

Binary Relations (4B)

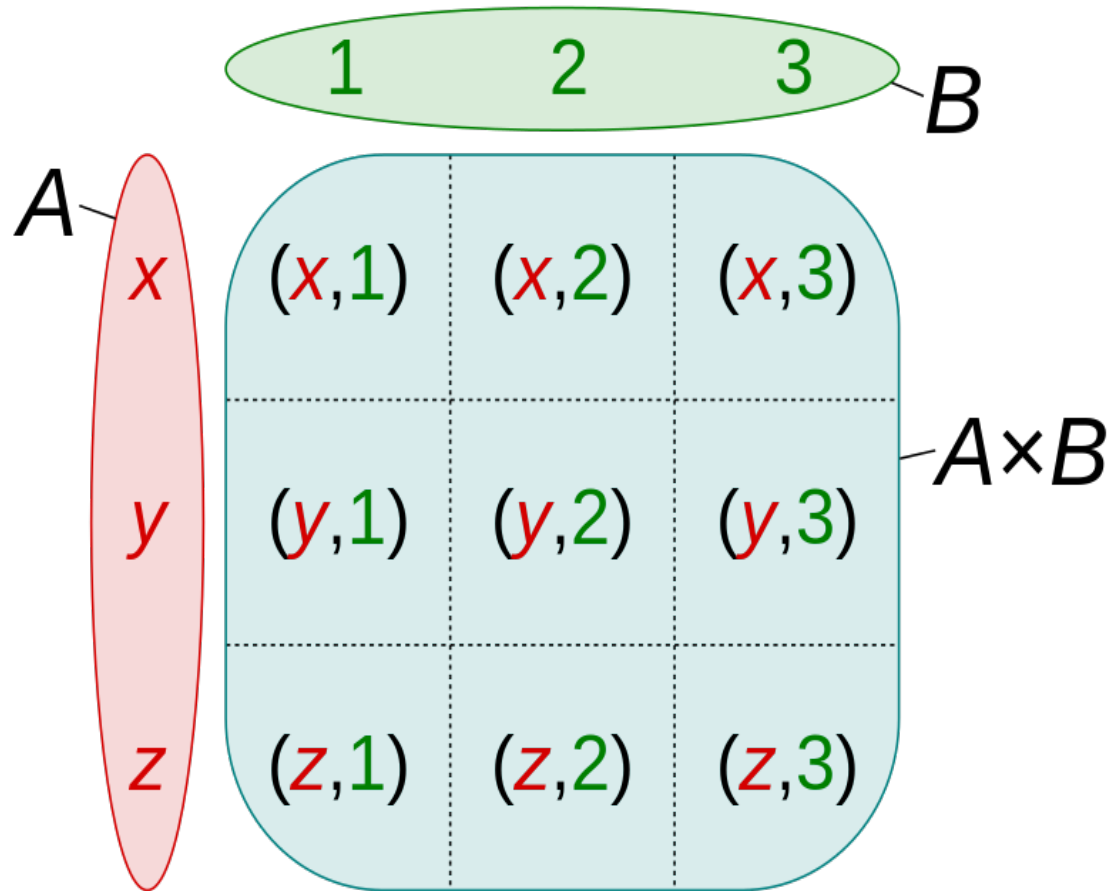
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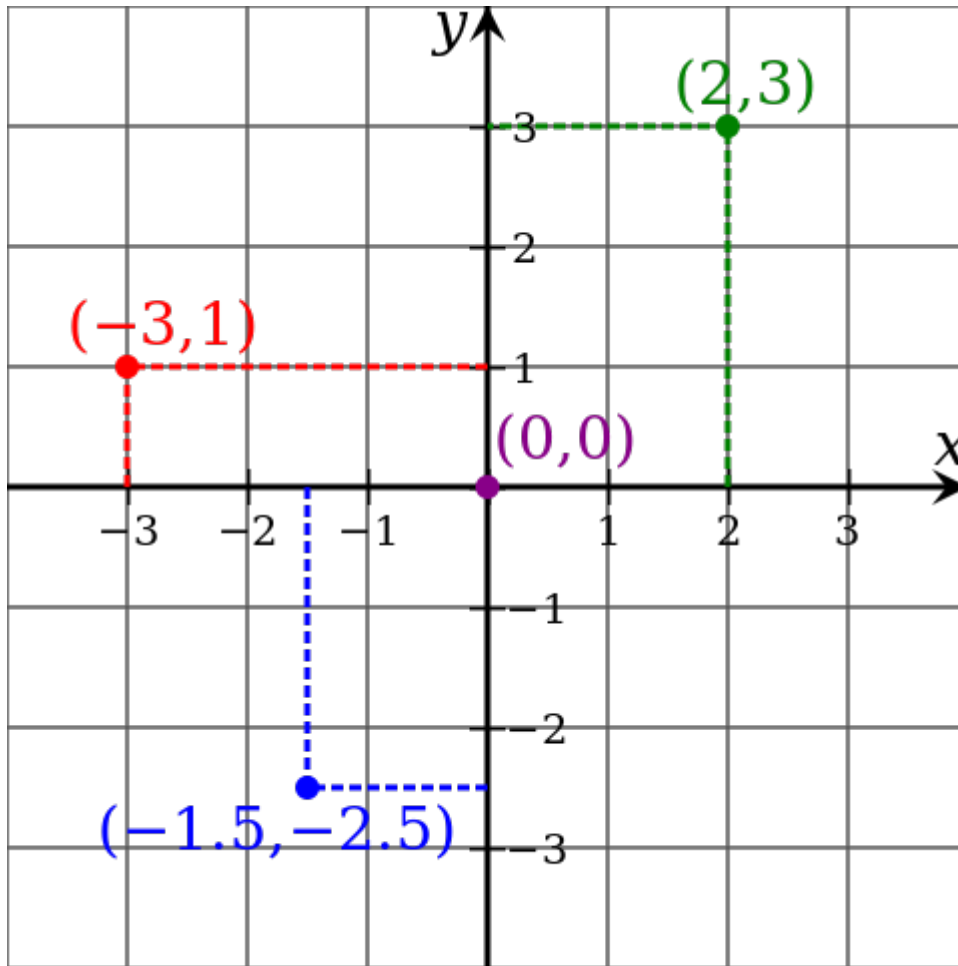
Cartesian Product



