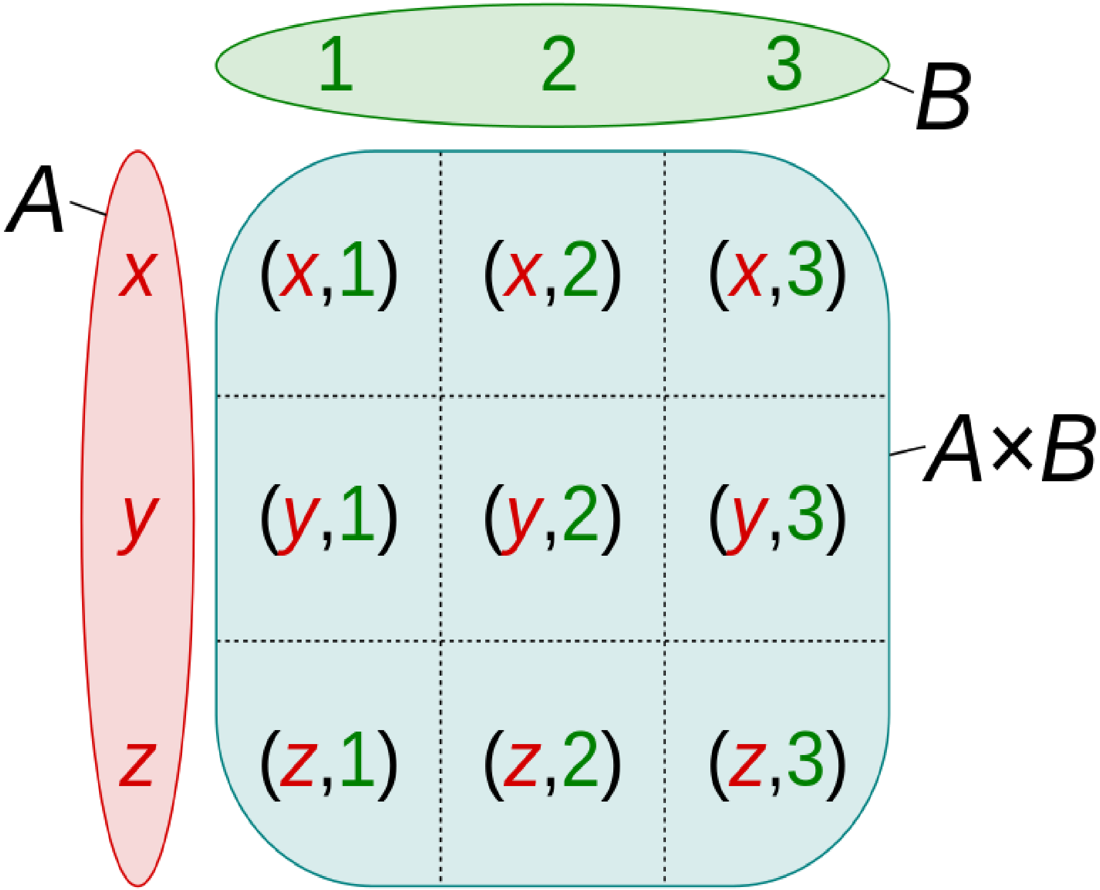
Binary Relations (4B)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

