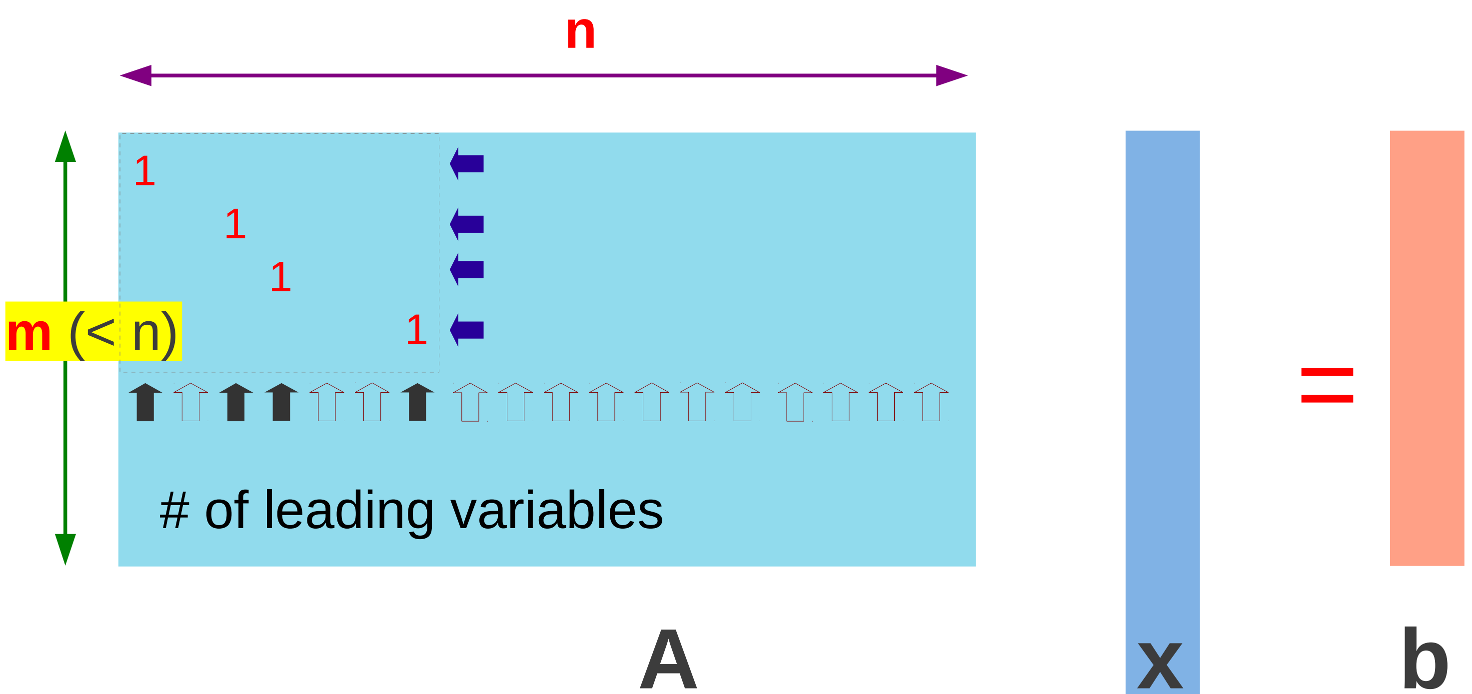
$$\left(\begin{array}{cc|c} 1 & 0 & c1 \\ 0 & 1 & c2 \\ 0 & 0 & c3 \\ 0 & 0 & c4 \\ 0 & 0 & c5 \end{array}\right)$$

