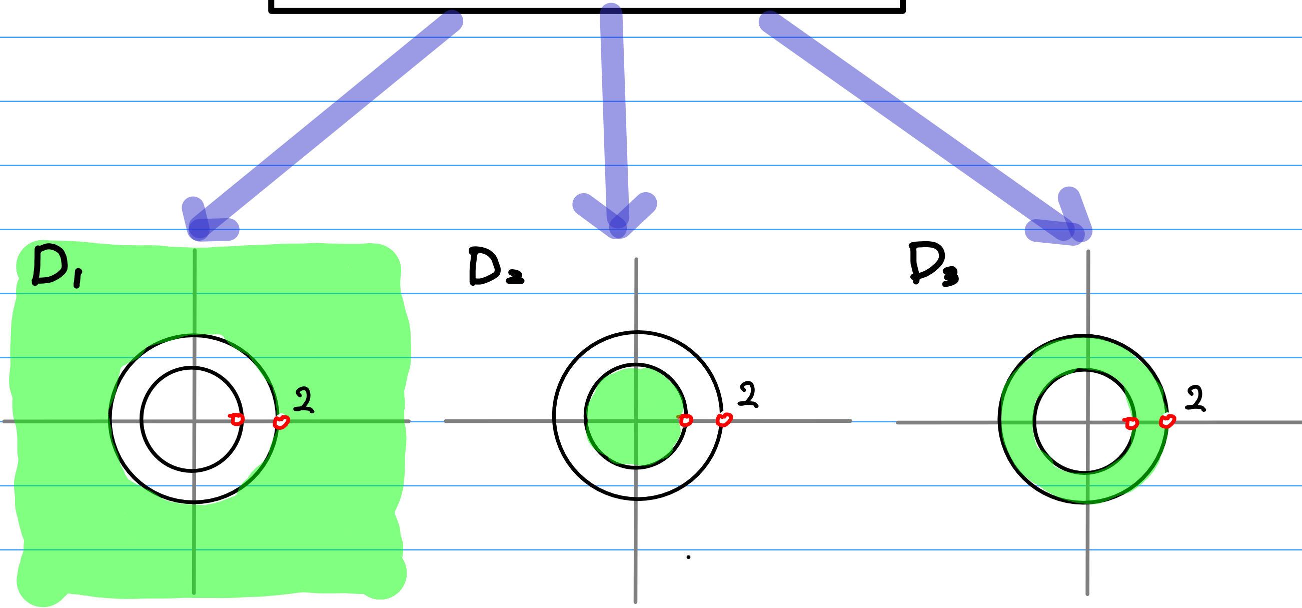
Laurent Series and z-Transform Examples case 1.B

20171012

Z.T. first

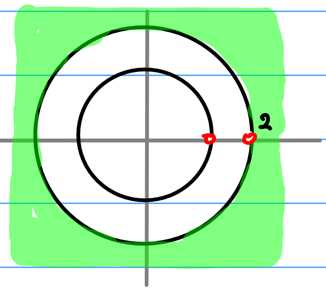
$$X(z) = \frac{-1}{(z-1)(z-2)}$$

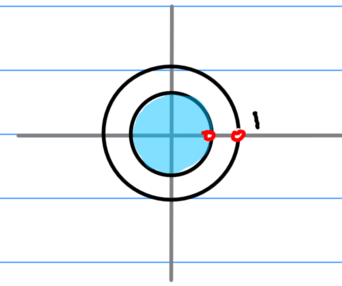
$$\boxed{X(z) = \frac{-1}{(z-1)(z-2)}}$$



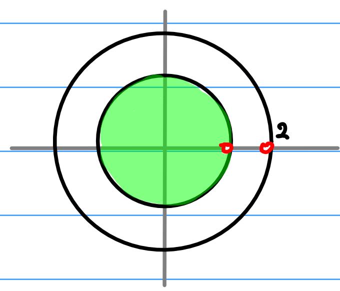
D_1	D_2	D_3
x_n ?	x_n ?	x_n ?
$\Vert$	$\Vert$	$\Vert$
a_n	a_n	a_n
$f(z)$?	$f(z)$?	$f(z)$?

