Laurent Series and z-Transform - Geometric Series Time Shift A

20180908 Sat

Causal Signal $a(n) \Rightarrow f(z), \quad X(z)$

$(n \geq 0) \qquad (|z| < p) \quad (|z| > p^{-1})$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^1 + \left(\frac{1}{2}\right)^2 z^2 + \cdots = \frac{1}{1 - \left(\frac{z}{2}\right)}$$

$$X(z) = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \cdots = \frac{1}{1 - \left(\frac{1}{2z}\right)}$$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1 - \left(\frac{z}{2}\right)} \Rightarrow \frac{2}{2 - z} \qquad (|z| < 2)$$

$$X(z) = \frac{1}{1 - \left(\frac{1}{2z}\right)} \Rightarrow \frac{z}{z - 0.5} \qquad (|z| > 0.5)$$

Anti-Causal Signal $\quad a(n) \Rightarrow f(z), \quad X(z)$

$(n<0) \qquad (|z|>p) \quad (|z|<p^{-1})$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n<0)$$

$$f(z) = \left(\frac{1}{2}\right)^{-1} z^{-1} + \left(\frac{1}{2}\right)^{-2} z^{-2} + \left(\frac{1}{2}\right)^{-3} z^{-3} + \cdots = \frac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)}$$

$$X(z) = \left(\frac{1}{2}\right)^{-1} z^{1} + \left(\frac{1}{2}\right)^{-2} z^{2} + \left(\frac{1}{2}\right)^{-3} z^{3} + \cdots = \frac{(2z)}{1-(2z)}$$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n<0)$$

$$f(z) = \frac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} \rightarrow \frac{2}{z-2} \quad (|z|>2)$$

$$X(z) = \frac{(2z)}{1-(2z)} \rightarrow \frac{z}{0.5-z} \quad (|z|<0.5)$$

$$a_n = -\left(\frac{1}{2}\right)^n \quad (n<0)$$

$$f(z) = \frac{-\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} \rightarrow \frac{2}{2-z} \quad (|z|>2)$$

$$X(z) = \frac{-(2z)}{1-(2z)} \rightarrow \frac{z}{z-0.5} \quad (|z|<0.5)$$

Inverse z

$z \leftarrow z^{-1}$, $ROC(z) \leftarrow ROC(z^{-1})$

Causal $\quad z^{-1} \quad$ anti-Causal

$$f(z) = \frac{2}{2-z} \quad (|z| < 2) \qquad f(z^{-1}) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5) \qquad X(z^{-1}) = \frac{2}{2-z} \quad (|z| < 2)$$

$$f(z^{-1}) = \frac{2}{2-z^{-1}} \quad (|z^{-1}| < 2) \;\Rightarrow\; f(z^{-1}) = X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

$$X(z^{-1}) = \frac{z^{-1}}{z^{-1}-0.5} \quad (|z^{-1}| > 0.5) \;\Rightarrow\; X(z^{-1}) = f(z) = \frac{2}{2-z} \quad (|z| < 2)$$

$f(z^{-1}) = X(z)$ Laurent Series (anti-causal signal)
with the same formula as causal X(z)

$X(z^{-1}) = f(z)$ z-Transform (anti-causal signal)
with the same formula as causal f(z)

Causal $\qquad$ anti-Causal

$$f(z) = \frac{2}{2-z} \quad (|z| < 2) \qquad X(z^{-1}) = \frac{2}{2-z} \quad (|z| < 2)$$

$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5) \qquad f(z^{-1}) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

Inverse z

$f(z^{-1})\,,\ ROC(z^{-1}) \Longrightarrow a_{-n}$

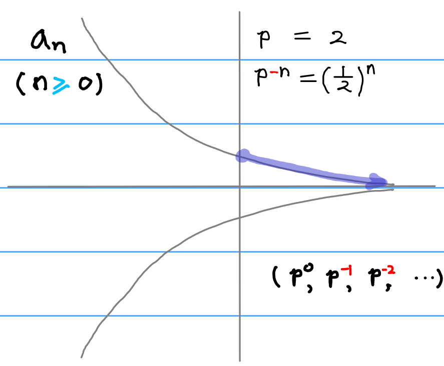
Causal	anti-Causal
$f(z) = \dfrac{2}{2-z} \quad (\lvert z\rvert < 2)$	$f(z^{-1}) = \dfrac{z}{z-0.5} \quad (\lvert z\rvert > 0.5)$
$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z\rvert > 0.5)$	$X(z^{-1}) = \dfrac{2}{2-z} \quad (\lvert z\rvert < 2)$

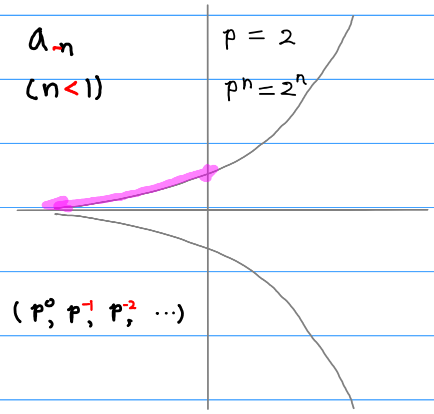
$$f(z) = \frac{2}{2-z} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \qquad n = 0, 1, 2, \cdots$$

$$f(z^{-1}) = \frac{2}{2-z^{-1}} = \frac{1}{1-\left(\frac{1}{2z}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \cdots \qquad a_{-n} = \left(\tfrac{1}{2}\right)^{-n}$$

$$= p^0 z^0 + p^{1} z^{-1} + p^{2} z^{-2} + \cdots \qquad n = 0, -1, -2, \cdots$$





Inverse z $X(z^{-1})$, $ROC(z^{-1}) \Longrightarrow a_{-n}$

Causal	anti-Causal
$f(z) = \dfrac{2}{2 - z}$ $(\lvert z\rvert < 2)$	$f(z^{-1}) = \dfrac{z}{z - 0.5}$ $(\lvert z\rvert > 0.5)$
$X(z) = \dfrac{z}{z - 0.5}$ $(\lvert z\rvert > 0.5)$	$X(z^{-1}) = \dfrac{2}{2 - z}$ $(\lvert z\rvert < 2)$

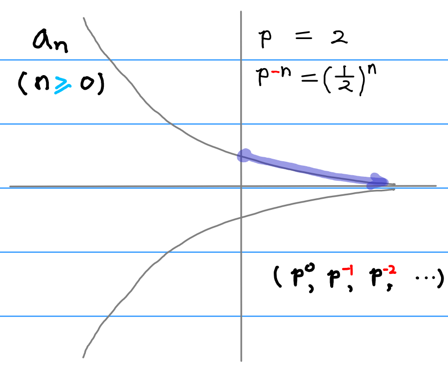
$$X(z) = \frac{z}{z - 0.5} = \frac{1}{1 - \left(\frac{1}{2z}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

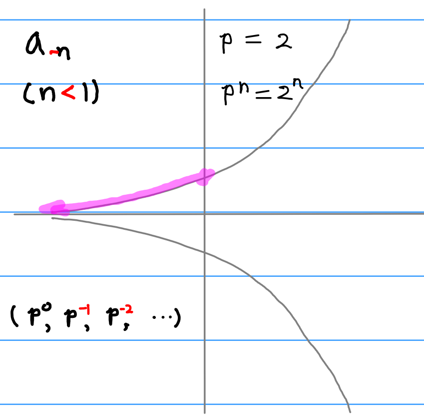
$$= p^0 z^0 + p^{-1} z^{-1} + p^{-2} z^{-2} + \cdots \qquad n = 0, 1, 2, \cdots$$

$$X(z^{-1}) = \frac{z^{-1}}{z^{-1} - 0.5} = \frac{1}{1 - \left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \qquad a_{-n} = \left(\tfrac{1}{2}\right)^{-n}$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \qquad n = 0, -1, -2, \cdots$$

z





Inverse ROC

$ROC(z) \leftarrow ROC(z^{-1})$

Causal | anti-Causal

$$f(z) = \frac{2}{2-z} \quad (|z| < 2) \quad \Rightarrow \quad -f(z) = -\frac{2}{2-z} \quad (|z| > 2)$$