Cartesian product $A \times B$
of the sets $A = \{x, y, z\}$
and $B = \{1, 2, 3\}$

https://en.wikipedia.org/wiki/Cartesian_product

Cartesian Coordinates



Cartesian coordinates of
example points

https://en.wikipedia.org/wiki/Cartesian_product

Reflexive Relation Examples

$$x \geq y$$

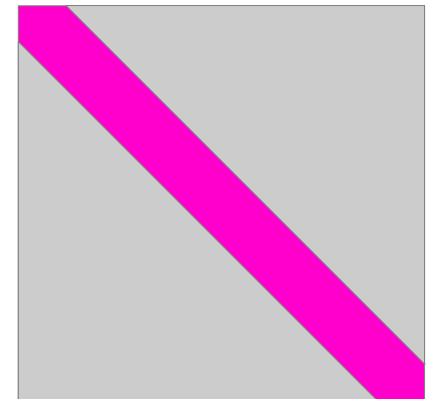
	1	2	3	4	5	6	7	8	x
1	●	✓	✓	✓	✓	✓	✓	✓	
2		●	✓	✓	✓	✓	✓	✓	
3			●	✓	✓	✓	✓	✓	
4				●	✓	✓	✓	✓	
5					●	✓	✓	✓	
6						●	✓	✓	
7							●	✓	
8								●	

- Must be true for every member of the set in any reflexive relation
- ✓ Is true for this case (need not be true for all cases)

$$x > y$$

	1	2	3	4	5	6	7	8	x
1	⊗	✓	✓	✓	✓	✓	✓	✓	
2		⊗	✓	✓	✓	✓	✓	✓	
3			⊗	✓	✓	✓	✓	✓	
4				⊗	✓	✓	✓	✓	
5					⊗	✓	✓	✓	
6						⊗	✓	✓	
7							⊗	✓	
8								⊗	

- ⊗ Must be false for every member of the set in any irreflexive relation
- ✓ Is true for this case (need not be true for all cases)

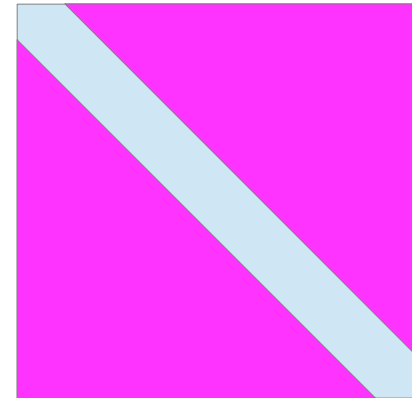


https://en.wikipedia.org/wiki/Reflexive_relation

Symmetric Relation Examples

x and **y** are odd

	1	2	3	4	5	6	7	8	x
1	✓		1✓		2✓		3✓		
2									
3	①		✓		4✓		5✓		
4									
5	②		④		✓		6✓		
6									
7	③		⑤		⑦		✓		
8									



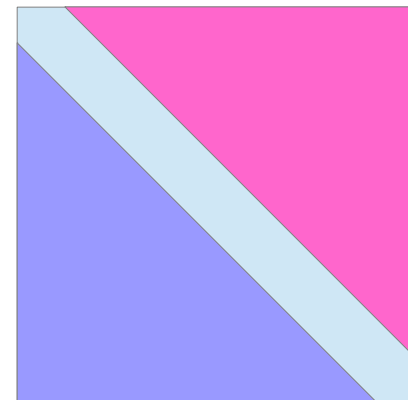
- ✓ Is true for this case (need not be true for all cases)
- ① Must be true if the check mark with the same number (z) is true for it to be a symmetric relation
- ② Is true for this case and requires the circle with the same number (z) to also be true for it to be a symmetric relation

https://en.wikipedia.org/wiki/Cartesian_product

Anti-Symmetric Relation Examples

x is even and **y** is odd

	1	2	3	4	5	6	7	8	x
1	✓			✓		✓		✓	
2	1	11		12		14			
3		✓		✓		✓		✓	
4	2		5	13		15			
5		✓		✓		✓		✓	
6	3		6	14		16			
7		✓		✓		✓		✓	
8	4		7	15		17		10	