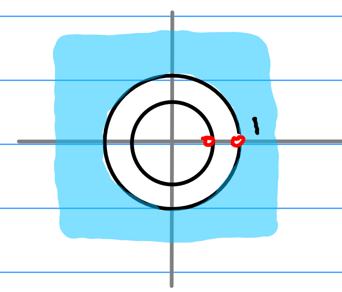
1. B $X(z) = \frac{-1}{(z-1)(z-2)} \xrightarrow{z^{-1}} f(z) = \frac{-0.5 z^2}{(z-1)(z-0.5)}$



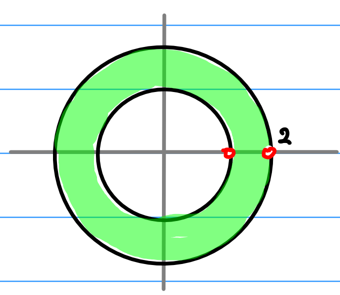


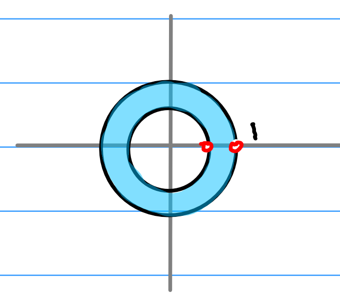
$$\sum_{n=1}^{\infty} \left[1 - 2^{-n+1} \right] z^{-n} = \sum_{n=0}^{\infty} \left[\left(\tfrac{1}{2}\right)^{n+1} - 1 \right] z^{n}$$





$$\sum_{n=-1}^{-\infty} \left[-1 + 2^{n-1} \right] z^{-n} = \sum_{n=-1}^{-\infty} \left(1 - \left(\tfrac{1}{2}\right)^{n+1} \right) z^{n}$$

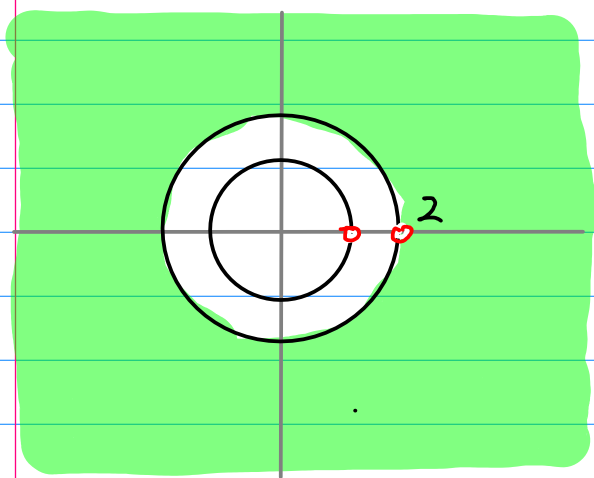




$$\sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^{-n} = \sum_{n=-1}^{-\infty} z^{n} + \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^{n+1} \cdot z^{n}$$

$$X(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$ Z.T. first

① D_1 $|z| > 2$ $\left[\left|\frac{1}{z}\right| < 1 , \quad \left|\frac{2}{z}\right| < 1 \right]$



ROC (Region of Convergence)

$|z| > \frac{1}{2} \Rightarrow \frac{1}{|2z|} < 1$

$\left(\frac{2}{z}\right)^0 + \left(\frac{2}{z}\right)^1 + \left(\frac{2}{z}\right)^2 + \cdots \xrightarrow{\text{converge}} \frac{1}{1-\frac{2}{z}}$

$|z| > 1 \Rightarrow \frac{1}{|z|} < 1$

$\left(\frac{1}{z}\right)^0 + \left(\frac{1}{z}\right)^1 + \left(\frac{1}{z}\right)^2 + \cdots \xrightarrow{\text{converge}} \frac{1}{1-\frac{1}{z}}$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{z}{z-1} - \frac{1}{z}\frac{z}{z-2} = \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

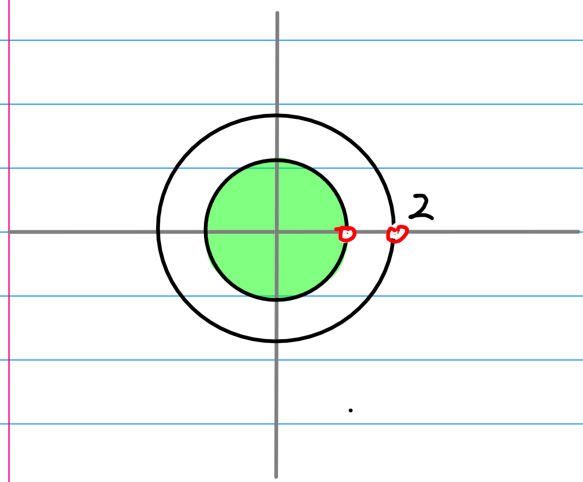
$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n} = \sum_{n=1}^{\infty} (1-2^{n-1}) z^{-n}$$

$$\left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots$$
$$+ \frac{1}{2}\left\{ \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots \right\} \xrightarrow{\text{converge}} \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{(z-1)(z-2)}$$

$$(1-2^0)z^{-1} + (1-2^1)z^{-2} + (1-2^2)z^{-3} + \cdots \xrightarrow{\text{converge}} \frac{-1}{(z-1)(z-2)} \quad (|z| > 2)$$

$$x[n] = 1 - 2^{n-1} \longleftrightarrow X(z) = \frac{-1}{(z-1)(z-2)} \quad (|z| > 2)$$