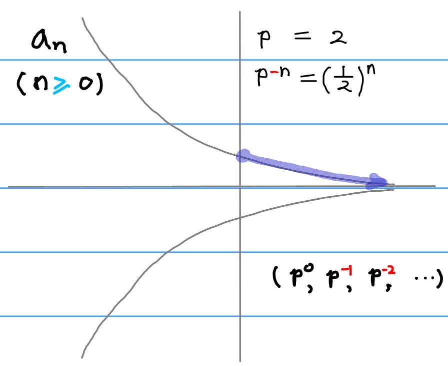
$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5) \quad \Rightarrow \quad -X(z) = -\frac{z}{z-0.5} \quad (|z| < 0.5)$$

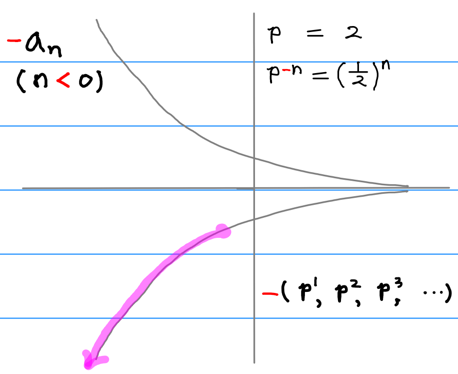
$$f(z) = \frac{2}{2-z} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \qquad n = 0, 1, 2, \cdots$$

$$-f(z) = \frac{2}{z-2} = \frac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} = \left(\tfrac{1}{2}\right)^{-1} z^{-1} + \left(\tfrac{1}{2}\right)^{-2} z^{-2} + \left(\tfrac{1}{2}\right)^{-3} z^{-3} + \cdots \qquad a_n = -\left(\tfrac{1}{2}\right)^n$$

$$= p^1 z^{-1} + p^2 z^{-2} + p^3 z^{-3} + \cdots \qquad n = -1, -2, -3, \cdots$$





Inverse ROC $f(z)$, ROC(z^{-1}) $\Rightarrow -a_n$

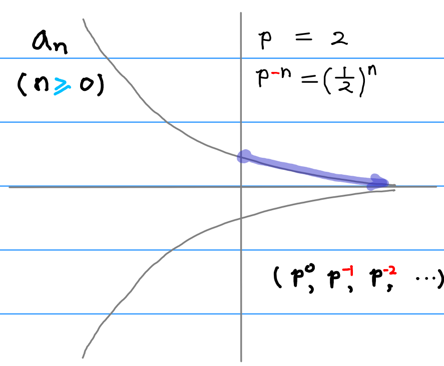
Causal	anti-Causal
$f(z) = \dfrac{2}{2-z} \quad (\lvert z\rvert < 2)$	$f(z) = \dfrac{2}{2-z} \quad (\lvert z\rvert > 0.5)$
$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z\rvert > 0.5)$	$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z\rvert < 2)$

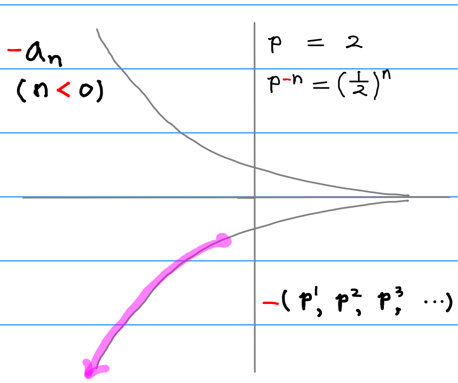
$$f(z) = \frac{2}{2-z} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \qquad n = 0, 1, 2, \cdots$$

$$f(z) = \frac{2}{2-z} = \frac{-\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} = -\left[(2)^1 z^{-1} + (2)^2 z^{-2} + \left(\tfrac{1}{2}\right)^3 z^{-3} + \cdots\right] \qquad -a_n = -\left(\tfrac{1}{2}\right)^n$$

$$= -\left[p^1 z^{-1} + p^2 z^{-2} + p^3 z^{-3} + \cdots\right] \qquad n = -1, -2, -3, \cdots$$





Inverse ROC $\quad X(z)\,,\ ROC(z^{-1}) \Rightarrow -a_n$

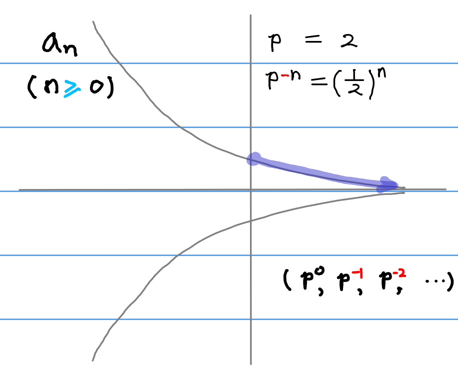
Causal	anti-Causal
$f(z) = \dfrac{2}{2-z} \quad (\lvert z\rvert < 2)$	$f(z) = \dfrac{2}{2-z} \quad (\lvert z\rvert > 0.5)$
$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z\rvert > 0.5)$	$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z\rvert < 2)$

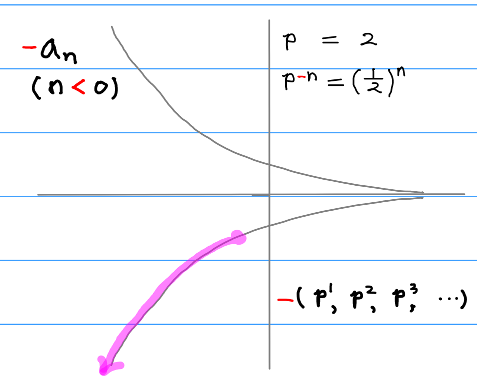
$$X(z) = \frac{z}{z-0.5} = \frac{1}{1-\left(\frac{1}{2z}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^{-1} + p^{-2} z^{-2} + \cdots \qquad n = 0, 1, 2, \cdots$$

$$X(z) = \frac{z}{z-0.5} = \frac{-(2z)}{1-(2z)} = -\left[(2)^1 z^1 + (2)^2 z^2 + \left(\tfrac{1}{2}\right)^3 z^3 + \cdots\right] \qquad -a_n = -\left(\tfrac{1}{2}\right)^n$$

$$= -\left[p^1 z^1 + p^2 z^2 + p^3 z^3 + \cdots\right] \qquad n = -1, -2, -3, \cdots$$





p=2

	causal	anti-causal
	$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$	$a_n = \left(\frac{1}{2}\right)^n \quad (n < 0)$
	$f(z) = \dfrac{1}{1-\left(\frac{z}{2}\right)} \quad (\lvert z \rvert < 2)$ $X(z) = \dfrac{1}{1-\left(\frac{1}{2z}\right)} \quad (\lvert z \rvert > 0.5)$	$f(z) = \dfrac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} \quad (\lvert z \rvert > 2)$ $X(z) = \dfrac{(2z)}{1-(2z)} \quad (\lvert z \rvert < 0.5)$
	$f(z) = \dfrac{2}{2-z} \quad (\lvert z \rvert < 2)$ $X(z) = \dfrac{z}{z-0.5} \quad (\lvert z \rvert > 0.5)$	$f(z) = \dfrac{2}{z-2} \quad (\lvert z \rvert > 2)$ $X(z) = \dfrac{z}{0.5-z} \quad (\lvert z \rvert < 0.5)$

p=1/2

	causal	anti-causal
	$a_n = (2)^n \quad (n \geq 0)$	$a_n = (2)^n \quad (n < 0)$
	$f(z) = \dfrac{1}{1-(2z)} \quad (\lvert z \rvert < 0.5)$ $X(z) = \dfrac{1}{1-\left(\frac{2}{z}\right)} \quad (\lvert z \rvert > 2)$	$f(z) = \dfrac{\left(\frac{1}{2z}\right)}{1-\left(\frac{1}{2z}\right)} \quad (\lvert z \rvert > 0.5)$ $X(z) = \dfrac{\left(\frac{z}{2}\right)}{1-\left(\frac{z}{2}\right)} \quad (\lvert z \rvert < 2)$
	$f(z) = \dfrac{0.5}{0.5-z} \quad (\lvert z \rvert < 0.5)$ $X(z) = \dfrac{z}{z-2} \quad (\lvert z \rvert > 2)$	$f(z) = \dfrac{0.5}{z-0.5} \quad (\lvert z \rvert > 0.5)$ $X(z) = \dfrac{z}{2-z} \quad (\lvert z \rvert < 2)$