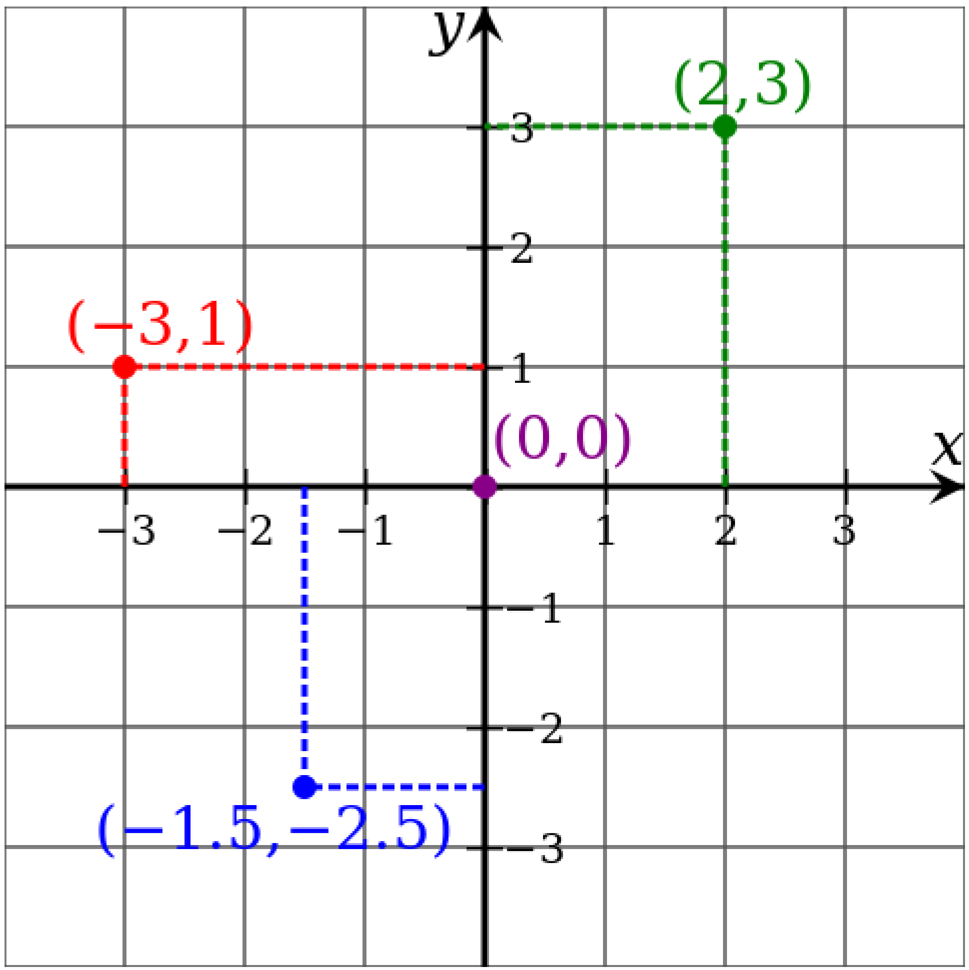
Cartesian Product



Cartesian product A × B
of the sets A = { x , y , z }
and B = { 1 , 2 , 3 }

https://en.wikipedia.org/wiki/Cartesian_product

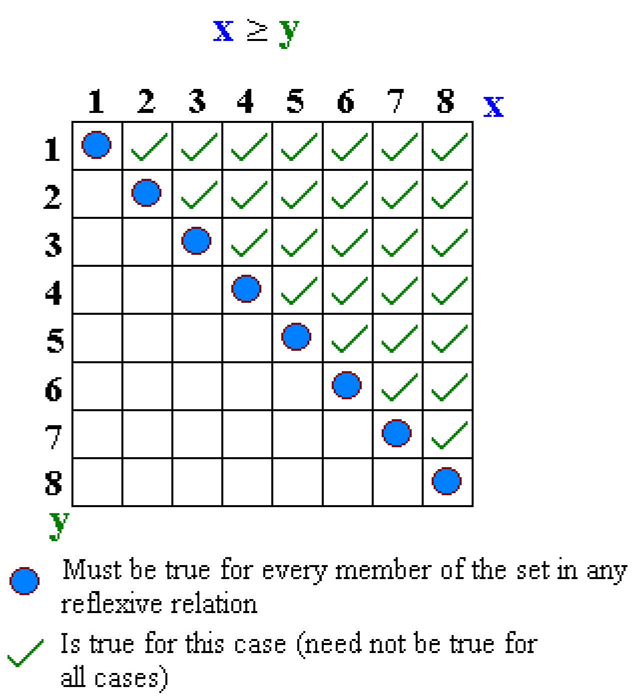
Cartesian Coordinates

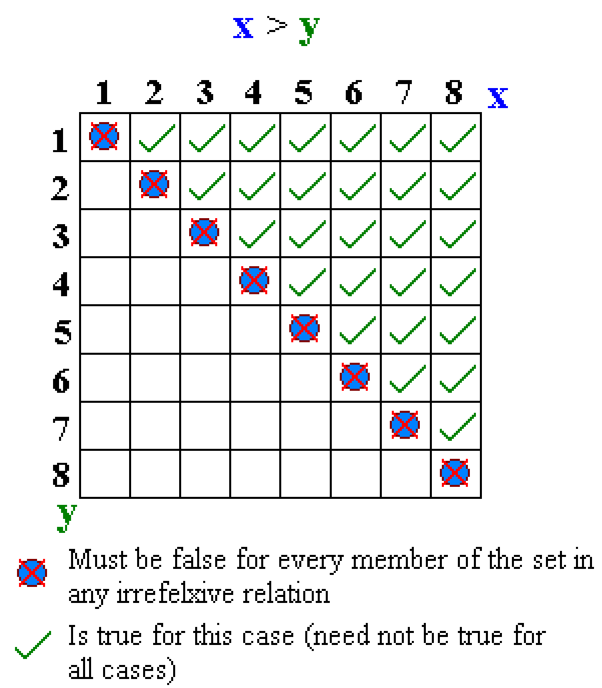