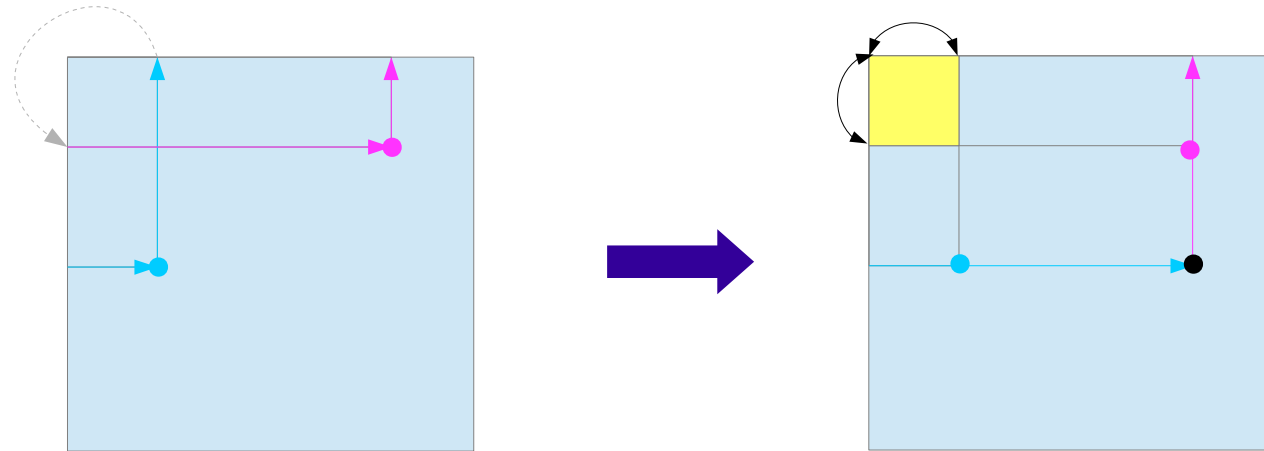
✓ Is true for this case (need not be true for all cases)

✗ Must be false if the check mark with the same number (z) is true for it to be an antisymmetric relation

z ✓ Is true for this case and requires the circle with the same number (z) to be false for it to be a symmetric relation

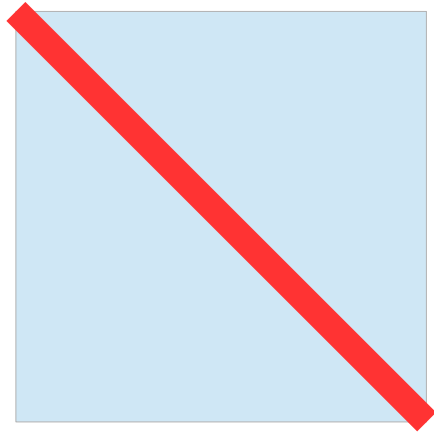
https://en.wikipedia.org/wiki/Cartesian_product