$f(z)$, $g(z^{-1})$	$f(z^{-1})$, $g(z)$	$X(z^{-1})$, $Y(z)$	$X(z)$, $Y(z^{-1})$

Laurent Series $a_n \leftrightarrow f(z)$ $b_n \leftrightarrow g(z)$

$n \geq 0$	$\lvert z\rvert < p$		$\lvert z\rvert > p^{-1}$	$n < 1$
causal a_n	$f(z)$	$\xleftrightarrow{z^{-1}}$	$f(z^{-1})$	anti-causal $-b_n$
	$\cdot \frac{p}{p} \frac{z^{-1}}{z^{-1}}$		$\cdot \frac{p}{p} \frac{z}{z}$	
$-a_n$ anti-causal	$g(z^{-1})$	$\xleftrightarrow{z^{-1}}$	$g(z)$	b_n causal
$n < 0$	$\lvert z\rvert > p$		$\lvert z\rvert < p^{-1}$	$n \geq 1$

Z- Transform $X(z) \leftrightarrow x_n$ $Y(z) \leftrightarrow y_n$

$n < 1$	$\lvert z\rvert < p$		$\lvert z\rvert > p^{-1}$	$n \geq 0$
anti- causal $-y_n$	$X(z^{-1})$	$\xleftrightarrow{z^{-1}}$	$X(z)$	causal x_n
	$\cdot \frac{p}{p} \frac{z^{-1}}{z^{-1}}$		$\cdot \frac{p}{p} \frac{z}{z}$	
causal y_n	$Y(z)$	$\xleftrightarrow{z^{-1}}$	$Y(z^{-1})$	anti- causal $-x_n$
$n \geq 1$	$\lvert z\rvert > p$		$\lvert z\rvert < p^{-1}$	$n < 0$

$$f(z),\quad X(z) \;=\; \frac{2}{2-z}\,,\quad \frac{z}{z-0.5}$$

f(z) Laurent Series

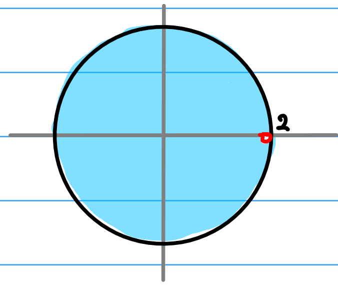
$n \in [0,\infty)$	$\lvert z\rvert < 2$		$\lvert z\rvert > 0.5$	$n \in (-\infty, 0]$
causal $a_n = 0.5^n$	$\frac{2}{2-z}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{z-0.5}$	anti-causal $a_{-n} = 0.5^{-n}$
	$\cdot\,\frac{p}{p}\,\frac{z^{-1}}{z^{-1}}$		$\cdot\,\frac{p}{p}\,\frac{z}{z}$	
$-a_n = -0.5^n$ anti-causal	$\frac{2}{2-z}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{z-0.5}$	$-a_{-n} = -0.5^{-n}$ causal
$n \in (-\infty, -1]$	$\lvert z\rvert > 2$		$\lvert z\rvert < 0.5$	$n \in [1,\infty)$

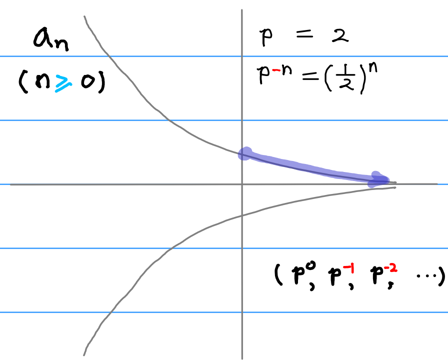
X(z) z-Transform

$n \in (-\infty, 0]$	$\lvert z\rvert < p$		$\lvert z\rvert > p^{-1}$	$n \in [0,\infty)$
anti-causal X(z) $a_{-n} = 0.5^{-n}$	$\frac{2}{2-z}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{z-0.5}$	causal X(z) $a_n = 0.5^n$
	$\cdot\,\frac{p}{p}\,\frac{z^{-1}}{z^{-1}}$		$\cdot\,\frac{p}{p}\,\frac{z}{z}$	
$-a_{-n} = -0.5^{-n}$ causal X(z)	$\frac{2}{2-z}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{z-0.5}$	$-a_n = -0.5^n$ anti-causal X(z)
$n \in [1,\infty)$	$\lvert z\rvert > p$		$\lvert z\rvert < p^{-1}$	$n \in (-\infty, -1]$

$n \in [0, \infty)$ $|z| < 2$

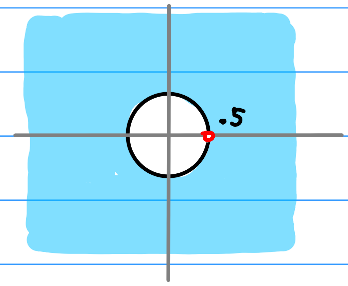
causal $a_n = 0.5^n$ | $\boxed{\dfrac{2}{2-z}}$

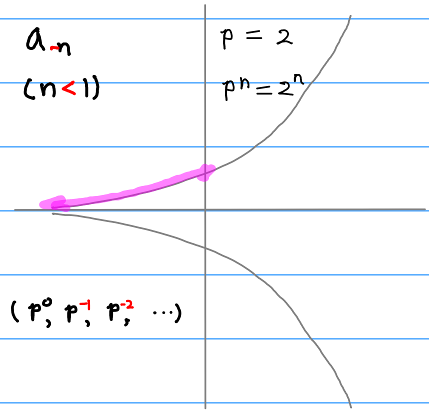




$|z| > 0.5$ $n \in (-\infty, 0]$

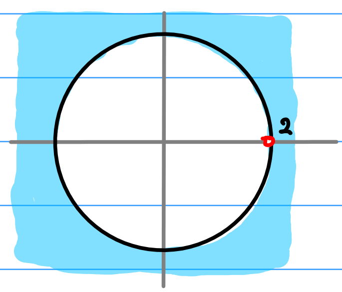
$\boxed{\dfrac{z}{z-0.5}}$ | anti-causal $a_{-n} = 0.5^{-n}$

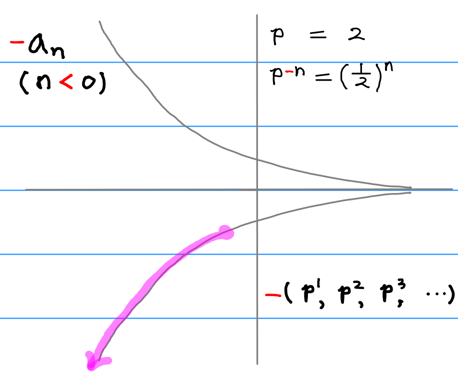




$-a_n = -0.5^n$ anti-causal | $\boxed{\dfrac{2}{2-z}}$

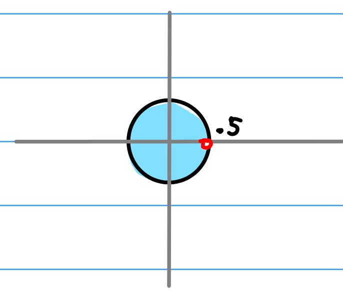
$n \in (-\infty, -1]$ $|z| > 2$

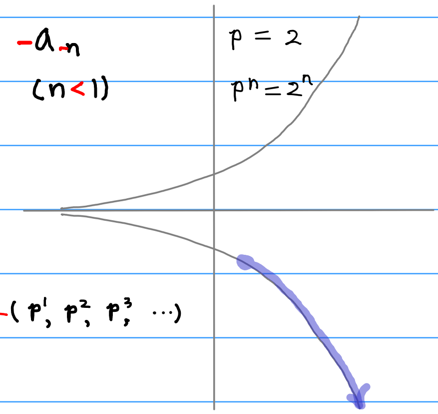




$\boxed{\dfrac{z}{z-0.5}}$ | $-a_{-n} = -0.5^{-n}$ causal

$|z| < 0.5$ $n \in [1, \infty)$





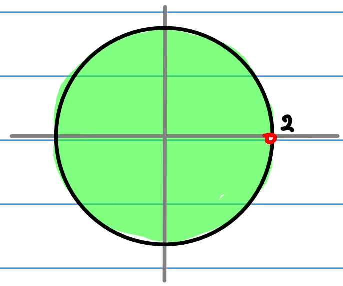
$n < 1$ $\quad |z| < 2$

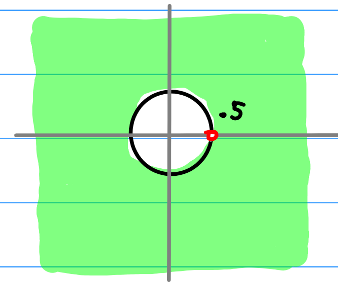
x_{-n}: $\boxed{\text{anti-causal} \quad x_{-n} = 0.5^{-n}}$ $\boxed{-\dfrac{0.5}{1 - 0.5z}}$