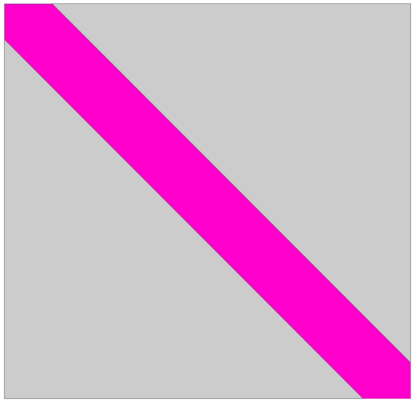
Cartesian coordinates of example points

https://en.wikipedia.org/wiki/Cartesian_product

Reflexive Relation Examples

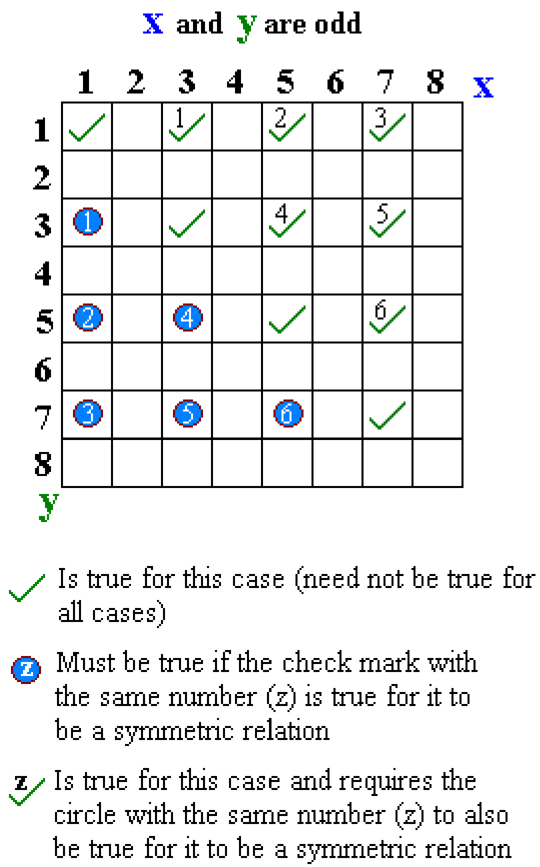


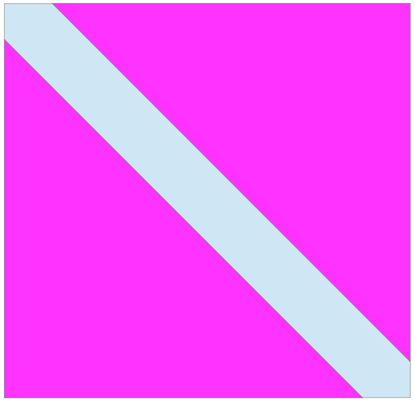




https://en.wikipedia.org/wiki/Reflexive_relation

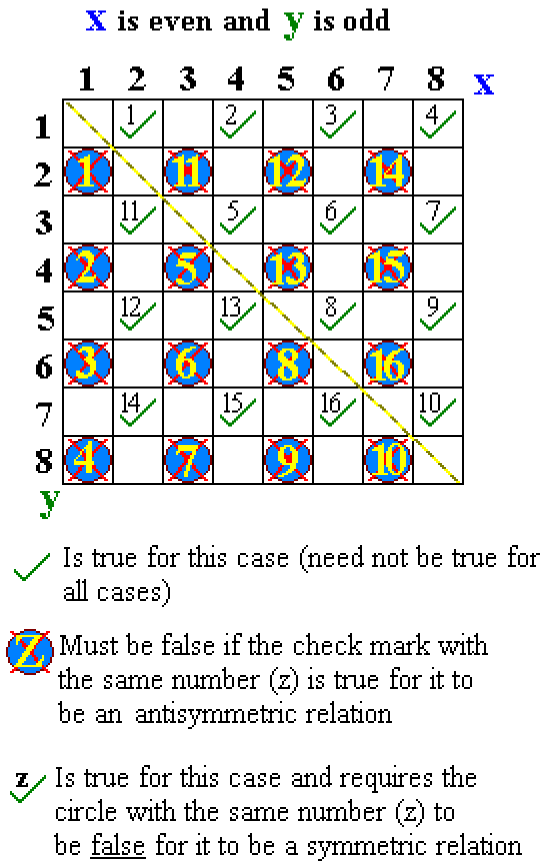