Transitive Relation Examples



Reflexive Relation

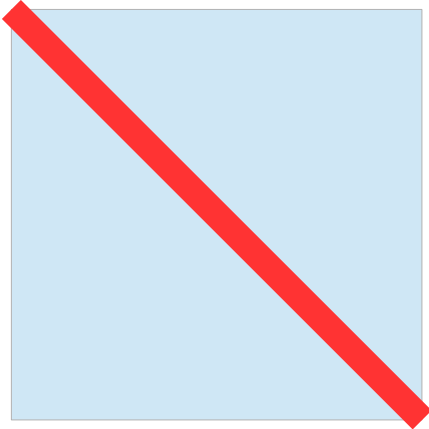
$$\forall x \quad (x, x) \in R$$



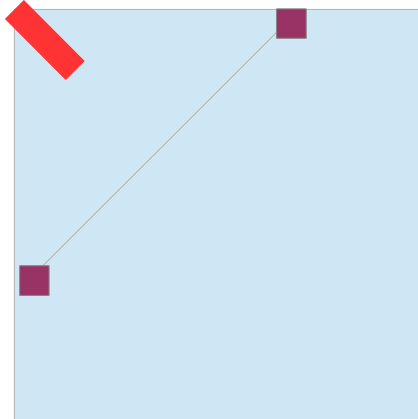
Symmetric Relation

$$\forall x, \forall y \left[(x, y) \in R \rightarrow (y, x) \in R \right]$$

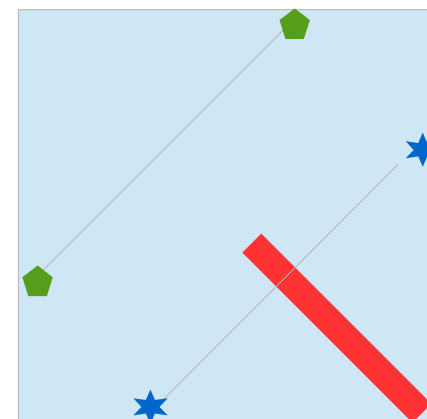
symmetric



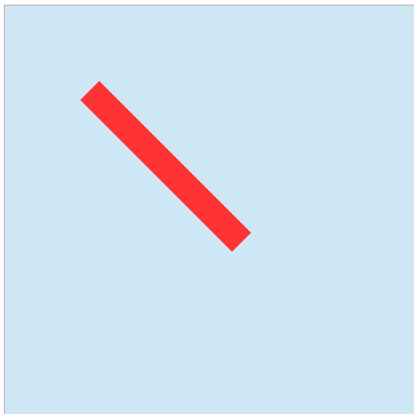
symmetric



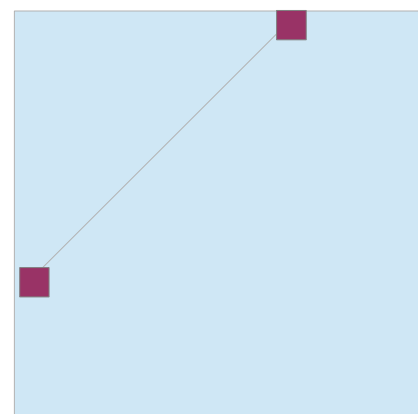
symmetric



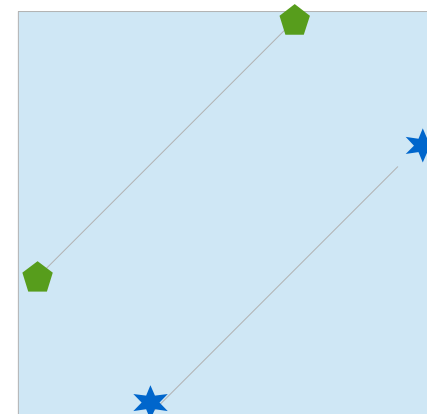
symmetric



symmetric



symmetric



Not Symmetric Relation

$$\neg \{ \forall x, \forall y [(x, y) \in R] \rightarrow [(y, x) \in R] \}$$

$$\exists x, \exists y \neg \{ [(x, y) \in R] \rightarrow [(y, x) \in R] \}$$

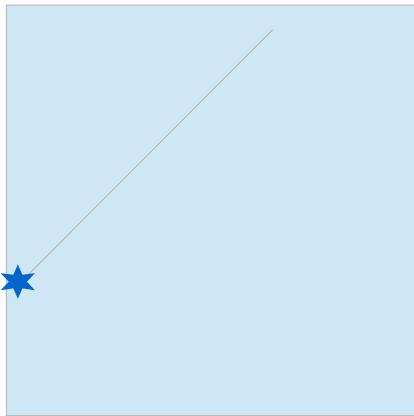
$$\exists x, \exists y \neg \{ \neg [(x, y) \in R] \vee [(y, x) \in R] \}$$

$$\exists x, \exists y [(x, y) \in R] \wedge \neg [(y, x) \in R]$$

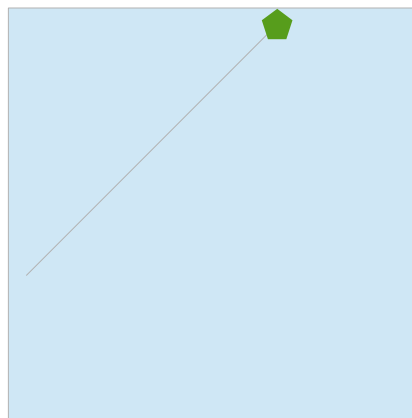
$$\exists x, \exists y [(x, y) \in R] \wedge [(y, x) \notin R]$$

counter example

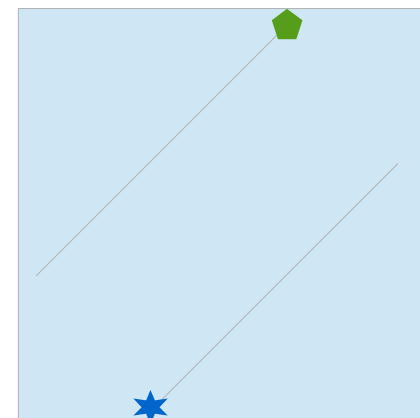
not symmetric



not symmetric



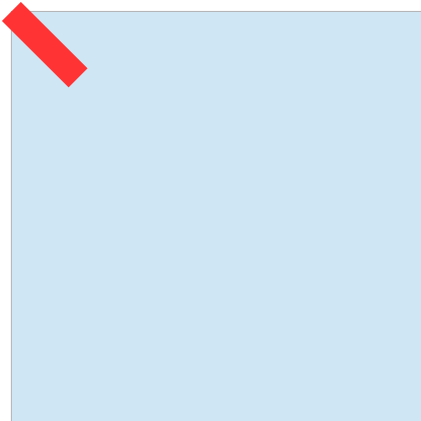
not symmetric



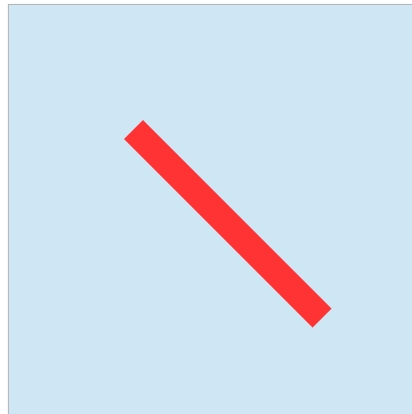
Anti-symmetric Relation

$$\forall x, \forall y \left[\left((x, y) \in R \wedge (y, x) \in R \right) \rightarrow x = y \right]$$

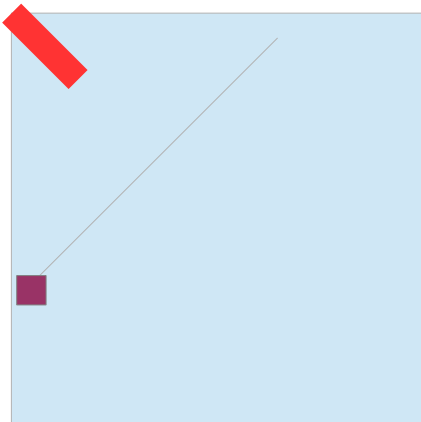
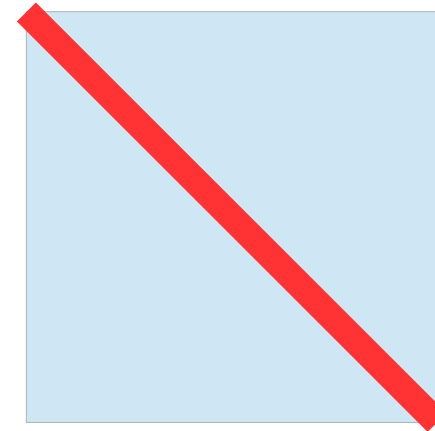
anti-symmetric