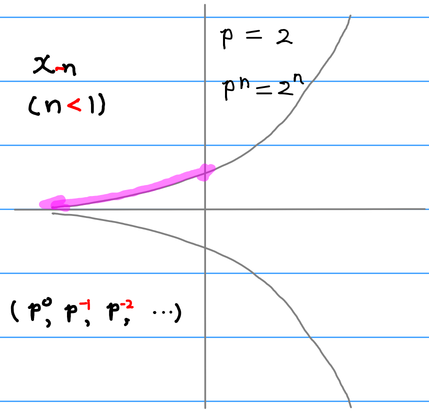
$\xleftrightarrow{z^{-1}}$

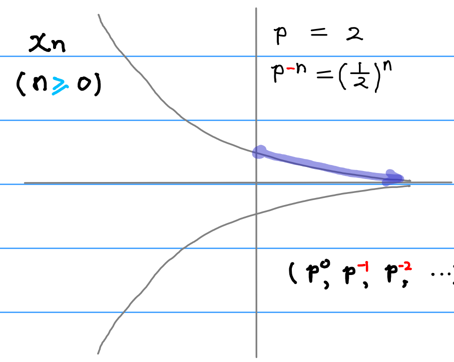
$|z| > 0.5$ $\quad n \geq 0$

$\boxed{-\dfrac{0.5}{1 - 0.5z^{-1}}}$ $\boxed{\text{causal} \quad x_n = 0.5^n}$ x_n









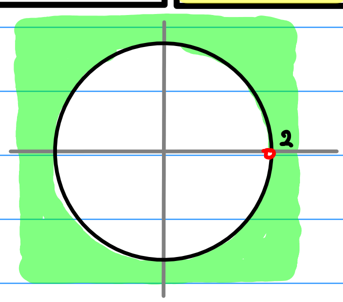
$n \geq 1$ $\quad |z| > 2$

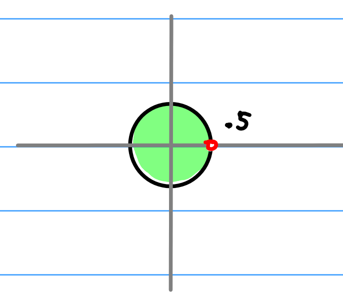
y_n: $\boxed{-x_{-n} = -0.5^{-n} \quad \text{causal}}$ $\boxed{\dfrac{z^{-1}}{1 - 2z^{-1}}}$

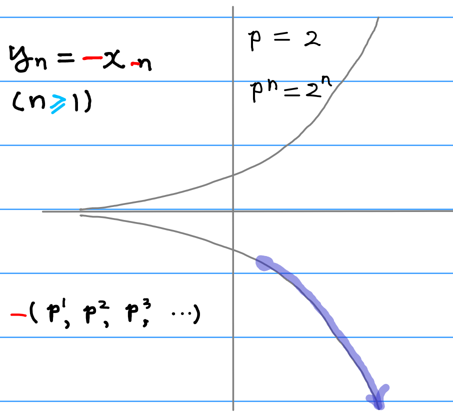
$\xleftrightarrow{z^{-1}}$

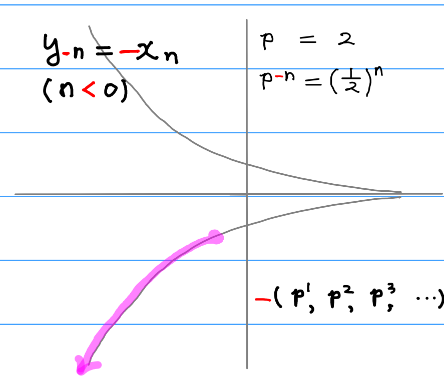
$|z| < 0.5$ $\quad n < 0$

$\boxed{\dfrac{z}{1 - 2z}}$ $\boxed{-x_n = -0.5^n \quad \text{anti-causal}}$ y_{-n}









2 formulas (A)

$$\boxed{\frac{1}{z-p}} \qquad \boxed{\frac{1}{z^{-1}-p}}$$

2 representations each

$$\boxed{\frac{1}{z-p}}$$

- $\boxed{-\frac{p^{-1}}{1-p^{-1}z}}$
- $\boxed{\frac{z^{-1}}{1-pz^{-1}}}$

$$\boxed{\frac{1}{z^{-1}-p}}$$

- $\boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}}$
- $\boxed{\frac{z}{1-pz}}$

2 formulas (B)

$$\boxed{\frac{p}{z-p}} \qquad \boxed{\frac{p}{z^{-1}-p}}$$

2 representations each

$$\boxed{\frac{p}{z-p}}$$

- $\boxed{-\frac{1}{1-p^{-1}z}}$
- $\boxed{\frac{pz^{-1}}{1-pz^{-1}}}$

$$\boxed{\frac{p}{z^{-1}-p}}$$

- $\boxed{-\frac{1}{1-p^{-1}z^{-1}}}$
- $\boxed{\frac{pz}{1-pz}}$

$$\xrightarrow{p}$$

$$\xleftarrow[1/p]{}$$

Ⓐ $f(z) \longrightarrow a_n$

2 formulas Ⓐ

$$\boxed{\frac{1}{z-p}} \qquad \boxed{\frac{1}{z^{-1}-p}}$$

2 representations each

$$\boxed{\frac{1}{z-p}} \to \begin{cases} -\dfrac{p^{-1}}{1-p^{-1}z} \triangleq f(z) & -(p^{-1})^{n+1} \quad (n \geq 0) \\ \dfrac{z^{-1}}{1-pz^{-1}} \triangleq g(z^{-1}) & (p^{-1})^{n+1} \quad (n < 0) \end{cases}$$

causal

$f(z) \parallel g(z^{-1})$

anti- causal

$$\boxed{\frac{1}{z^{-1}-p}} \to \begin{cases} -\dfrac{p^{-1}}{1-p^{-1}z^{-1}} \triangleq f(z^{-1}) & -(p^{-1})^{-n+1} \quad (n < 1) \\ \dfrac{z}{1-pz} \triangleq g(z) & (p^{-1})^{-n+1} \quad (n \geq 1) \end{cases}$$

anti- causal

$f(z^{-1}) \parallel g(z)$

causal

$$\boxed{-\frac{1}{2-z}} \to \begin{cases} \boxed{\dfrac{-0.5}{1-0.5z}} & \text{causal} \quad -0.5^{n+1} \; (n \geq 0) \\ \boxed{\dfrac{z^{-1}}{1-2z^{-1}}} & \text{anti} \quad 0.5^{n+1} \; (n<1) \end{cases}$$

$$\boxed{-\frac{0.5z}{z-0.5}} \to \begin{cases} \boxed{\dfrac{-0.5}{1-0.5z^{-1}}} & \text{anti} \quad -0.5^{-n+1} \; (n<0) \\ \boxed{\dfrac{z}{1-2z}} & \text{causal} \quad 0.5^{-n+1} \; (n \geq 1) \end{cases}$$

Ⓐ $f(z) \longrightarrow a_n$ method 1

p^{-1} $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z}}$ $\quad -(p^{-1}) \Rightarrow -(p^{-1})^{n+1} \Rightarrow -(p^{-1})^{n+1}$

$n=0,1,2,\cdots$

$-(p^{-1}z^{0}+p^{-2}z^{1}+p^{-3}z^{2}+\cdots)$

p $\boxed{\dfrac{z^{-1}}{1-pz^{-1}}}$ $\quad (p) \Rightarrow (p)^{-n-1} \Rightarrow (p^{-1})^{n+1}$

$n=-1,-2,-3,\cdots$

$(p^{0}z^{-1}+p^{1}z^{-2}+p^{2}z^{-3}+\cdots)$

p^{-1} $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}$ $\quad -(p^{-1}) \Rightarrow -(p^{-1})^{-n+1} \Rightarrow -(p^{-1})^{-n+1}$

$n=0,-1,-2,\cdots$

$-(p^{-1}z^{0}+p^{-2}z^{-1}+p^{-3}z^{-2}+\cdots)$

p $\boxed{\dfrac{z}{1-pz}}$ $\quad (p) \Rightarrow (p)^{n-1} \Rightarrow (p^{-1})^{-n+1}$

