## Stationarity

Young W Lim

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Based on Probability, Random Variables and Random Signal Principles, P.Z. Peebles, Jr. and B. Shi





#### 2 Correlation and Covariance Functions

# First Order Stationary

#### $f_X(x;t)$

if X(t) is to be a first-order stationary

 $f_{\boldsymbol{X}}(\boldsymbol{x}_1;\boldsymbol{t}_1) = f_{\boldsymbol{X}}(\boldsymbol{x}_1;\boldsymbol{t}_1 + \Delta)$ 

must be true for any time  $t_1$  and any real number  $\Delta$ 

the first order density function does not change with a shift in time origin

Consequences of stationarity

#### $f_X(x;t)$

- f<sub>X</sub>(x, t<sub>1</sub>) is independent of t<sub>1</sub>
   the first order density function
   does not change with a shift in time origin
- the process mean value is a constant

$$m_X(t) = \overline{X} = constant$$

### the process mean value

$$\boxed{m_X(t) = \overline{X} = constant}$$

$$m_X(t_1) = \int_{-\infty}^{\infty} x f_X(x; t_1) dx$$

$$m_X(t_2) = \int_{-\infty}^{\infty} x f_X(x; t_2) dx$$
let  $t_2 = t_1 + \Delta$ 

$$m_X(t_1) = m_X(t_1 + \Delta)$$

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Image: A matrix and a matrix

# Second-Order Stationary Process

#### $f_X(x_1, x_2; t_1, t_2)$

if the second order density function does not change with a shift in time origin

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta)$$

must be true for any time  $t_1$ ,  $t_2$  and any real number  $\Delta$  if X(t) is to be a second-order stationary

Auto-correlation function

$$R_{XX}(t,t+\tau) = E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

# N<sup>th</sup>-order Stationary Processes

#### $f_X(x_1,\cdots,x_N;t_1,\cdots,t_N)$

if the second order density function does not change with a shift in time origin

$$f_X(x_1,\cdots,x_N;t_1,\cdots,t_N)=f_X(x_1,\cdots,x_N;t_1+\Delta,\cdots,t_N+\Delta)$$

must be true for any time  $t_1, ..., t_N$  and any real number  $\Delta$  if X(t) is to be a second-order stationary

Wide Sense Stationary Process

#### $m_X(t), R_{XX}(\tau)$

$$m_X(t) = \overline{X} = constant$$

$$E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

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The properties of autocorrelation functions (1)

 $|R_{XX}(\tau)|, R_{XX}(-\tau), R_{XX}(0)|$ 

 $|R_{XX}(\tau)| \leq R_{XX}(0)$ 

 $R_{XX}(-\tau) = R_{XX}(\tau)$ 

 $R_{XX}(0) = E\left[X^2(t)\right]$ 

$$P[|X(t+\tau) - X(t)| > \varepsilon] = \frac{2}{\varepsilon^2} (R_{XX}(0) - R_{XX}(\tau))$$

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# The properties of autocorrelation functions (2)

### $R_{NN}(\tau), R_{XX}(\tau)$

if  $X(t) = \overline{X} + N(t)$  where N(t) is WSS, is zero-mean, and has autocorrelation function  $R_{NN}(\tau) \rightarrow 0$  as  $|\tau| \rightarrow \infty$ , then

$$\lim_{\tau|\to\infty}R_{XX}(\tau)=\overline{X}^2$$

if X(t) is mean square periodic, i.e, there exists a  $T \neq 0$ such that  $E[(X(t+T) - X(t))^2] = 0$  for all t, then  $R_{XX}(t)$  will have a **periodic** component with the same period  $R_{XX}(\tau)$  cannot have an arbitrary shape

Crosscorrelation functions (1)

#### $R_{XY}(t_1,t_2),R_{XY}(t,t+\tau)$

if

 $R_{XY}(t_1,t_2) = E\left[X(t_1)Y(t_2)\right]$ 

$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)] = R_{XY}(\tau)$$
$$R_{XY}(t, t+\tau) = 0$$

then X(t) and Y(t) are called **orthogonal processes** 

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Crosscorrelation functions (2)