Symmetric Relation Examples

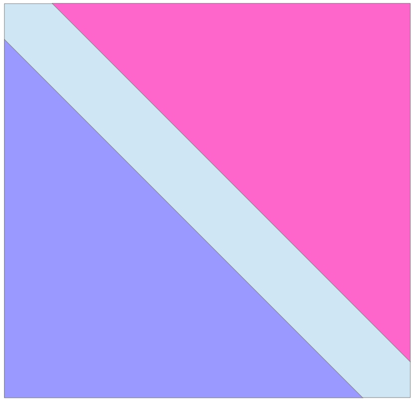




https://en.wikipedia.org/wiki/Cartesian_product

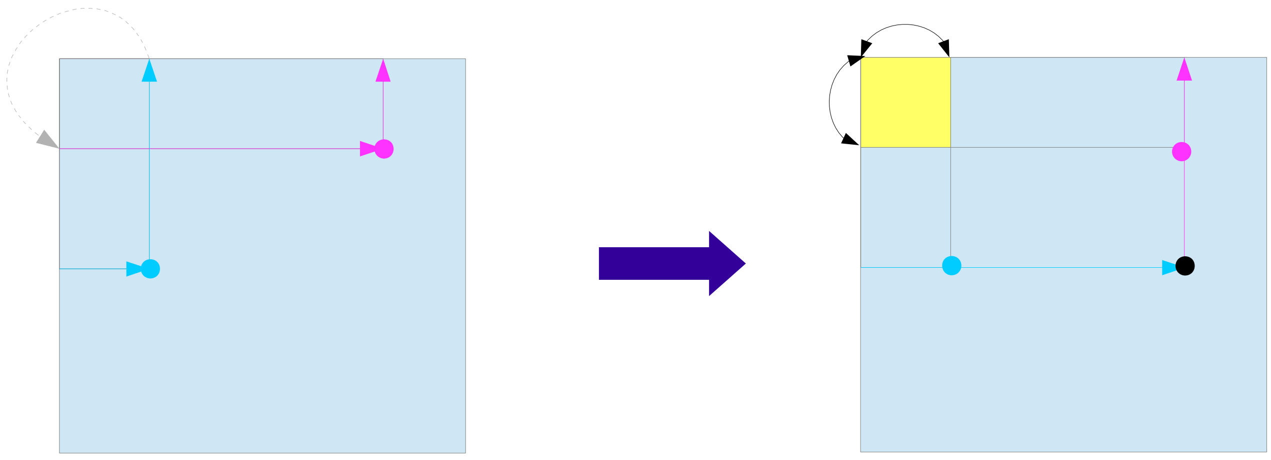
Anti-Symmetric Relation Examples





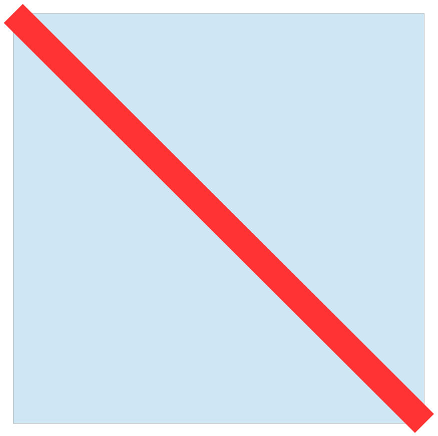
https://en.wikipedia.org/wiki/Cartesian_product