$n=1,2,3,\cdots$

$(p^{0}z^{1}+p^{1}z^{2}+p^{2}z^{3}+\cdots)$

Ⓐ $f(z) \longrightarrow a_n$ method 2

p^{-1} $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z}}$ $\dfrac{-(p^{-1})}{z} \rightarrow n=0,1,2,\cdots \rightarrow \dfrac{-(p^{-1})^{n+1}}{(p)^{-n-1}}$

p $\boxed{\dfrac{z^{-1}}{1-pz^{-1}}}$ $\dfrac{(p)}{z^{-1}/z^{-1}} \rightarrow n=-1,-2,-3,\cdots \rightarrow \dfrac{(p^{-1})^{n+1}}{(p)^{-n-1}}$

p $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}$ $\dfrac{-(p^{-1})}{z^{-1}} \rightarrow n=0,-1,-2,\cdots \rightarrow \dfrac{-(p^{-1})^{-n+1}}{(p)^{n-1}}$

p^{-1} $\boxed{\dfrac{z}{1-pz}}$ $\dfrac{(p)}{z/z} \rightarrow n=1,2,3,\cdots \rightarrow \dfrac{(p^{-1})^{-n+1}}{(p)^{n-1}}$

B $f(z) \longrightarrow a_n$

2 formulas

$$\boxed{\frac{p}{z-p}} \qquad \boxed{\frac{p}{z^{-1}-p}}$$

2 representations each

$$\boxed{\frac{p}{z-p}}$$

causal

$$\boxed{-\frac{1}{1-p^{-1}z}} \triangleq f(z) \qquad -(p^{-1})^n \qquad (n \geq 0)$$

$$\|$$

$$\boxed{\frac{pz^{-1}}{1-pz^{-1}}} \triangleq g'(z^{-1}) \qquad (p^{-1})^n \qquad (n<0)$$

anti- causal

$$\boxed{\frac{p}{z^{-1}-p}}$$

anti- causal

$$\boxed{-\frac{1}{1-p^{-1}z^{-1}}} \triangleq f'(z^{-1}) \qquad -(p^{-1})^{-n} \qquad (n<1)$$

$$\|$$

$$\boxed{\frac{pz}{1-pz}} \triangleq g'(z) \qquad (p^{-1})^{-n} \qquad (n \geq 1)$$

causal

$$\boxed{\frac{2}{2-z}}$$

$$\boxed{\frac{1}{1-0.5z}}$$ causal 0.5^n $(n \geq 0)$

$$\boxed{\frac{-2z^{-1}}{1-2z^{-1}}}$$ anti -0.5^n $(n<0)$

$$\boxed{\frac{z}{z-0.5}}$$

$$\boxed{\frac{1}{1-0.5z^{-1}}}$$ anti 0.5^{-n} $(n<1)$

$$\boxed{\frac{-2z}{1-2z}}$$ causal -0.5^{-n} $(n \geq 1)$

(B) $f(z) \longrightarrow a_n$ method 1

p^{-1} $\boxed{-\dfrac{1}{1-p^{-1}z}}$ $-(p^{-1}) \Rightarrow -(p^{-1})^n \Rightarrow -(p^{-1})^n$

$n = 0, 1, 2, \cdots$

$-(p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots)$

p $\boxed{\dfrac{pz^{-1}}{1-pz^{-1}}}$ $(p) \Rightarrow (p)^{-n} \Rightarrow (p^{-1})^n$

$n = -1, -2, -3, \cdots$

$(p^1 z^{-1} + p^2 z^{-2} + p^3 z^{-3} + \cdots)$

p $\boxed{-\dfrac{1}{1-p^{-1}z^{-1}}}$ $-(p^{-1}) \Rightarrow -(p^{-1})^{-n} \Rightarrow -(p^{-1})^{-n}$

$n = 0, -1, -2, \cdots$

$-(p^0 z^0 + p^{-1} z^{-1} + p^{-2} z^{-2} + \cdots)$

p^{-1} $\boxed{\dfrac{pz}{1-pz}}$ $(p) \Rightarrow (p)^n \Rightarrow (p^{-1})^{-n}$

$n = 1, 2, 3, \cdots$

$(p^1 z^1 + p^2 z^2 + p^3 z^3 + \cdots)$

(B) $f(z) \longrightarrow a_n$ method 2

p^{-1} $\boxed{-\dfrac{1}{1-p^{-1}z}}$ $\dfrac{-(p^{-1})}{z}$ → $n=0,1,2,\cdots$ → $\dfrac{-p^{-n}}{(p)^{-n}}$

p $\boxed{\dfrac{pz^{-1}}{1-pz^{-1}}}$ $\dfrac{(p)}{z^{-1}/z^{-1}}$ → $n=-1,-2,-3,\cdots$ → $\dfrac{p^{-n}}{(p)^{-n}}$

p $\boxed{-\dfrac{1}{1-p^{-1}z^{-1}}}$ $\dfrac{-(p^{-1})}{z^{-1}}$ → $n=0,-1,-2,\cdots$ → $\dfrac{-(p^{-1})^{-n}}{(p)^{n}}$

p^{-1} $\boxed{\dfrac{pz}{1-pz}}$ $\dfrac{(p)}{z/z}$ → $n=1,2,3,\cdots$ → $\dfrac{(p^{-1})^{-n}}{(p)^{n}}$

2 formulas

$$\boxed{\frac{p}{z - p}} \qquad \boxed{\frac{p}{z^{-1} - p}}$$

2 representations each

$$\boxed{\frac{p}{z - p}}$$

- $$\boxed{-\frac{1}{1 - p^{-1} z}} \triangleq \underset{\text{causal}}{f'(z)} = \underset{\text{anti- causal}}{X'(z^{-1})}$$
- $$\boxed{\frac{p z^{-1}}{1 - p z^{-1}}} \triangleq \underset{\text{causal}}{Y'(z)} = \underset{\text{anti- causal}}{g'(z^{-1})}$$

$$f'(z) = Y'(z), \qquad X'(z^{-1}) = g'(z^{-1})$$

$$\boxed{\frac{p}{z^{-1} - p}}$$

- $$\boxed{-\frac{1}{1 - p^{-1} z^{-1}}} \triangleq \underset{\text{causal}}{X'(z)} = \underset{\text{anti- causal}}{f'(z^{-1})}$$
- $$\boxed{\frac{p z}{1 - p z}} \triangleq \underset{\text{causal}}{g'(z)} = \underset{\text{anti- causal}}{Y'(z^{-1})}$$

$$X'(z) = g'(z), \qquad f'(z^{-1}) = Y'(z^{-1})$$

$$
\begin{tikzcd}[column sep=small]
\boxed{\begin{array}{c} n \in [0,\infty) \\ \text{causal} \\ a_n = 0.5^n \end{array}} & \boxed{\begin{array}{c} |z| < 2 \\ \dfrac{2}{2-z} \end{array}} \arrow[d, equal, "\cdot\frac{p}{p}\,\frac{z^{-1}}{z^{-1}}"] \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{c} |z| > 0.5 \\ \dfrac{z}{z-0.5} \end{array}} \arrow[d, equal, "\cdot\frac{p}{p}\,\frac{z}{z}"'] & \boxed{\begin{array}{c} n \in (-\infty,0] \\ \text{anti-causal} \\ a_{-n} = 0.5^{-n} \end{array}} \\
\boxed{\begin{array}{c} -a_n = -0.5^n \\ \text{anti-causal} \\ n \in (-\infty,-1] \end{array}} & \boxed{\begin{array}{c} \dfrac{2}{2-z} \\ |z| > 2 \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{c} \dfrac{z}{z-0.5} \\ |z| < 0.5 \end{array}} & \boxed{\begin{array}{c} -a_{-n} = -0.5^{-n} \\ \text{causal} \\ n \in [1,\infty) \end{array}}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\boxed{\dfrac{2}{2-z}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, "-\frac{1}{2}"] & \boxed{\dfrac{z}{z-0.5}} \arrow[d, "-\frac{1}{2}"] \\
\boxed{-\dfrac{1}{2-z}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{-\dfrac{0.5z}{z-0.5}}
\end{tikzcd}
$$

$$0.5^{-n+1} = \left(\frac{1}{2}\right)^{-n+1}$$

$$= 2^{+n-1}$$

$$\frac{2}{2-z}$$

- $\frac{1}{1-0.5z}$ — causal: $0.5^{n}\ (n \geq 0)$
- $\frac{-2z^{-1}}{1-2z^{-1}}$ — anti: $-0.5^{n}\ (n<0)$

$$\frac{z}{z-0.5}$$

- $\frac{1}{1-0.5z^{-1}}$ — anti: $0.5^{-n}\ (n<1)$
- $\frac{-2z}{1-2z}$ — causal: $-0.5^{-n}\ (n \geq 1)$

$$-\frac{1}{2-z}$$

- $\frac{-0.5}{1-0.5z}$ — causal: $-0.5^{n+1}\ (n \geq 0)$
- $\frac{z^{-1}}{1-2z^{-1}}$ — anti: $0.5^{n+1}\ (n<1)$

$$-\frac{0.5z}{z-0.5}$$

- $\frac{-0.5}{1-0.5z^{-1}}$ — anti: $-0.5^{-n+1}\ (n<0)$
- $\frac{z}{1-2z}$ — causal: $0.5^{-n+1}\ (n \geq 1)$