n = 2 **r** = 2

# of parameters = n - r = 0 unique

c3 = 0 & c4 = 0 & c5 = 0

Underdetermined System



Ab = b

- inconsistent
- consistent but infinitely many solutions

rank(A) = r $\leq$ m

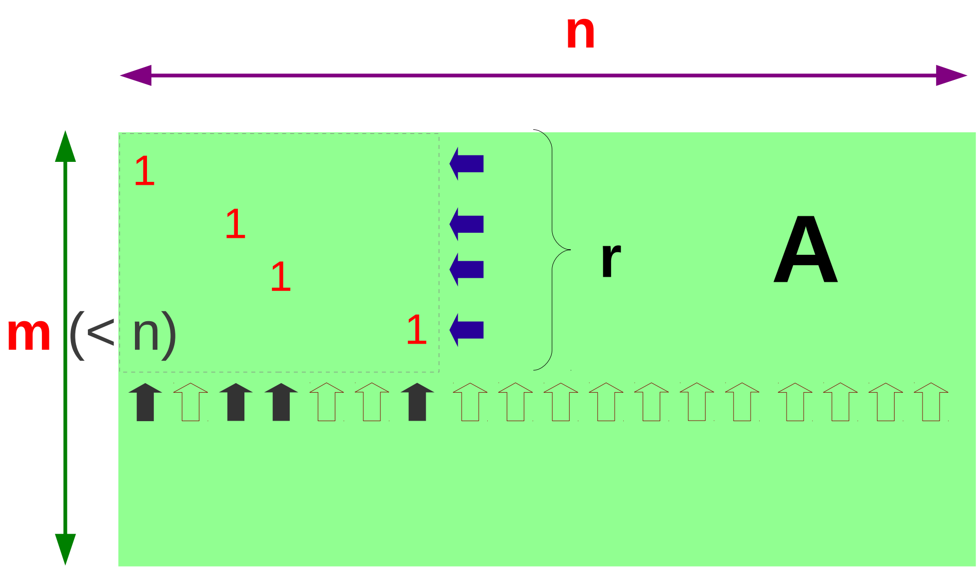
n – r parameters

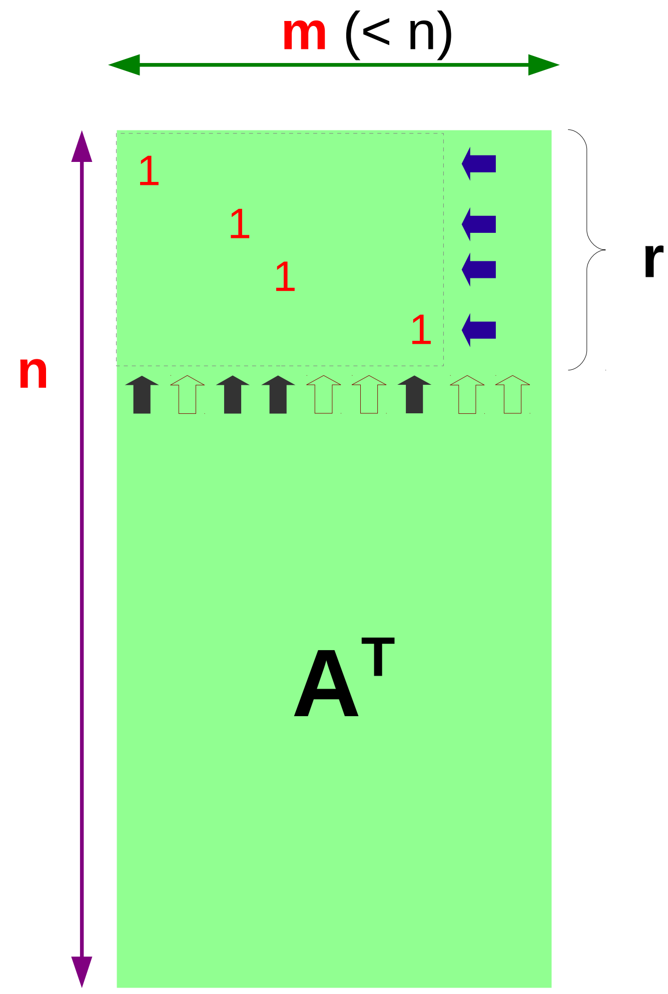
n – m > 0 parameters

at least one parameter

➡ infinitely many solutions

Fundamental Matrix Spaces (1)