$\boxed{\text{II} \quad D_2 \quad |z|<1 \qquad \left[\,\left|\frac{z}{1}\right|<1, \quad \left|\frac{z}{2}\right|<1\,\right]}$



$|z|<2 \Rightarrow \frac{|z|}{2}<1$

$$\left(\frac{z}{2}\right)^0 + \left(\frac{z}{2}\right)^1 + \left(\frac{z}{2}\right)^2 + \cdots \xrightarrow[\text{converge}]{} \frac{1}{1-\frac{z}{2}}$$

$|z|<1 \Rightarrow \frac{|z|}{1}<1$

$$\left(\frac{z}{1}\right)^0 + \left(\frac{z}{1}\right)^1 + \left(\frac{z}{1}\right)^2 + \cdots \xrightarrow[\text{converge}]{} \frac{1}{1-\frac{z}{1}}$$

ROC (Region of onvergence)

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = -\frac{1}{1}\frac{1}{1-z} + \frac{1}{2}\frac{2}{2-z} = -\frac{1}{1}\frac{1}{1-\left(\frac{z}{1}\right)} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} = \sum_{n=0}^{\infty}\left(-1+2^{-n-1}\right)z^n$$

$$= \sum_{k=0}^{-\infty}\left(-1+2^{k-1}\right)z^{-k} = \sum_{n=0}^{-\infty}\left(-1+2^{n-1}\right)z^{-n}$$

$$-\left\{\left(\frac{z}{1}\right) + \left(\frac{z}{1}\right)^2 + \left(\frac{z}{1}\right)^3 + \cdots\right\} + \frac{1}{2}\left\{\left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right\} \xrightarrow[\text{converge}]{} \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{(z-1)(z-2)}$$

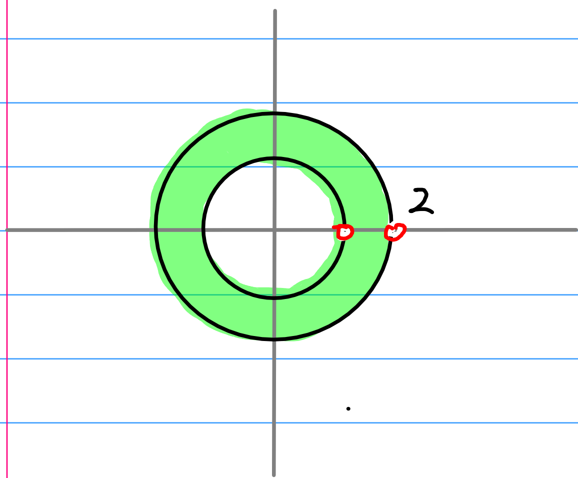
$$(-1+2^{-2})z^1 + (-1+2^{-3})z^2 + (-1+2^{-4})z^3 + \cdots \xrightarrow[\text{converge}]{} \frac{-1}{(z-1)(z-2)} \quad (|z|<2)$$

$$x[n] = -1+2^{n-1} \quad \Longleftrightarrow \quad X(z) = \frac{-1}{(z-1)(z-2)} \quad (|z|>2)$$

$n \le 0$

Z.T. first

III D_3 $\quad 1 < |z| < 2 \quad \left[\left|\frac{1}{z}\right| < 1 , \quad \left|\frac{z}{2}\right| < 1 \right]$



$|z| < 2 \Rightarrow \frac{|z|}{2} < 1$

$\left(\frac{z}{2}\right)^0 + \left(\frac{z}{2}\right)^1 + \left(\frac{z}{2}\right)^2 + \cdots \xrightarrow{} \frac{1}{1-\frac{z}{2}}$

converge

$|z| > 1 \Rightarrow \frac{1}{|z|} < 1$

$\left(\frac{1}{z}\right)^0 + \left(\frac{1}{z}\right)^1 + \left(\frac{1}{z}\right)^2 + \cdots \xrightarrow{} \frac{1}{1-\frac{1}{z}}$

converge

ROC (Region of Convergence)