$\frac{2}{2-z}, \frac{z}{z-0.5}$ vs. $-\frac{0.5}{1-0.5z}, \frac{z}{1-2z}$

$n \in [0,\infty)$ causal $a_n = 0.5^n$

$|z| < 2$ $\quad \frac{2}{2-z}$

$\xleftrightarrow{z^{-1}}$

$|z| > 0.5$ $\quad \frac{z}{z-0.5}$

$n \in (-\infty, 0]$ anti-causal $a_{-n} = 0.5^{-n}$

$\cdot \frac{p}{p} \frac{z^{-1}}{z^{-1}}$ $\qquad \cdot \frac{p}{p} \frac{z}{z}$

$-a_n = -0.5^n$ anti-causal $n \in (-\infty, -1]$

$\frac{2}{2-z}$ $\quad |z| > 2$

$\xleftrightarrow{z^{-1}}$

$\frac{z}{z-0.5}$ $\quad |z| < 0.5$

$-a_{-n} = -0.5^{-n}$ causal $n \in [1, \infty)$

$n \in [0,\infty)$

a_n causal -0.5^{n+1}

$|z| < 2$ $\quad -\frac{0.5}{1-0.5z}$

$\xleftrightarrow{z^{-1}}$

$|z| > 0.5$ $\quad -\frac{0.5}{1-0.5z^{-1}}$

$n \in (-\infty, 0]$ anti-causal -0.5^{-n+1}

-2^{+n-1}

2^{+n-1}

b. 0.5^{n+1} anti-causal $n \in (-\infty, -1]$

$\frac{z^{-1}}{1-2z^{-1}}$ $\quad |z| > 2$

$\xleftrightarrow{z^{-1}}$

$\frac{z}{1-2z}$ $\quad |z| < 0.5$

0.5^{-n+1} causal $n \in [1, \infty)$

$\dfrac{2}{2-z}$

$\dfrac{1}{1-0.5z}$ causal 0.5^{n} $(n \geq 0)$

$\dfrac{-2z^{-1}}{1-2z^{-1}}$ anti -2^{-n} $(n<0)$

$\dfrac{z}{z-0.5}$

$\dfrac{1}{1-0.5z^{-1}}$ anti 0.5^{-n} $(n<1)$

$\dfrac{-2z}{1-2z}$ causal -0.5^{-n} $(n \geq 1)$

$n \in [0, \infty)$ $|z| < 2$

causal $a_n = 0.5^{n}$ | $\boxed{\dfrac{2}{2-z}}$ $\overset{z^{-1}}{\longleftrightarrow}$ $\boxed{\dfrac{z}{z-0.5}}$ | anti-causal $a_{-n} = 0.5^{-n}$

$|z| > 0.5$ $n \in (-\infty, 0]$

$\cdot \dfrac{p}{p} \dfrac{z^{-1}}{z^{-1}}$ $\cdot \dfrac{p}{p} \dfrac{z}{z}$

$-a_n = -0.5^{n}$ anti-causal | $\boxed{\dfrac{2}{2-z}}$ $\overset{z^{-1}}{\longleftrightarrow}$ $\boxed{\dfrac{z}{z-0.5}}$ | $-a_{-n} = -0.5^{-n}$ causal

$n \in (-\infty, -1]$ $|z| > 2$ $|z| < 0.5$ $n \in [1, \infty)$

$-\frac{1}{2-z}$

- $\frac{-0.5}{1-0.5z}$ causal -0.5^{n+1} $(n\geq 0)$
- $\frac{z^{-1}}{1-2z^{-1}}$ anti 0.5^{n+1} $(n<1)$

$-\frac{0.5z}{z-0.5}$

- $\frac{-0.5}{1-0.5z^{-1}}$ anti -0.5^{-n+1} $(n<1)$
- $\frac{z}{1-2z}$ causal 0.5^{-n+1} $(n\geq 1)$

	$n \in [0,\infty)$	$\|z\| < 2$		$\|z\| > 0.5$	$n \in (-\infty, 0]$
a_n	causal -0.5^{n+1}	$-\frac{0.5}{1-0.5z}$	$\xleftrightarrow{z^{-1}}$	$-\frac{0.5}{1-0.5z^{-1}}$	anti-causal -0.5^{-n+1}

b_n	0.5^{n+1} anti-causal	$\frac{z^{-1}}{1-2z^{-1}}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{1-2z}$	0.5^{-n+1} causal
	$n \in (-\infty,-1]$	$\|z\| > 2$		$\|z\| < 0.5$	$n \in [1,\infty)$

	$n \in [0,\infty)$	$\|z\| < 2$		$\|z\| > 0.5$	$n \in (-\infty, 0]$
a_n	causal -0.5^{n+1}	$-\frac{0.5}{1-0.5z}$	$\xleftrightarrow{z^{-1}}$	$-\frac{0.5}{1-0.5z^{-1}}$	anti-causal -2^{+n-1}

b_n	0.5^{n+1} anti-causal	$\frac{z^{-1}}{1-2z^{-1}}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{1-2z}$	2^{+n-1} causal
	$n \in (-\infty,-1]$	$\|z\| < 0.5$		$\|z\| < 0.5$	$n \in [1,\infty)$

Time Shift

$p = 2$

① $(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = \frac{2}{2-z}$ $X(z) = \frac{z}{z-0.5}$

③ $(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = -\frac{2}{2-z}$ $X(z) = -\frac{z}{z-0.5}$

⑤ $(n \geq 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = \frac{2z}{2-z}$ $X(z) = \frac{1}{z-0.5}$

⑦ $(n < 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = -\frac{2z}{2-z}$ $X(z) = -\frac{1}{z-0.5}$

⑨ $(n \geq -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = \frac{2}{(2-z)z}$ $X(z) = \frac{z^2}{z-0.5}$

⑪ $(n < -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = -\frac{2}{(2-z)z}$ $X(z) = -\frac{z^2}{z-0.5}$

Time Shift

$p = \frac{1}{2}$

2 $(n \geq 0)$ $\quad a_n = (2)^n \qquad f(z) = \frac{0.5}{0.5 - z} \qquad X(z) = \frac{z}{z - 2}$

4 $(n < 0)$ $\quad a_n = (2)^n \qquad f(z) = -\frac{0.5}{0.5 - z} \qquad X(z) = -\frac{z}{z - 2}$

6 $(n \geq 1)$ $\quad a_{n-1} = (2)^{n-1} \qquad f(z) = \frac{0.5z}{0.5 - z} \qquad X(z) = \frac{1}{z - 2}$

8 $(n < 1)$ $\quad a_{n-1} = (2)^{n-1} \qquad f(z) = -\frac{0.5z}{0.5 - z} \qquad X(z) = -\frac{1}{z - 2}$

10 $(n \geq -1)$ $\quad a_{n+1} = (2)^{n+1} \qquad f(z) = \frac{0.5}{(0.5 - z)z} \qquad X(z) = \frac{z^2}{z - 2}$

12 $(n < -1)$ $\quad a_{n+1} = (2)^{n+1} \qquad f(z) = -\frac{0.5}{(0.5 - z)z} \qquad X(z) = -\frac{z^2}{z - 2}$

Time Shift

$2 \leftrightarrow \frac{1}{2}$

1. $(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = \frac{2}{2-z}$ $X(z) = \frac{z}{z-0.5}$

2. $(n \geq 0)$ $a_n = (2)^n$ $f(z) = \frac{0.5}{0.5-z}$ $X(z) = \frac{z}{z-2}$

3. $(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = -\frac{2}{2-z}$ $X(z) = -\frac{z}{z-0.5}$