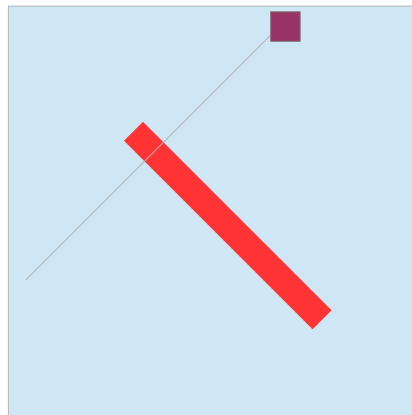
anti-symmetric



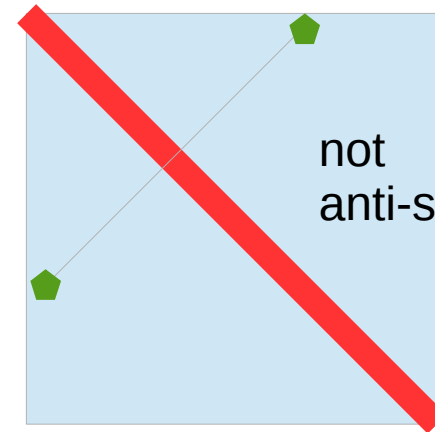
anti-symmetric



anti-symmetric



anti-symmetric



not
anti-symmetric

Not Anti-symmetric Relation

$$\forall x, \forall y \left[(x, y) \in R \wedge (y, x) \in R \right] \rightarrow [x = y]$$

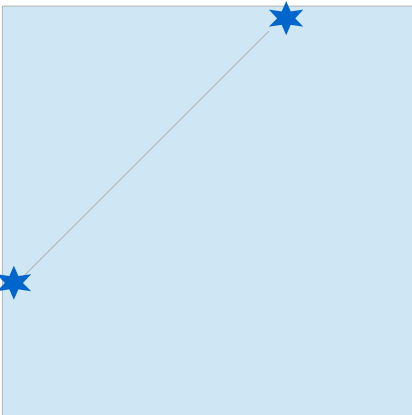
$$\exists x, \exists y \neg \{ [(x, y) \in R \wedge (y, x) \in R] \rightarrow [x = y] \}$$

$$\exists x, \exists y \neg \{ \neg [(x, y) \in R \wedge (y, x) \in R] \vee [x = y] \}$$

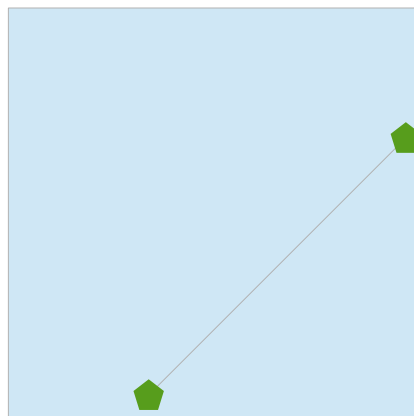
$$\exists x, \exists y \{ [(x, y) \in R \wedge (y, x) \in R] \wedge \neg [x = y] \}$$

$$\exists x, \exists y \{ [(x, y) \in R \wedge (y, x) \in R] \wedge [x \neq y] \} \quad \text{counter example}$$

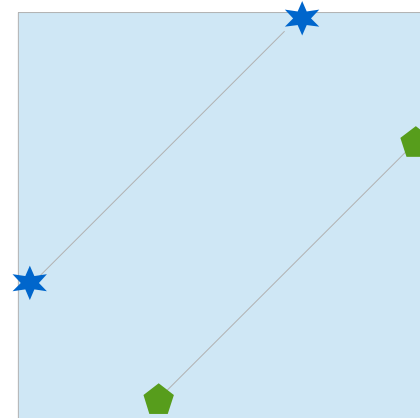
not anti-symmetric



not anti-symmetric



not anti-symmetric



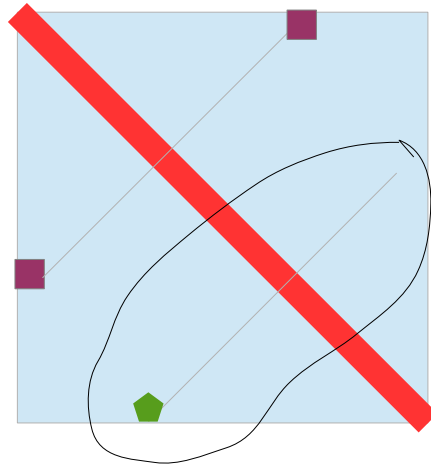
Equivalent Anti-symmetric Relation

$$\forall x, \forall y \left[(x, y) \in R \wedge (y, x) \in R \right] \rightarrow \left[x = y \right]$$

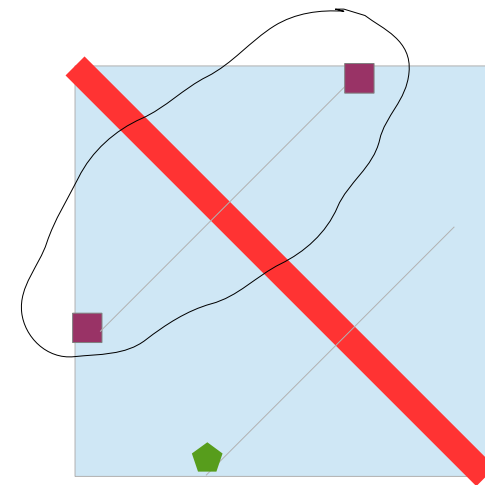
$$\forall x, \forall y \left[x \neq y \right] \rightarrow \left[(x, y) \notin R \vee (y, x) \notin R \right]$$