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{z}{z-1} + \frac{1}{2}\frac{2}{z-2} = \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{k=0}^{-\infty}\frac{z^{-k}}{2^{-k+1}} = \sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{-\infty} 2^{n-1} z^{-n}$$

$$x_n = \begin{cases} 1 & (n > 0) \\ 2^{n-1} & (n \leq 0) \end{cases}$$

$\left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots$

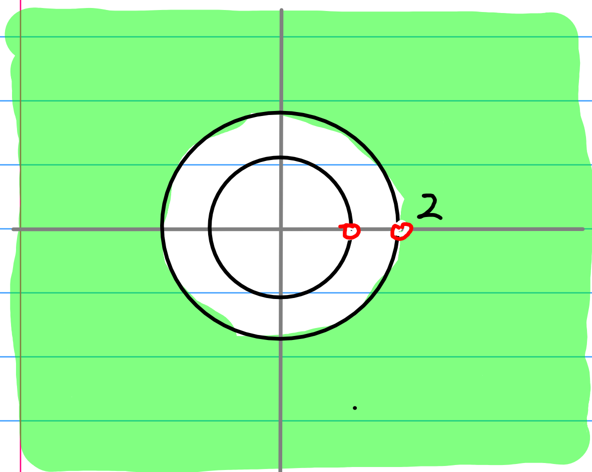
$+ \frac{1}{2}\left\{\left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right\} \xrightarrow{} \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{(z-1)(z-2)}$

converge

Z.T. first

$$X(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

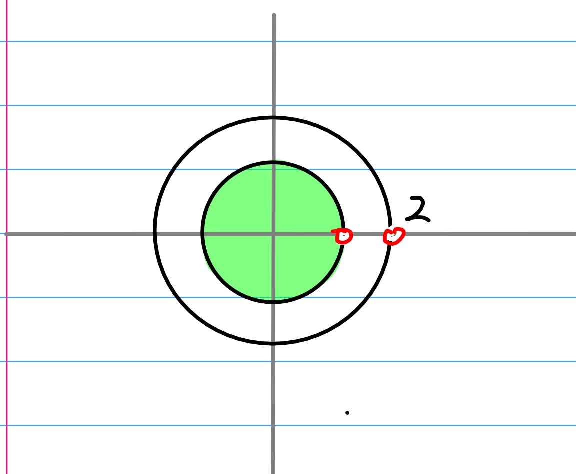
Ⅰ D_1 $|z| > 2$ causal $x_n = 0 \quad (n \leq 0)$



$$\boxed{X(z) = \frac{-1}{(z-1)(z-2)}}$$

$$x_n = \begin{cases} 1 - 2^{n-1} & (n > 0) \\ 0 & (n \leq 0) \end{cases}$$

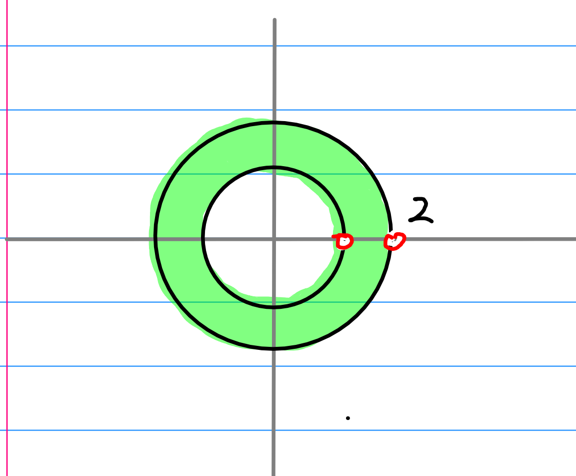
Ⅱ D_3 $|z| < 1$ anti-causal $x_n = 0 \quad (n \geq 0)$



$$\boxed{X(z) = \frac{-1}{(z-1)(z-2)}}$$

$$x_n = \begin{cases} 0 & (n > 0) \\ 2^{n-1} - 1 & (n \leq 0) \end{cases}$$

Ⅲ D_2 $1 < |z| < 2$ two-sided



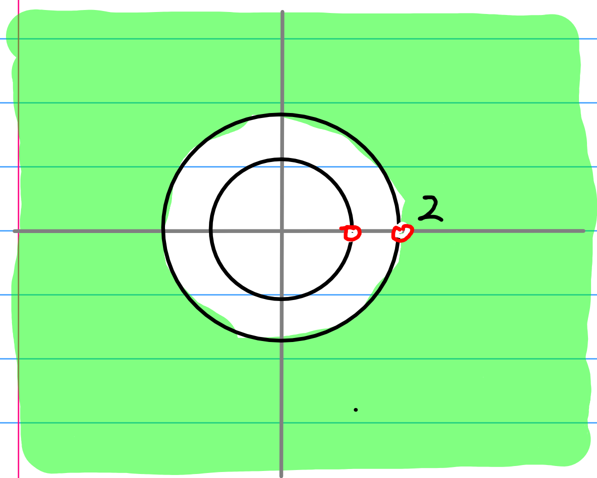
$$\boxed{X(z) = \frac{-1}{(z-1)(z-2)}}$$