4. $(n < 0)$ $a_n = (2)^n$ $f(z) = -\frac{0.5}{0.5-z}$ $X(z) = -\frac{z}{z-2}$

5. $(n \geq 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = \frac{2z}{2-z}$ $X(z) = \frac{1}{z-0.5}$

6. $(n \geq 1)$ $a_{n-1} = (2)^{n-1}$ $f(z) = \frac{0.5z}{0.5-z}$ $X(z) = \frac{1}{z-2}$

7. $(n < 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = -\frac{2z}{2-z}$ $X(z) = -\frac{1}{z-0.5}$

8. $(n < 1)$ $a_{n-1} = (2)^{n-1}$ $f(z) = -\frac{0.5z}{0.5-z}$ $X(z) = -\frac{1}{z-2}$

9. $(n \geq -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = \frac{2}{(2-z)z}$ $X(z) = \frac{z^2}{z-0.5}$

10. $(n \geq -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = \frac{0.5}{(0.5-z)z}$ $X(z) = \frac{z^2}{z-2}$

11. $(n < -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = -\frac{2}{(2-z)z}$ $X(z) = -\frac{z^2}{z-0.5}$

12. $(n < -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = -\frac{0.5}{(0.5-z)z}$ $X(z) = -\frac{z^2}{z-2}$

Shift to the right

shift to the right →		$\times z$	$\times z^{-1}$
delete a_0			
1	$(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$	$f(z) = \frac{2}{2-z}$	$X(z) = \frac{z}{z-0.5}$
5	$(n \geq 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	$f(z) = \frac{2z}{2-z}$	$X(z) = \frac{1}{z-0.5}$
2	$(n \geq 0)$ $a_n = (2)^n$	$f(z) = \frac{0.5}{0.5-z}$	$X(z) = \frac{z}{z-2}$
6	$(n \geq 1)$ $a_{n-1} = (2)^{n-1}$	$f(z) = \frac{0.5z}{0.5-z}$	$X(z) = \frac{1}{z-2}$

shift to the right →		$\times z$	$\times z^{-1}$
insert a_0			
3	$(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$	$f(z) = -\frac{2}{2-z}$	$X(z) = -\frac{z}{z-0.5}$
7	$(n < 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	$f(z) = -\frac{2z}{2-z}$	$X(z) = -\frac{1}{z-0.5}$
4	$(n < 0)$ $a_n = (2)^n$	$f(z) = -\frac{0.5}{0.5-z}$	$X(z) = -\frac{z}{z-2}$
8	$(n < 1)$ $a_{n-1} = (2)^{n-1}$	$f(z) = -\frac{0.5z}{0.5-z}$	$X(z) = -\frac{1}{z-2}$

Shift to the left

Shift to the left ← $\times z^{-1}$ $\times z$

delete a_0

(1) $(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = \frac{2}{2-z}$ $X(z) = \frac{z}{z-0.5}$

(9) $(n \geq -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = \frac{2}{(2-z)z}$ $X(z) = \frac{z}{z-0.5}$

(2) $(n \geq 0)$ $a_n = (2)^n$ $f(z) = \frac{0.5}{0.5-z}$ $X(z) = \frac{z}{z-2}$

(10) $(n \geq -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = \frac{0.5}{(0.5-z)z}$ $X(z) = \frac{z}{z-2}$

Shift to the left ← $\times z^{-1}$ $\times z$

insert a_0

(3) $(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = -\frac{2}{2-z}$ $X(z) = -\frac{z}{z-0.5}$

(11) $(n < -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = -\frac{2}{(2-z)z}$ $X(z) = -\frac{z}{z-0.5}$

(4) $(n < 0)$ $a_n = (2)^n$ $f(z) = -\frac{0.5}{0.5-z}$ $X(z) = -\frac{z}{z-2}$

(12) $(n < -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = -\frac{0.5}{(0.5-z)z}$ $X(z) = -\frac{z}{z-2}$

$n=-4$	$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n+1} $n=-2,-3,-4,\cdots$ | b^{n+1} $n=-1,0,1,\cdots$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$n=3$	
b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n} $n=-1,-2,-3,\cdots$ | b^{n} $n=0,1,2,\cdots$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$n=3$		
	b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n-1} $n=0,-1,-2,\cdots$ | b^{n-1} $n=1,2,3,\cdots$

$$1 \longleftrightarrow \frac{1}{1}$$

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1 $(n \geq 0)$ $\quad a_n = (1)^n \qquad f(z) = \frac{1}{1-z} \qquad X(z) = \frac{z}{z-1}$

2 $(n \geq 0)$ $\quad a_n = (1^{-1})^n \qquad f(z) = \frac{1}{1-z} \qquad X(z) = \frac{z}{z-1}$

3 $(n < 0)$ $\quad a_n = (1)^n \qquad f(z) = -\frac{1}{1-z} \qquad X(z) = -\frac{z}{z-1}$

4 $(n < 0)$ $\quad a_n = (1^{-1})^n \qquad f(z) = -\frac{1}{1-z} \qquad X(z) = -\frac{z}{z-1}$

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5 $(n \geq 1)$ $\quad a_{n-1} = (1)^{n-1} \qquad f(z) = \frac{z}{1-z} \qquad X(z) = \frac{1}{z-1}$

6 $(n \geq 1)$ $\quad a_{n-1} = (1^{-1})^{n-1} \qquad f(z) = \frac{z}{1-z} \qquad X(z) = \frac{1}{z-1}$

7 $(n < 1)$ $\quad a_{n-1} = (1)^{n-1} \qquad f(z) = -\frac{z}{1-z} \qquad X(z) = -\frac{1}{z-1}$

8 $(n < 1)$ $\quad a_{n-1} = (1^{-1})^{n-1} \qquad f(z) = -\frac{z}{1-z} \qquad X(z) = -\frac{1}{z-1}$

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9 $(n \geq -1)$ $\quad a_{n+1} = (1)^{n+1} \qquad f(z) = \frac{1}{(1-z)z} \qquad X(z) = \frac{z}{z-1}$

10 $(n \geq -1)$ $\quad a_{n+1} = (1^{-1})^{n+1} \qquad f(z) = \frac{1}{(1-z)z} \qquad X(z) = \frac{z}{z-1}$

11 $(n < -1)$ $\quad a_{n+1} = (1)^{n+1} \qquad f(z) = -\frac{1}{(1-z)z} \qquad X(z) = -\frac{z}{z-1}$

12 $(n < -1)$ $\quad a_{n+1} = (1^{-1})^{n+1} \qquad f(z) = -\frac{1}{(1-z)z} \qquad X(z) = -\frac{z}{z-1}$

1. $(n \geq 0) \quad a_n = (1)^n \qquad f(z) = \dfrac{1}{1-z} \qquad X(z) = \dfrac{z}{z-1}$

2. $(n \geq 0) \quad a_n = (1^{-1})^n \qquad f(z) = \dfrac{1}{1-z} \qquad X(z) = \dfrac{z}{z-1}$

shift to the right $\longrightarrow$ $\quad \times z \quad \times z^{-1}$

delete a_0

5. $(n \geq 1) \quad a_{n-1} = (1)^{n-1} \qquad f(z) = \dfrac{z}{1-z} \qquad X(z) = \dfrac{1}{z-1}$

6. $(n \geq 1) \quad a_{n-1} = (1^{-1})^{n-1} \qquad f(z) = \dfrac{z}{1-z} \qquad X(z) = \dfrac{1}{z-1}$

3. $(n < 0) \quad a_n = (1)^n \qquad f(z) = -\dfrac{1}{1-z} \qquad X(z) = -\dfrac{z}{z-1}$

4. $(n < 0) \quad a_n = (1^{-1})^n \qquad f(z) = -\dfrac{1}{1-z} \qquad X(z) = -\dfrac{z}{z-1}$

shift to the right $\longrightarrow$ $\quad \times z \quad \times z^{-1}$

insert a_0

7. $(n < 1) \quad a_{n-1} = (1)^{n-1} \qquad f(z) = -\dfrac{z}{1-z} \qquad X(z) = -\dfrac{1}{z-1}$

8. $(n < 1) \quad a_{n-1} = (1^{-1})^{n-1} \qquad f(z) = -\dfrac{z}{1-z} \qquad X(z) = -\dfrac{1}{z-1}$

1 $(n \geq 0) \quad a_n = (1)^n \qquad f(z) = \frac{1}{1-z} \qquad X(z) = \frac{z}{z-1}$