Transitive Relation Examples



Reflexive Relation

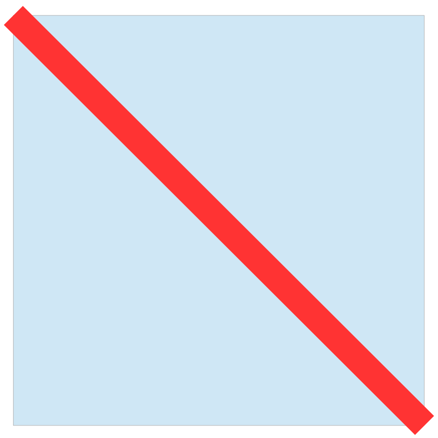
$$\forall x \quad (x, x) \in R$$



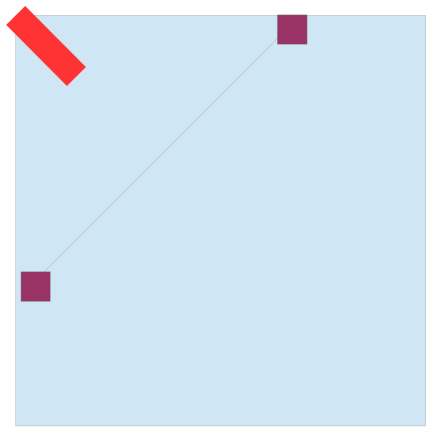
Symmetric Relation

$$\forall x, \forall y \; [\; (x, y) \in R \rightarrow (y, x) \in R \;]$$

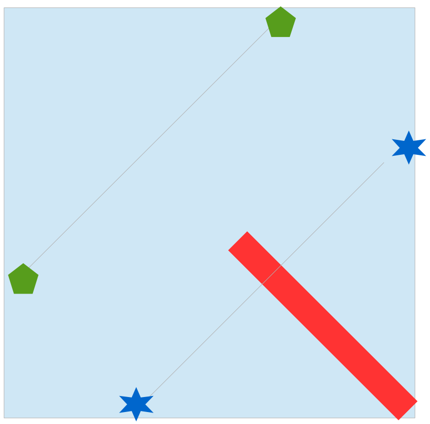
symmetric

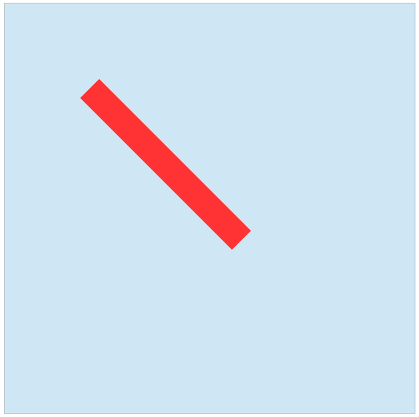


symmetric

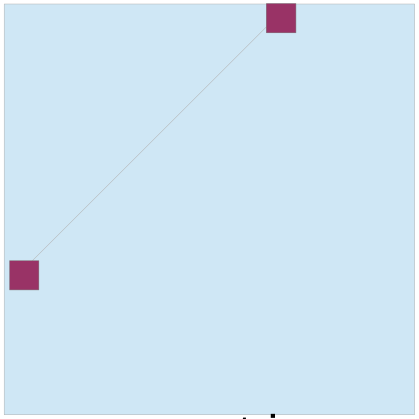


symmetric

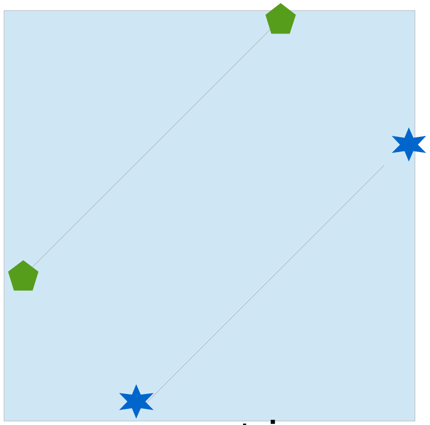




symmetric



symmetric



symmetric