$$x_n = \begin{cases} 1 & (n > 0) \\ 2^{n-1} & (n \leq 0) \end{cases}$$

Z.T. first

$$X(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

① D_1 $|z| > 2$ causal $x_n = 0 \quad (n<0)$



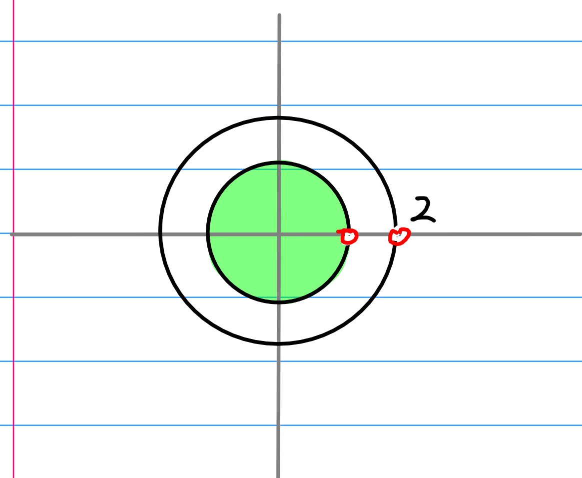
$\left|\frac{1}{z}\right| < 1 \qquad \left|\frac{2}{z}\right| < 1$

$$X(z) = \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} - \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{2}{z}\right)}$$

$$= \sum_{n=0}^{\infty} 1 \cdot z^{-n-1} - \sum_{n=0}^{\infty} 2^n z^{-n-1}$$

$$= \sum_{n=1}^{\infty} \left[1 - 2^{n-1}\right] z^{-n}$$

② D_3 $|z| < 1$ anti-causal $x_n = 0 \quad (n>0)$



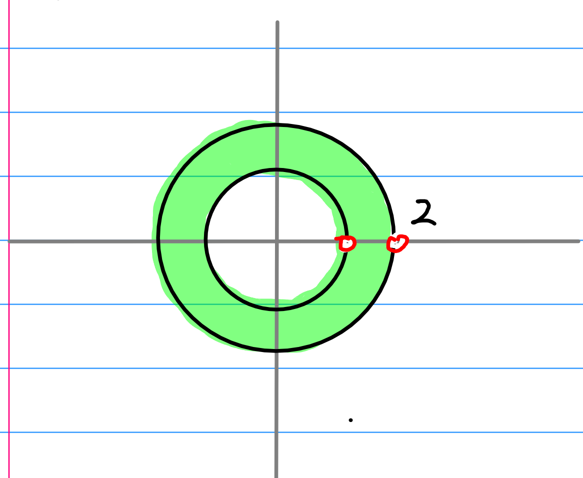
$\left|\frac{z}{1}\right| < 1 \qquad \left|\frac{z}{2}\right| < 1$

$$X(z) = \frac{-1}{1-\left(\frac{z}{1}\right)} + \frac{\frac{1}{2}}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} 1 \; z^n - \sum_{n=0}^{\infty} 2^{-n-1} z^n$$

$$= \sum_{n=-1}^{-\infty} \left[-1 + 2^{n-1}\right] z^{-n}$$

③ D_2 $1 < |z| < 2$ two-sided



$\left|\frac{1}{z}\right| < 1 \qquad \left|\frac{z}{2}\right| < 1$

$$X(z) = \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} + \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{z}{2}\right)}$$

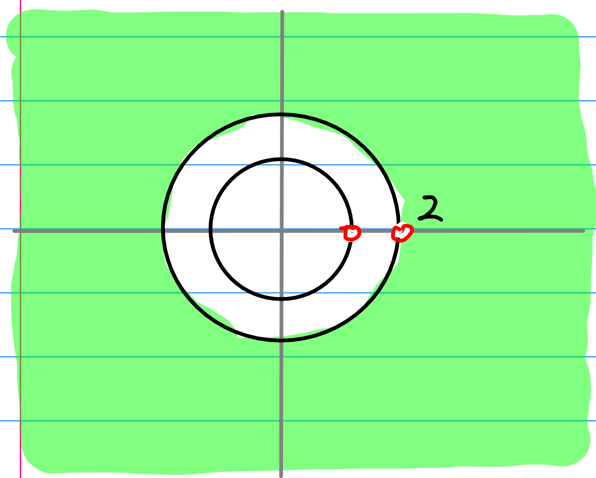
$$= \sum_{n=0}^{\infty} 1 \cdot z^{-n-1} + \sum_{n=0}^{\infty} 2^{-n-1} z^n$$