### $R_{XY}(t,t+\tau),R_{XY}(\tau)$

if X(t) and Y(t) are statistically independent

$$R_{XY}(t,t+\tau) = E[X(t)Y(t+\tau)] = m_X(t)m_Y(t+\tau)$$

if X(t) and Y(t) are stistically independent and are at least WSS,

$$R_{XY}(\tau) = \overline{XY}$$

which is constant

# The properties of crosscorrelation functions (1)

### $R_{XY}(\tau), |R_{XY}(\tau)|$

$$R_{\mathbf{X}\mathbf{Y}}(\boldsymbol{\tau}) = R_{\mathbf{X}\mathbf{Y}}(-\boldsymbol{\tau})$$

$$|R_{XY}(\tau)| = \sqrt{R_{XX}(0)R_{YY}(0)}$$

$$|R_{XY}(\tau)| \leq rac{1}{2} [R_{XX}(0) + R_{YY}(0)]$$

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The properties of crosscorrelation functions (2)

### $R_{YX}(-\tau)$

 $R_{YX}(-\tau) = E[Y(t)X(t-\tau)] = E[Y(s+\tau)X(s)] = R_{XY}(\tau)$ 

$$E\left[\{\mathbf{Y}(t+\tau)+\alpha \mathbf{X}(t)\}^2\right]\geq 0$$

the **geometric mean** of two positive numbers cannot exceed their **arithmetic mean** 

The properties of crosscorrelation functions (3)

### $|R_{XY}(\tau)|$

$$|R_{XY}(\tau)| \leq \frac{1}{2} [R_{XX}(0) + R_{YY}(0)]$$
$$\sqrt{R_{XX}(0)R_{YX}(0)} \leq \frac{1}{2} [R_{XX}(0) + R_{YY}(0)]$$

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### **Covariance Functions**

#### $C_{XX}(t,t+\tau), C_{XY}(t,t+\tau)$

$$C_{XX}(t,t+\tau) = E\left[\left\{X(t) - m_X(t)\right\}\left\{X(t+\tau) - m_X(t+\tau)\right\}\right]$$

$$C_{XY}(t,t+\tau) = E\left[\left\{X(t) - m_X(t)\right\}\left\{Y(t+\tau) - m_Y(t+\tau)\right\}\right]$$

$$C_{XX}(t,t+\tau) = R_{XX}(t,t+\tau) - m_X(t)m_X(t+\tau)$$

$$C_{XY}(t,t+\tau) = R_{XY}(t,t+\tau) - m_X(t)m_Y(t+\tau)$$
at least jointly WSS

$$C_{XX}(\tau) = R_{XX}(\tau) - \overline{X}^2$$

$$C_{XY}(\tau) = R_{XY}(\tau) - \overline{XY}$$

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# The properties of covariance functions

 $C_{XX}(0)$ 

For a WSS process, variance does not depend on time and if au=0

 $C_{XX}(0) = R_{XX}(0) - \overline{X}^2$ 

$$\sigma_{\mathbf{X}}^{2} = E\left[\left\{\mathbf{X}(t) - E\left[\mathbf{X}(t)\right]\right\}^{2}\right] = C_{\mathbf{X}\mathbf{X}}(0)$$

it the two random processes uncorrelated

$$C_{XY}(t,t+\tau) = R_{XY}(t,t+\tau) - m_X(t)m_Y(t+\tau) = 0$$

$$R_{XY}(t,t+\tau) = m_X(t)m_Y(t+\tau)$$

### Discrete-Time Processes and Sequences (1)

 $R_{XX}[n, n+k], R_{YY}[n, n+k], C_{XX}[n, n+k], C_{YY}[n, n+k]$  $m_X[n] = \overline{X}, m_Y[n] = \overline{Y}$  $R_{XX}[n, n+k] = R_{XX}[k]$  $R_{YY}[n, n+k] = R_{YY}[k]$  $C_{\mathbf{X}\mathbf{X}}[n, n+k] = R_{\mathbf{X}\mathbf{X}}[k] - \overline{X}^2$  $C_{YY}[n, n+k] = R_{YY}[k] - \overline{Y}^2$ 

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Discrete-Time Processes and Sequences (2)

 $R_{XY}[n, n+k], R_{YX}[n, n+k], C_{XY}[n, n+k], C_{YX}[n, n+k]$  $m_X[n] = \overline{X}, m_Y[n] = \overline{Y}$  $R_{\mathbf{X}\mathbf{Y}}[n, n+k] = R_{\mathbf{X}\mathbf{Y}}[k]$  $R_{\mathbf{Y}\mathbf{X}}[n, n+k] = R_{\mathbf{Y}\mathbf{X}}[k]$  $C_{XY}[n, n+k] = R_{XY}[k] - \overline{XY}$  $C_{YX}[n, n+k] = R_{YX}[k] - \overline{YX}$ 

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