2 $(n \geq 0) \quad a_n = (1^{-1})^n \qquad f(z) = \frac{1}{1-z} \qquad X(z) = \frac{z}{z-1}$

shift to the left $\leftarrow$ $\qquad \times z^{-1} \qquad \times z$

delete a_0

9 $(n \geq -1) \quad a_{n+1} = (1)^{n+1} \qquad f(z) = \frac{1}{(1-z)z} \qquad X(z) = \frac{z}{z-1}$

10 $(n \geq -1) \quad a_{n+1} = (1^{-1})^{n+1} \qquad f(z) = \frac{1}{(1-z)z} \qquad X(z) = \frac{z}{z-1}$

3 $(n < 0) \quad a_n = (1)^n \qquad f(z) = -\frac{1}{1-z} \qquad X(z) = -\frac{z}{z-1}$

4 $(n < 0) \quad a_n = (1^{-1})^n \qquad f(z) = -\frac{1}{1-z} \qquad X(z) = -\frac{z}{z-1}$

shift to the left $\leftarrow$ $\qquad \times z^{-1} \qquad \times z$

insert a_0

11 $(n < -1) \quad a_{n+1} = (1)^{n+1} \qquad f(z) = -\frac{1}{(1-z)z} \qquad X(z) = -\frac{z}{z-1}$

12 $(n < -1) \quad a_{n+1} = (1^{-1})^{n+1} \qquad f(z) = -\frac{1}{(1-z)z} \qquad X(z) = -\frac{z}{z-1}$

Causality

$f(z)\ (|z|<p) \leftrightarrow a_n\ (n \geq 0)\quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$X(z^{-1})\ (|z|<p) \leftrightarrow x_{-n}\ (n < 1)\quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$f(z^{-1})\ (|z|>p^{-1}) \leftrightarrow a_{-n}\ (n < 1)\quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$X(z)\ (|z|>p^{-1}) \leftrightarrow x_n\ (n \geq 0)\quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$f(z)\ (|z|>p) \leftrightarrow -a_n\ (n < 0)\quad (p^{0}, p^{1}, p^{2}, \cdots)$

$X(z^{-1})\ (|z|>p) \leftrightarrow -x_{-n}\ (n \geq 1)\quad (p^{0}, p^{1}, p^{2}, \cdots)$

$f(z^{-1})\ (|z|<p^{-1}) \leftrightarrow -a_{-n}\ (n \geq 1)\quad (p^{0}, p^{1}, p^{2}, \cdots)$

$X(z)\ (|z|<p^{-1}) \leftrightarrow -x_n\ (n < 0)\quad (p^{0}, p^{1}, p^{2}, \cdots)$

$$\boxed{\begin{array}{cc} \boxed{\begin{array}{c} f(z) \\ g(z^{-1}) \end{array}} & \boxed{\begin{array}{c} f(z^{-1}) \\ g(z) \end{array}} \end{array}} \quad \boxed{\begin{array}{cc} \boxed{\begin{array}{c} X(z^{-1}) \\ Y(z) \end{array}} & \boxed{\begin{array}{c} X(z) \\ Y(z^{-1}) \end{array}} \end{array}} \quad \boxed{\begin{array}{cc} a_n & a_{-n} \\ b_{-n} & b_n \end{array}} \quad \boxed{\begin{array}{cc} x_{-n} & x_n \\ y_n & y_{-n} \end{array}}$$

$$\boxed{\begin{array}{cc} f(z) & f(z^{-1}) \\ f(z) & f(z^{-1}) \end{array}} \quad \boxed{\begin{array}{cc} X(z^{-1}) & X(z) \\ X(z^{-1}) & X(z) \end{array}}$$

$$\boxed{\begin{array}{cc} -(p^{-1}, p^{-2}, p^{-3}, \dots) & -(p^{-1}, p^{-2}, p^{-3}, \dots) \\ (p^0, p^1, p^2, \dots) & (p^0, p^1, p^2, \dots) \end{array}} \quad \boxed{\begin{array}{cc} -(p^{-1}, p^{-2}, p^{-3}, \dots) & -(p^{-1}, p^{-2}, p^{-3}, \dots) \\ (p^0, p^1, p^2, \dots) & (p^0, p^1, p^2, \dots) \end{array}}$$

$$\boxed{\begin{array}{cc} -\dfrac{p^{-1}}{1-p^{-1}z} & -\dfrac{p^{-1}}{1-p^{-1}z^{-1}} \\ \dfrac{z^{-1}}{1-pz^{-1}} & \dfrac{z}{1-pz} \end{array}} \quad \boxed{\begin{array}{cc} -\dfrac{p^{-1}}{1-p^{-1}z} & -\dfrac{p^{-1}}{1-p^{-1}z^{-1}} \\ \dfrac{z^{-1}}{1-pz^{-1}} & \dfrac{z}{1-pz} \end{array}}$$

$$\boxed{\begin{array}{ll} f(z) & g(z) \\ f(z) & g(z) \end{array}} \quad \boxed{\begin{array}{ll} Y(z) & X(z) \\ Y(z) & X(z) \end{array}} \quad \boxed{\begin{array}{ll} a_n & a_{-n} \\ -a_n & -a_{-n} \end{array}} \quad \boxed{\begin{array}{ll} x_{-n} & x_n \\ -x_{-n} & -x_n \end{array}}$$

$$\boxed{\begin{array}{ll} |z|<p & |z|>p^{-1} \\ |z|>p & |z|<p^{-1} \end{array}} \quad \boxed{\begin{array}{ll} |z|<p & |z|>p^{-1} \\ |z|>p & |z|<p^{-1} \end{array}}$$

$$\boxed{\begin{array}{ll} [0,\infty) & (-\infty,0] \\ (-\infty,-1] & [1,\infty) \end{array}} \quad \boxed{\begin{array}{ll} (-\infty,0] & [0,\infty) \\ [1,\infty) & (-\infty,-1] \end{array}}$$

$$\boxed{\begin{array}{l} -(p^{-1}, p^{-2}, p^{-3}, \cdots) \\ (p^0, p^1, p^2, \cdots) \end{array}} \quad \boxed{\begin{array}{l} -(p^{-1}, p^{-2}, p^{-3}, \cdots) \\ (p^0, p^1, p^2, \cdots) \end{array}}$$

a_n	a_{-n}
$-a_n$	$-a_{-n}$

2^{-n}	2^{n}
-2^{-n}	-2^{n}

$a_n = -2^{-n}$

x_{-n}	x_n
$-x_{-n}$	$-x_n$

2^{-n}	2^{n}
-2^{-n}	-2^{n}

$x_n = -2^{n}$

$-(p^{-1}, p^{-2}, p^{-3}, \dots)$	$-(p^{-1}, p^{-2}, p^{-3}, \dots)$
$(p^{0}, p^{1}, p^{2}, \dots)$	$(p^{0}, p^{1}, p^{2}, \dots)$

$-(-2^{-1}, -2^{-2}, -2^{-3}, \dots)$	$-(-2^{-1}, -2^{-2}, -2^{-3}, \dots)$
$(-2^{0}, -2^{1}, -2^{2}, \dots)$	$(-2^{0}, -2^{1}, -2^{2}, \dots)$

$-\dfrac{p^{-1}}{1-p^{-1}z}$	$-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}$
$\dfrac{z^{-1}}{1-pz^{-1}}$	$\dfrac{z}{1-pz}$

$\dfrac{2^{-1}}{1-2^{-1}z}$	$\dfrac{2^{-1}}{1-2^{-1}z^{-1}}$
$-\dfrac{z^{-1}}{1-2z^{-1}}$	$-\dfrac{z}{1-2z}$

$\dfrac{\left(\frac{1}{2}\right)}{1-\left(\frac{z}{2}\right)}$	$\dfrac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2z}\right)}$
$-\dfrac{\left(\frac{1}{z}\right)}{1-\left(\frac{2}{z}\right)}$	$-\dfrac{z}{1-2z}$

$\lvert z\rvert < p$	$\lvert z\rvert > p^{-1}$
$\lvert z\rvert > p$	$\lvert z\rvert < p^{-1}$

$\lvert z\rvert < 2$	$\lvert z\rvert > 2^{-1}$
$\lvert z\rvert > 2$	$\lvert z\rvert < 2^{-1}$

$[0, \infty)$	$(-\infty, 0]$
$(-\infty, -1]$	$[1, \infty)$

$[0, \infty)$	$(-\infty, 0]$
$(-\infty, -1]$	$[1, \infty)$