$$= \sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^{-n}$$

Z.T. first

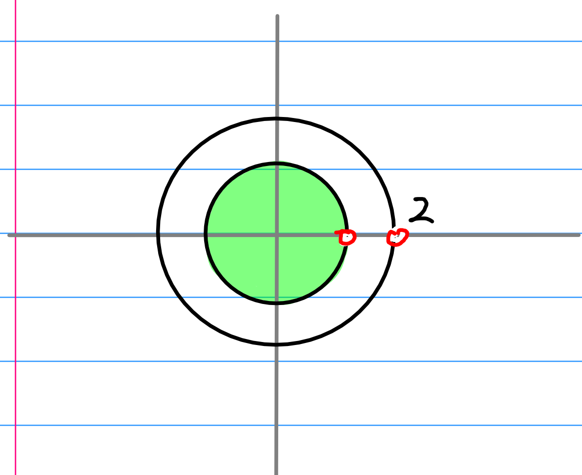
$$X(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

Ⓘ D_1 $|z| > 2$ causal $x_n = 0$ $(n < 0)$



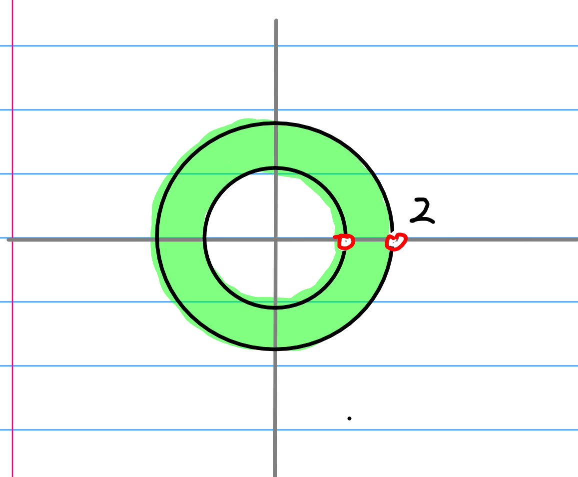
$$X(z) = \sum_{n=1}^{\infty} \left[1 - 2^{-n+1} \right] z^{-n}.$$

Ⓘⓘ D_3 $|z| < 1$ anti-causal $x_n = 0$ $(n > 0)$



$$X(z) = \sum_{n=-1}^{\infty} \left[-1 + 2^{n-1} \right] z^{-n}$$

Ⓘⓘⓘ D_2 $1 < |z| < 2$ two-sided



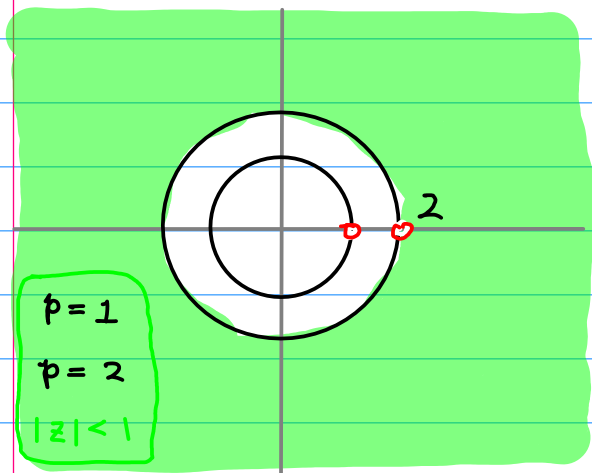
$$X(z) = \sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^{-n}$$

$$X(z) = \frac{-1}{(z-1)(z-2)} \qquad f(z) =$$

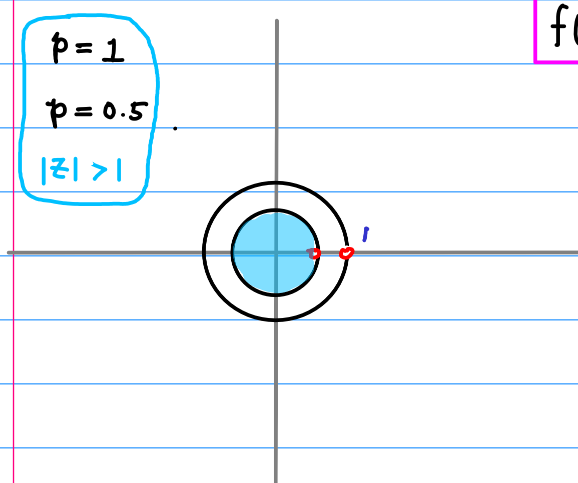
Z.T. first

Ⅰ − 1

Ⅰ D_1 $|z| > 2$ $\left[\ \left|\frac{1}{z}\right| < 1\ ,\ \left|\frac{2}{z}\right| < 1\ \right]$