Not Symmetric Relation

$$\neg\{\ \forall x, \forall y\ [\ (x,\ y) \in R\ ] \rightarrow [\ (y,\ x) \in R\ ]\ \}$$

$$\exists x, \exists y\ \neg\{\ [\ (x,\ y) \in R\ ] \rightarrow [\ (y,\ x) \in R\ ]\ \}$$

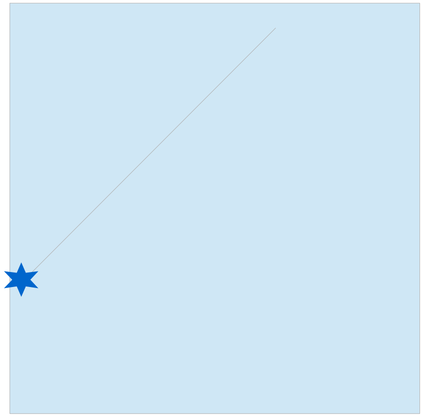
$$\exists x, \exists y\ \neg\{\ \neg[\ (x,\ y) \in R\ ] \lor [\ (y,\ x) \in R\ ]\ \}$$

$$\exists x, \exists y\ [\ (x,\ y) \in R\ ] \land \neg[\ (y,\ x) \in R\ ]$$

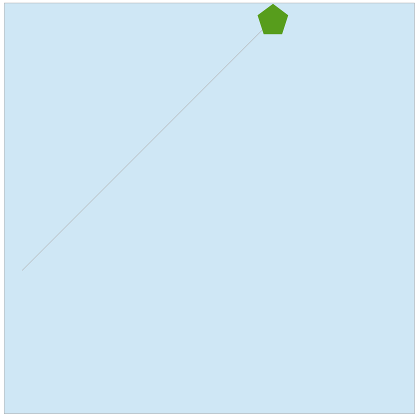
$$\exists x, \exists y\ [\ (x,\ y) \in R\ ] \land [\ (y,\ x) \notin R\ ]$$