No symmetric relation is allowed

neither
symmetric



nor
anti-symmetric

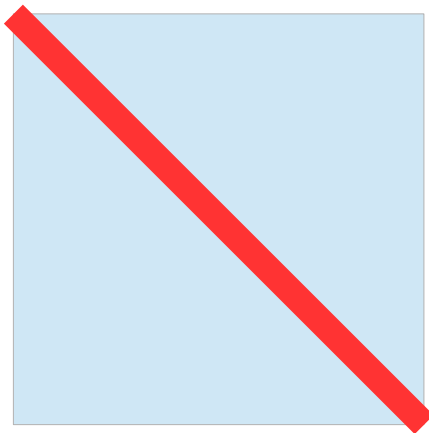


Reflexive, Symmetric, Anti-symmetric

$$\forall x \quad (x, x) \in R$$

$$\forall x, \forall y \quad [(x, y) \in R] \rightarrow [(y, x) \in R]$$

$$\forall x, \forall y \quad [(x, y) \in R \wedge (y, x) \in R] \rightarrow [x = y]$$



Reflexive

Also, symmetric

Also, anti-symmetric

(no relation for (x, y) where $x \neq y$)

(no relation for (x, y) where $x \neq y$)

not symmetric

\neq anti-symmetric

not anti-symmetric

\neq symmetric

Matrix of a Relation

$$R_1 \in \{(1, a), (2, b), (3, a), (3, b)\}$$

$$A_1 = \begin{matrix} & \begin{matrix} a & b \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{matrix}$$

$$R_2 \in \{(a, x), (a, y), (b, y), (b, z)\}$$

$$A_2 = \begin{matrix} & \begin{matrix} x & y & z \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{matrix}$$

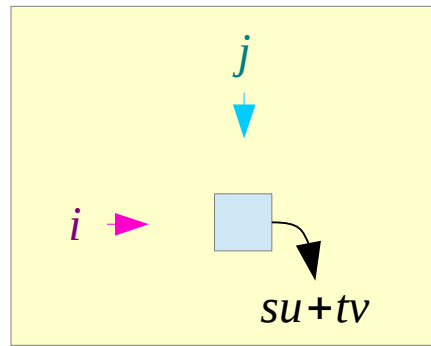
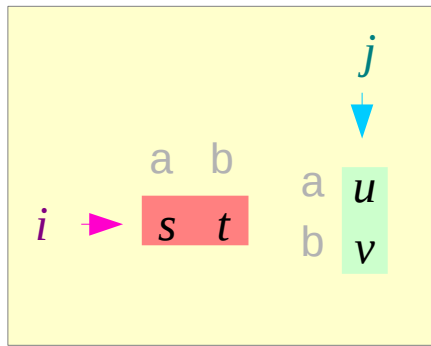
$$A_1 A_2 = \begin{matrix} & \begin{matrix} x & y & z \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{matrix} & \begin{matrix} x & y & z \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \end{matrix}$$

Sufficient Part

$$A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_1 = \begin{matrix} & a & b \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{matrix}$$

$$A_2 = \begin{matrix} & x & y & z \\ \begin{matrix} a \\ b \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{matrix}$$



$$i \in \{1, 2, 3\}$$

$$j \in \{x, y, z\}$$

$$s \in \{0, 1\}$$

$$t \in \{0, 1\}$$

$$u \in \{0, 1\}$$

$$v \in \{0, 1\}$$

$$su+tv \neq 0$$



$$su = 1$$

$$\begin{matrix} (s = 1) \\ (u = 1) \end{matrix}$$

$$\begin{matrix} (i, a) \in R_1 \\ (a, j) \in R_2 \end{matrix}$$

$$(i, j) \in R_2 \circ R_1$$

$$tv = 1$$

$$\begin{matrix} (t = 1) \\ (v = 1) \end{matrix}$$

$$\begin{matrix} (i, b) \in R_1 \\ (b, j) \in R_2 \end{matrix}$$

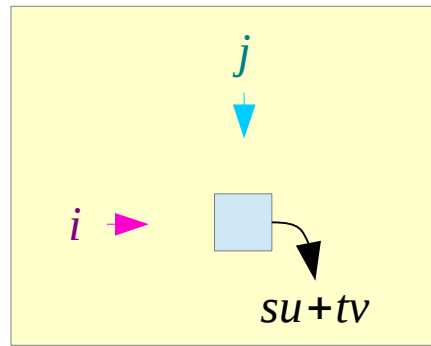
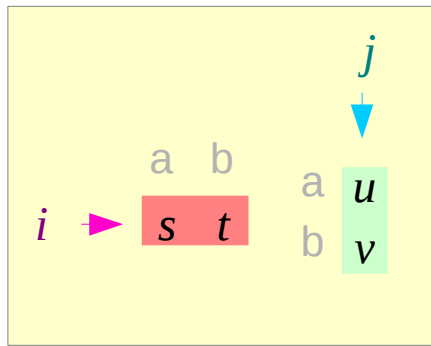
$$(i, j) \in R_2 \circ R_1$$

Necessary Part

$$A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_1 = \begin{matrix} & a & b \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{matrix}$$

$$A_2 = \begin{matrix} & x & y & z \\ \begin{matrix} a \\ b \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{matrix}$$