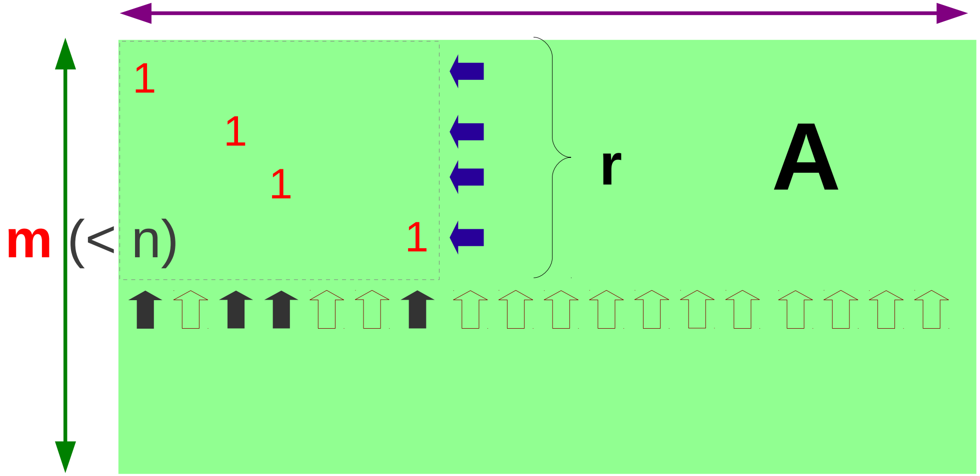
row(A) = col(A^T)

col(A) = row(A^T)

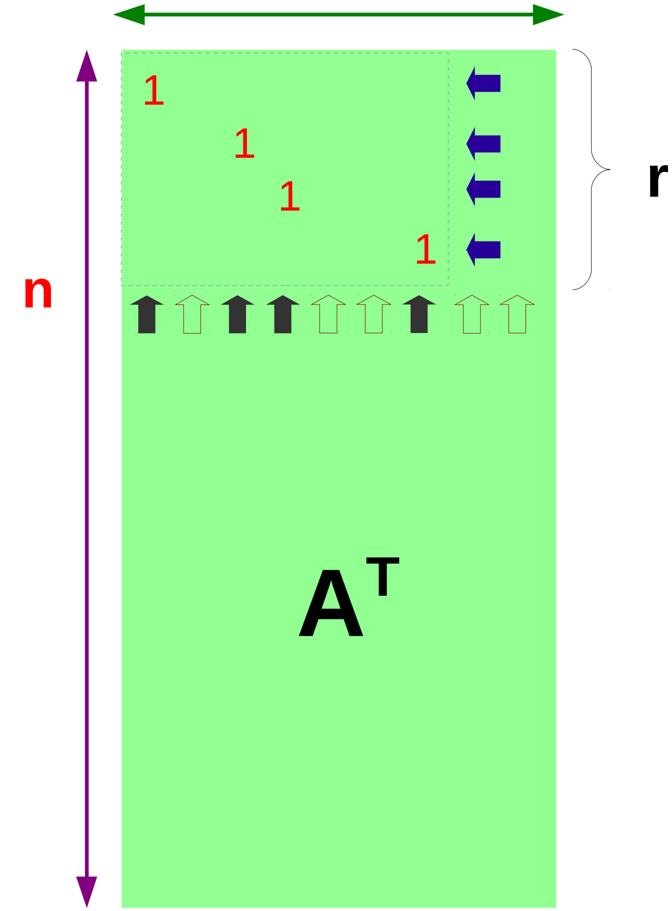
null(A) null(A^T)

Fundamental Matrix Spaces (2)

n = rank(A) + nullity(A)



m = rank(A) + nullity(A^T)



row(A)	=	**col**(A^T)
col(A)	=	**row**(A^T)
null(A)		**null**(A^T)

nullity(A) = **n – r**

nullity(A^T) = **m – r**

rank(A) = **rank**(A^T) = **r**

Orthogonal Complement

m = rank(A) + **nullity**(A^T)

W a subspace of R^n

The **orthogonal complement** of W ⟺ The set of all vectors in R^n that are orthogonal to *every vector* in W