$$
\begin{aligned}
X(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\
&= \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} - \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{2}{z}\right)} \\
&= \sum_{n=0}^{\infty} 1\, z^{-n-1} - \sum_{n=0}^{\infty} 2^n z^{-n-1} \\
&= \sum_{n=1}^{\infty} \left[1 - 2^{n-1}\right] z^{-n}
\end{aligned}
$$



$$
\begin{aligned}
\boxed{f(z) = X(z^{-1})} &= \sum_{n=1}^{\infty} \left[1 - 2^{n-1}\right] z^{n} \\
&= \sum_{n=1}^{\infty} 1 \cdot z^n - 2^{n-1} \cdot z^n \\
&= \frac{z}{1-\left(\frac{z}{1}\right)} - \frac{z}{1-\left(\frac{2z}{1}\right)} \\
&= -\frac{z}{z-1} + \frac{0.5z}{(z-0.5)} \\
&= \frac{-z+0.5+0.5z-0.5}{(z-1)(z-0.5)}\, z \\
&= \frac{-0.5z^2}{(z-1)(z-0.5)}
\end{aligned}
$$

$$X(z) = \frac{-1}{(z-1)(z-2)}$$

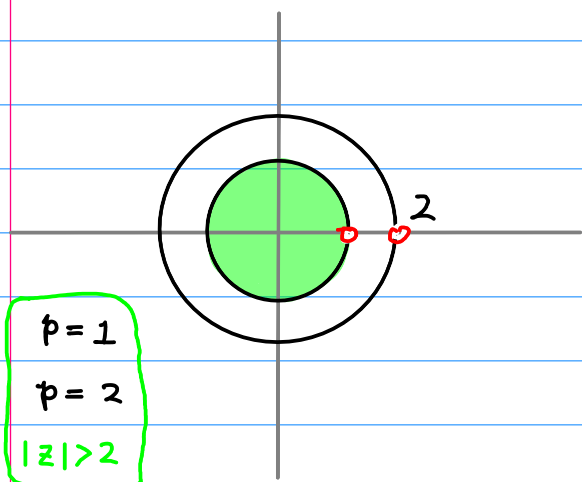
$$X(z^{-1}) = \frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \frac{-z^2}{(1-z)(1-2z)} = \frac{-0.5z^2}{(z-1)(z-0.5)} = f(z)$$

Z.T. first

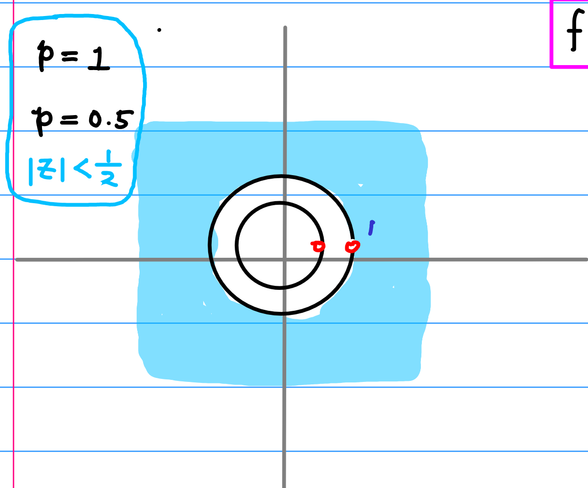
Ⅱ -1

$$X(z) = \frac{-1}{(z-1)(z-2)} \qquad f(z) =$$

Ⅱ D_2 $\quad |z| < 1 \quad \left[\left|\frac{z}{1}\right| < 1, \quad \left|\frac{z}{2}\right| < 1 \right]$



$$X(z) = \frac{1}{z-1} - \frac{1}{z-2}$$
$$= \frac{-1}{1-\left(\frac{z}{1}\right)} + \frac{\frac{1}{2}}{1-\left(\frac{z}{2}\right)}$$
$$= \sum_{n=0}^{\infty} (-1)\left(\frac{z}{1}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{z}{2}\right)^n$$
$$= -\sum_{n=0}^{\infty} 1 \cdot z^n + \sum_{n=0}^{\infty} 2^{-n-1} z^n$$
$$= \sum_{n=0}^{-\infty} \left[-1 + 2^{n-1}\right] z^{-n}$$