$i \in \{1, 2, 3\}$
 $j \in \{x, y, z\}$

$s \in \{0, 1\}$
 $t \in \{0, 1\}$
 $u \in \{0, 1\}$
 $v \in \{0, 1\}$

$(i, j) \in R_2 \circ R_1 \Rightarrow$

$(i, a) \in R_1$
 $(a, j) \in R_2$

$(i, b) \in R_1$
 $(b, j) \in R_2$

$(s = 1)$
 $(u = 1)$

$(t = 1)$
 $(v = 1)$

$su = 1$

$tv = 1$

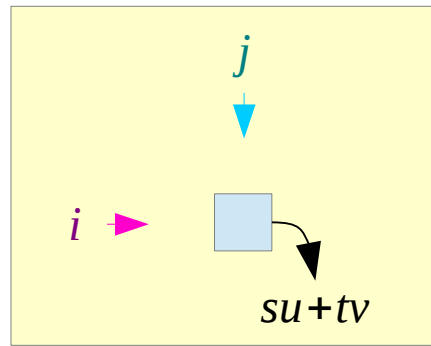
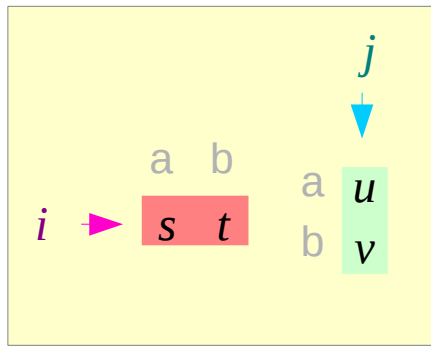
$su+tv \neq 0$

Necessary Part

$$A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_1 = \begin{matrix} & a & b \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{matrix}$$

$$A_2 = \begin{matrix} & x & y & z \\ \begin{matrix} a \\ b \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{matrix}$$



$$\begin{aligned} i &\in \{1, 2, 3\} & s &\in \{0, 1\} \\ j &\in \{x, y, z\} & t &\in \{0, 1\} \\ & & u &\in \{0, 1\} \\ & & v &\in \{0, 1\} \end{aligned}$$

$su+tv \neq 0$ *nonzero* $(i, j)^{th}$ element of $A_1 A_2$

$$\iff (i, j) \in R_2 \circ R_1$$

Transitivity Test

$$A = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \end{matrix}$$

$$A^2 = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} * & * & * & * \\ a & b & c & d \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \end{matrix} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} * & e & * & * \\ * & f & * & * \\ * & g & * & * \\ * & h & * & * \end{bmatrix} \end{matrix} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

nonzero $(i, j)^{\text{th}}$ element of $A^2 \Rightarrow$

nonzero $(i, j)^{\text{th}}$ element of A

$$\begin{matrix} ae = 1 \\ (2, 1) \in R \\ (1, 2) \in R \end{matrix}$$

$$\begin{matrix} bf = 1 \\ (2, 2) \in R \\ (2, 2) \in R \end{matrix}$$

$$\begin{matrix} cg = 1 \\ (2, 3) \in R \\ (3, 2) \in R \end{matrix}$$

$$\begin{matrix} dh = 1 \\ (2, 4) \in R \\ (4, 2) \in R \end{matrix}$$

$\Rightarrow (2, 2) \in R$

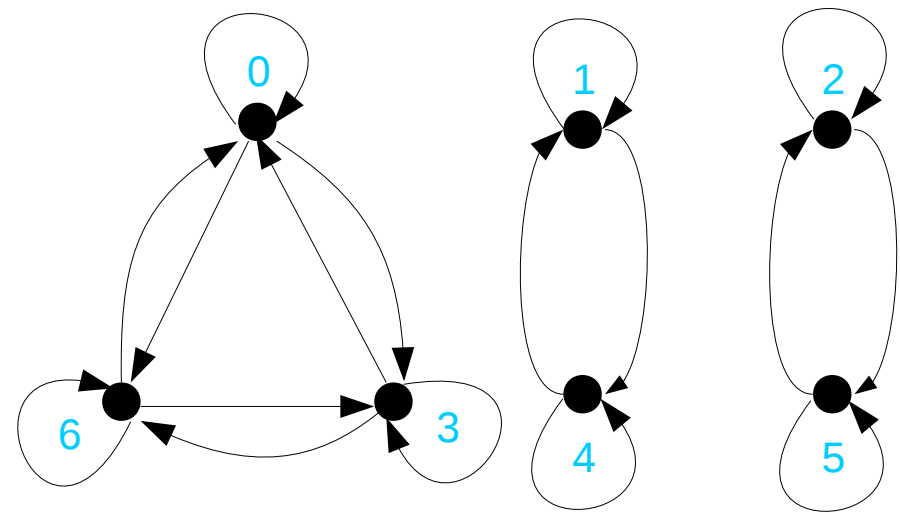
Binary Relations and Digraphs

$$A = \{0, 1, 2, 3, 4, 5, 6\}$$

$$R \subset A \times A$$

$$R = \{(a, b) \mid a \equiv b \pmod{3}\}$$

$$R = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right. \end{array}$$



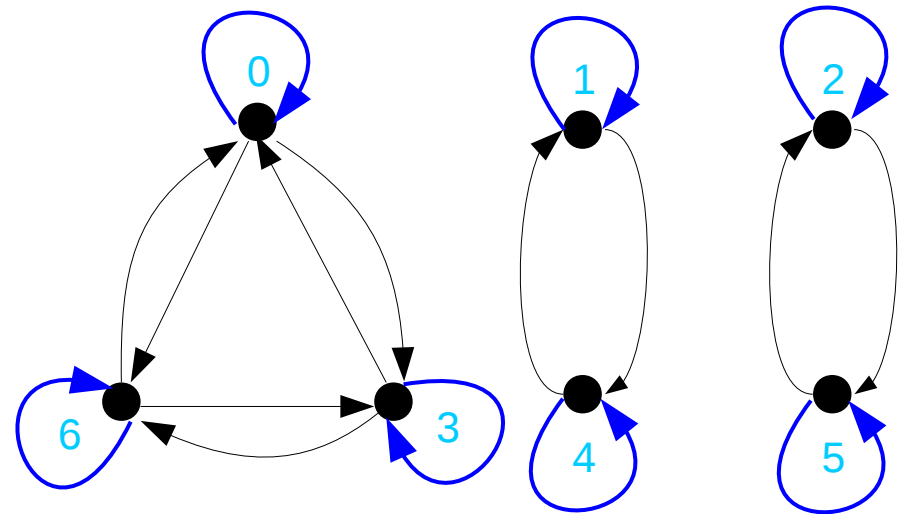
http://www.math.fsu.edu/~pkirby/mad2104/SlideShow/s7_1.pdf