$W^\perp$

$W^\perp$ a subspace of R^n

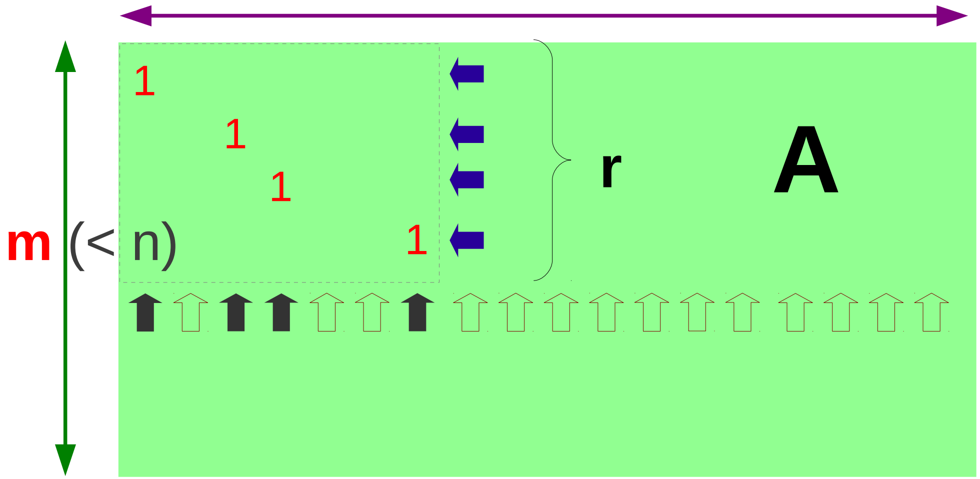
$$W^\perp \cap W = \{\mathbf{0}\}$$

The orthogonal complement of W $W^\perp$

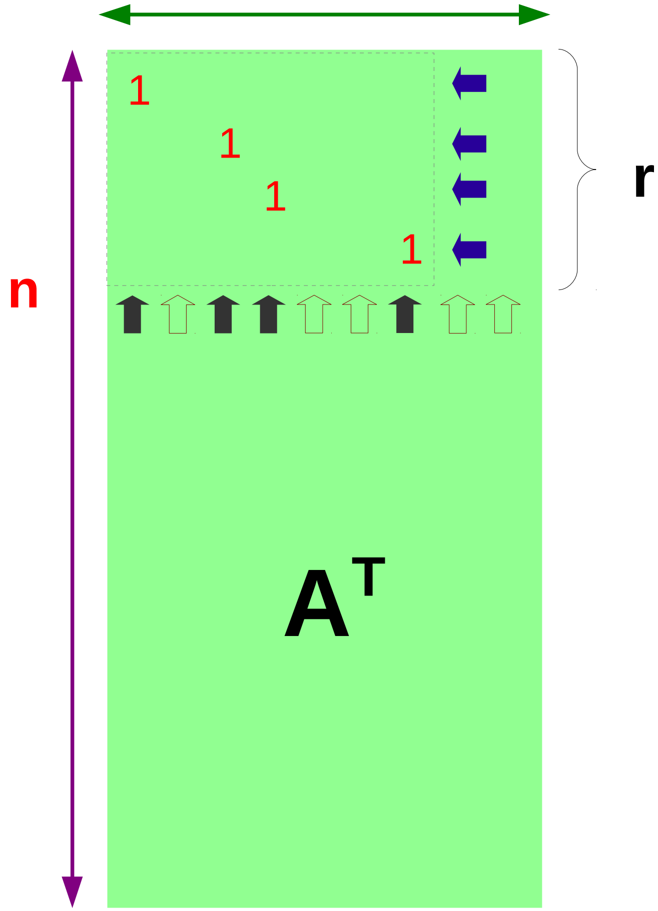
The orthogonal complement of $W^\perp$ W

Fundamental Matrix Spaces (3)

n = rank(A) + **nullity**(A)



m = rank(A) + **nullity**(A^T)



The orthogonal complements

row(A) $\perp$ **null**(A)

row(A^T) $\perp$ **null**(A^T)

col(A) $\perp$ **null**(A^T)

A nxn Matrix A

1. **A** is invertible
2. **Ax** = **0** has only the trivial solution
3. The RREF(**A**) = $\mathbf{I}_n$
4. **A** can be written as a product of elementary matrix
5. **Ax** = **b** is consistent for every n x 1 **b**
6. **Ax** = **b** has exactly one solution for every n x 1 **b**
7. det(**A**) $\neq$ 0
8. The column vectors are linearly independent
9. The row vectors are linearly independent
10. The column vectors span R^n
11. The row vectors span R^n
12. The column vectors form a basis for R^n
13. The row vectors form a basis for R^n
14. rank(**A**) = n
15. nullity(**A**) = 0
16. The orthogonal complement of the null space is R^n
17. The orthogonal complement of the row space is $\{\mathbf{0}\}$

References

[1] http://en.wikipedia.org/
[2] Anton, et al., Elementary Linear Algebra, 10th ed, Wiley, 2011
[3] Anton, et al., Contemporary Linear Algebra,