counter example

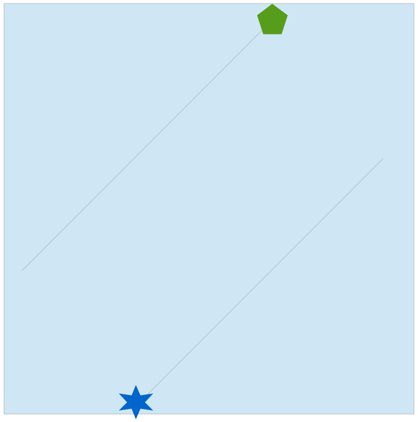
not symmetric



not symmetric



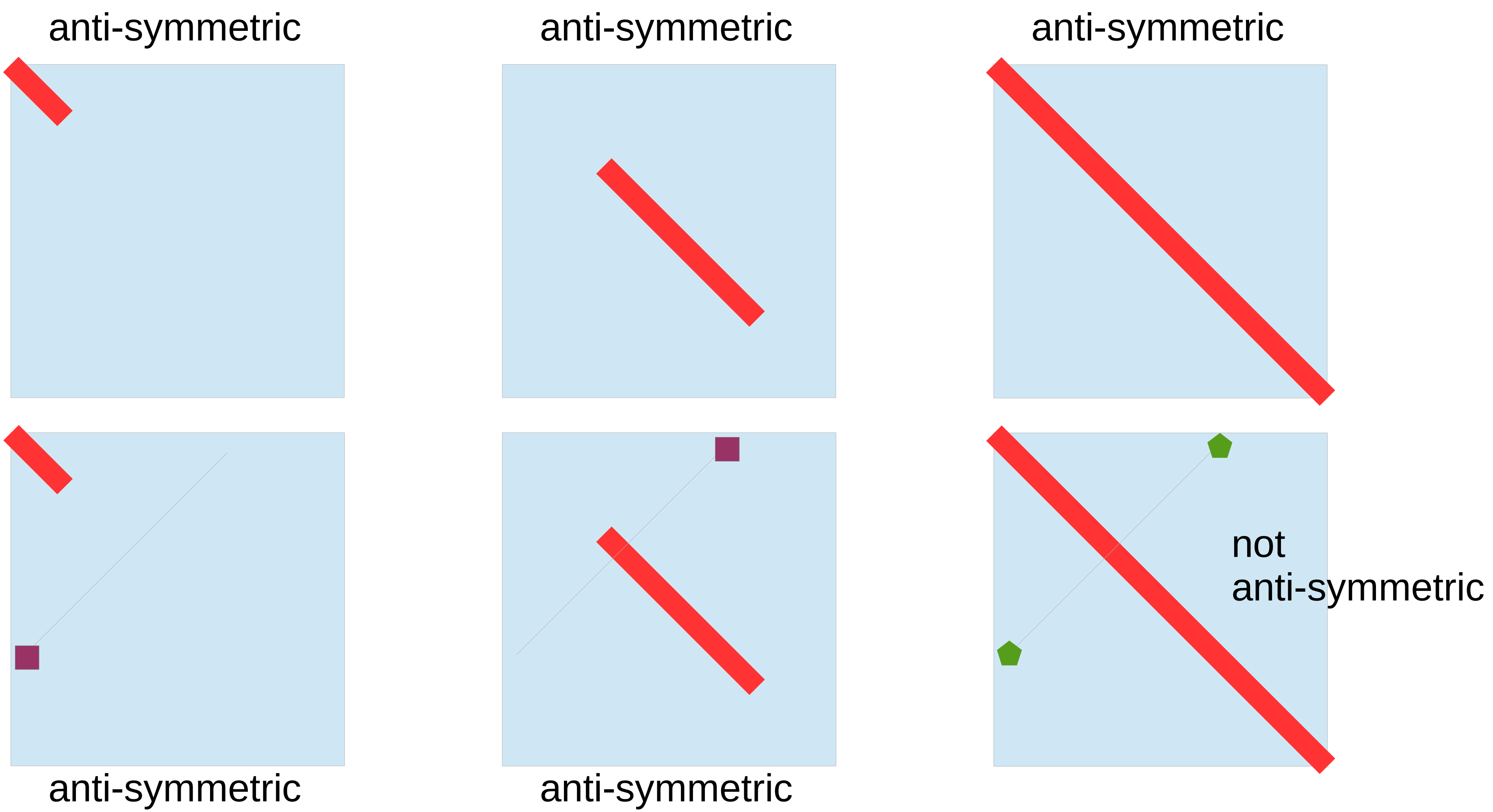
not symmetric



Young Won Lim
5/10/17

Anti-symmetric Relation

$$\forall x, \forall y \ [((x, y) \in R \land (y, x) \in R) \rightarrow x = y]$$



Young Won Lim
5/10/17

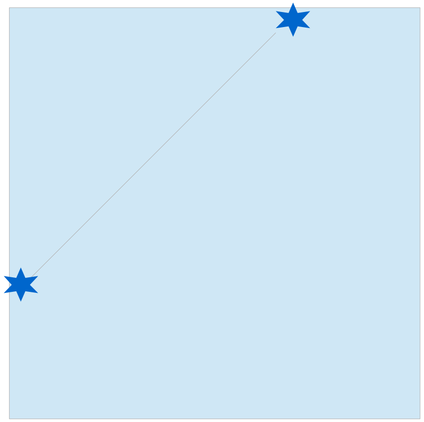
Not Anti-symmetric Relation

$$\forall x, \forall y \ \left[\ (x,\ y) \in R \ \land \ (y,\ x) \in R \ \right] \rightarrow \left[\ x = y \ \right]$$