$$\boxed{f(z) = X(z^{-1})} = \sum_{n=0}^{-\infty} \left[-1 + 2^{n-1}\right] z^n$$
$$= \sum_{n=0}^{\infty} -1 \cdot z^{-n} + \sum_{n=0}^{\infty} 2^{-n-1} \cdot z^{-n}$$
$$= -\frac{1}{1-\left(\frac{1}{z}\right)} + \frac{\frac{1}{2}}{1-\left(\frac{1}{2z}\right)}$$
$$= -\frac{z}{z-1} + \frac{0.5z}{z-0.5}$$
$$= \frac{-z+0.5+0.5z-0.5}{(z-1)(z-0.5)} z$$
$$= \frac{-0.5z^2}{(z-1)(z-0.5)}$$

$$X(z) = \frac{-1}{(z-1)(z-2)}$$

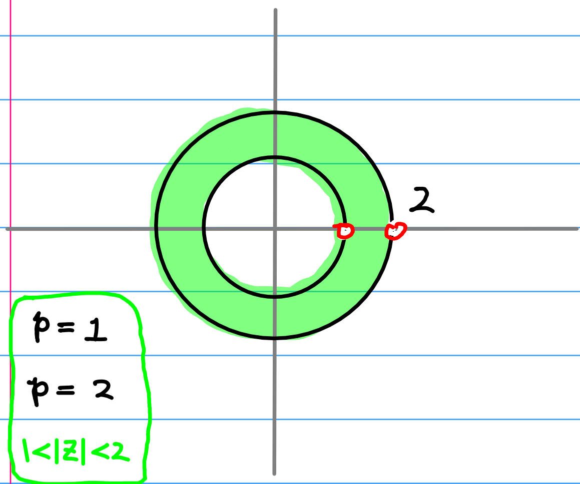
$$X(z^{-1}) = \frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \frac{-z^2}{(1-z)(1-2z)} = \frac{-0.5z^2}{(z-1)(z-0.5)} = f(z)$$

$$X(z) = \frac{-1}{(z-1)(z-2)} \qquad f(z) =$$

Z.T. first

Ⅲ -1

Ⅲ D_3 $\quad 1 < |z| < 2 \quad \left[\left|\frac{1}{z}\right| < 1 , \quad \left|\frac{z}{2}\right| < 1 \right]$



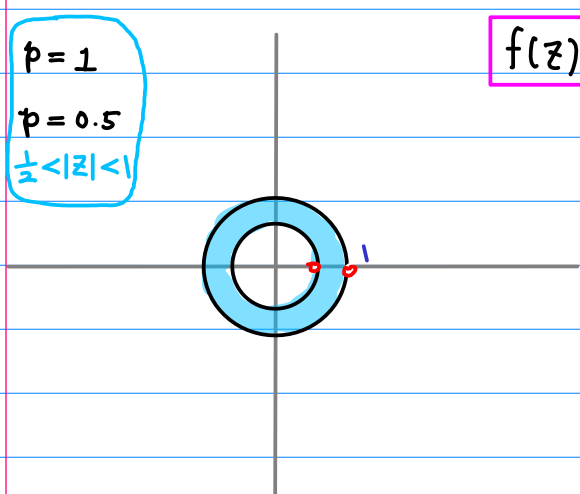
$$X(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} + \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)\left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{z}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} z^{-n-1} + \sum_{n=0}^{\infty} 2^{-n-1} z^n$$

$$= \sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^{-n}$$



$$\boxed{f(z) = X(z^{-1})} = \sum_{n=1}^{\infty} z^n + \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^n$$

$$= \sum_{n=1}^{\infty} z^n + \sum_{n=0}^{\infty} 2^{-n-1} \cdot z^{-n}$$

$$= \frac{\left(\frac{z}{1}\right)}{1-\left(\frac{z}{1}\right)} + \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2z}\right)}$$

$$= \frac{-z}{z-1} + \frac{0.5z}{z-0.5}$$

$$= \frac{-z+0.5+0.5z-0.5}{(z-1)(z-0.5)} z$$

$$= \frac{-0.5z^2}{(z-1)(z-0.5)}$$

$$X(z) = \frac{-1}{(z-1)(z-2)}$$

$$X(z^{-1}) = \frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \frac{-z^2}{(1-z)(1-2z)} = \frac{-0.5z^2}{(z-1)(z-0.5)} = f(z)$$