Reflexive Relation

$$A = \{0, 1, 2, 3, 4, 5, 6\}$$

$$R \subset A \times A$$

$$R = \{(a, b) \mid a \equiv b \pmod{3}\}$$

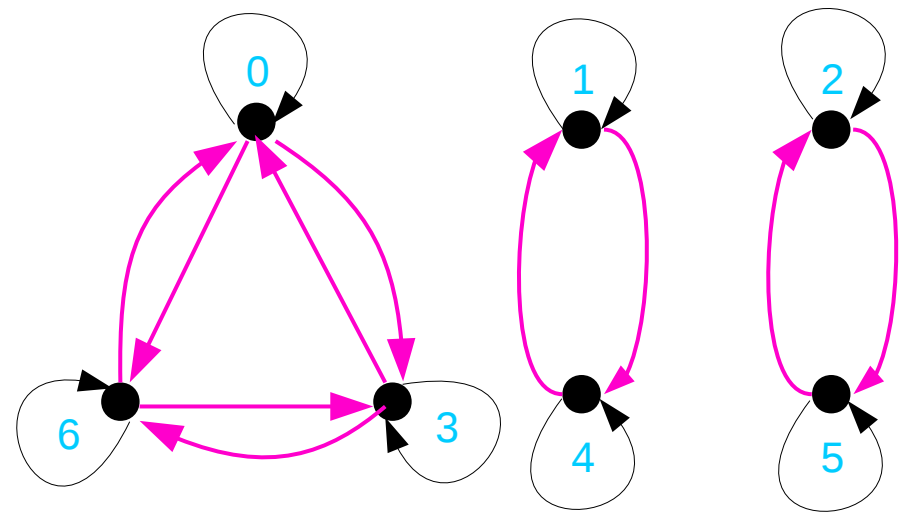
$$R = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 4 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 6 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}$$


Symmetric Relation

$$A = \{0, 1, 2, 3, 4, 5, 6\}$$

$$R \subset A \times A$$

$$R = \{(a, b) \mid a \equiv b \pmod{3}\}$$

$$R = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$


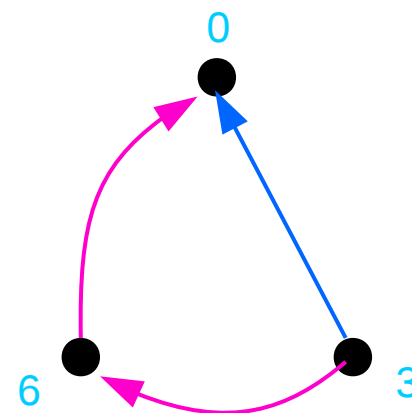
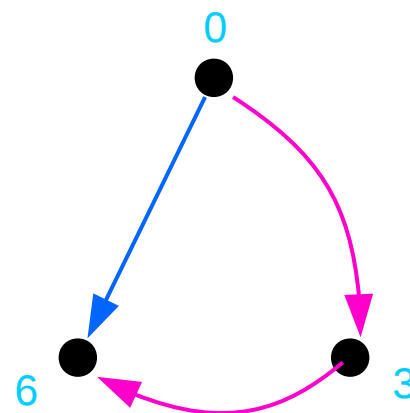
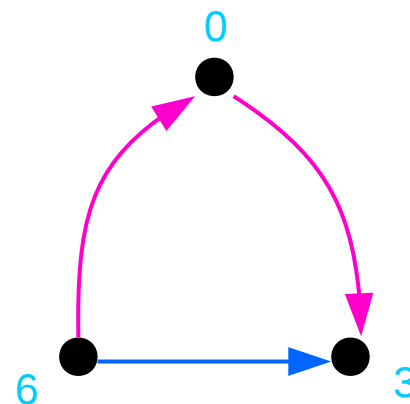
Transitive Relation

$$A = \{0, 1, 2, 3, 4, 5, 6\}$$

$$R \subset A \times A$$

$$R = \{(a, b) \mid a \equiv b \pmod{3}\}$$

$$RR = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \left[\begin{array}{cccccccc} 3 & 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 \\ 3 & 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 \\ 3 & 0 & 0 & 3 & 0 & 0 & 3 \end{array} \right. \end{array}$$



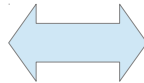
Equivalence Relation

$$A = \{0, 1, 2, 3, 4, 5, 6\}$$

$$R \subset A \times A$$

$$R = \{(a, b) \mid a \equiv b \pmod{3}\}$$

Equivalence Relation



Reflexive Relation &
Symmetric Relation &
Transitive Relation

Equivalence Class

$$A = \mathbf{Z}^+ = \{0, 1, 2, 3, 4, 5, 6, \dots\}$$

$$R \subset A \times A$$

$$R = \{(a, b) \mid a \equiv b \pmod{3}\}$$

$\{0, 3, 6, 9, \dots\}$	$[0]$	$[33]$
$\{1, 4, 7, 10, \dots\}$	$[1]$	$[331]$
$\{2, 5, 8, 11, \dots\}$	$[2]$	$[3332]$

<https://www.cse.iitb.ac.in/~nutan/courses/cs207-12/notes/lec7.pdf>

References

- [1] <http://en.wikipedia.org/>
- [2]