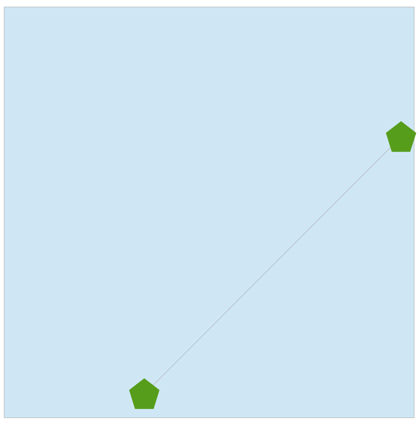
$$\exists x, \exists y \neg \{ \left[\ (x,\ y) \in R \ \land \ (y,\ x) \in R \ \right] \rightarrow \left[\ x = y \ \right] \}$$
$$\exists x, \exists y \neg \{ \neg \left[\ (x,\ y) \in R \ \land \ (y,\ x) \in R \ \right] \lor \left[\ x = y \ \right] \}$$
$$\exists x, \exists y \{ \left[\ (x,\ y) \in R \ \land \ (y,\ x) \in R \ \right] \land \neg \left[\ x = y \ \right] \}$$
$$\exists x, \exists y \{ \left[\ (x,\ y) \in R \ \land \ (y,\ x) \in R \ \right] \land \left[\ x \neq y \ \right] \}$$

counter example

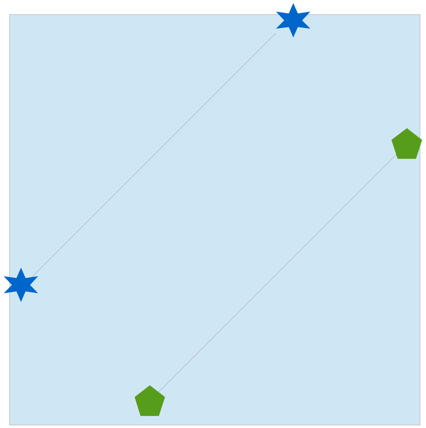
not anti-symmetric



not anti-symmetric



not anti-symmetric



Equivalent Anti-symmetric Relation

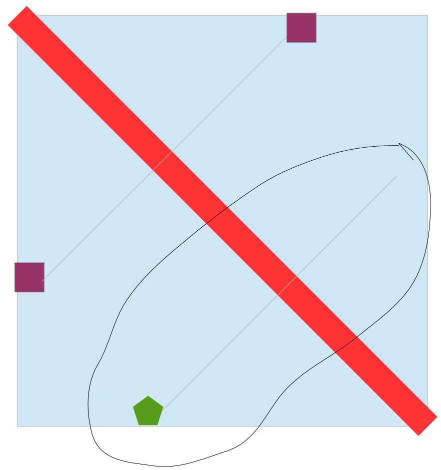
$$\forall x, \forall y \; [\; (x, y) \in R \;\wedge\; (y, x) \in R \;] \rightarrow [\; x = y \;]$$

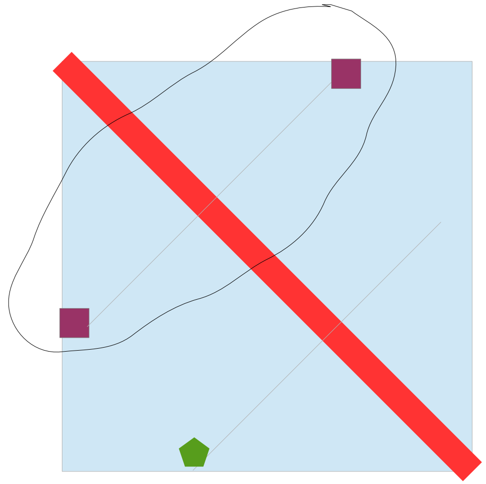
$$\forall x, \forall y \; [\; x \neq y \;] \rightarrow [\; (x, y) \notin R \;\vee\; (y, x) \notin R \;]$$

No symmetric relation is allowed

neither symmetric

nor anti-symmetric



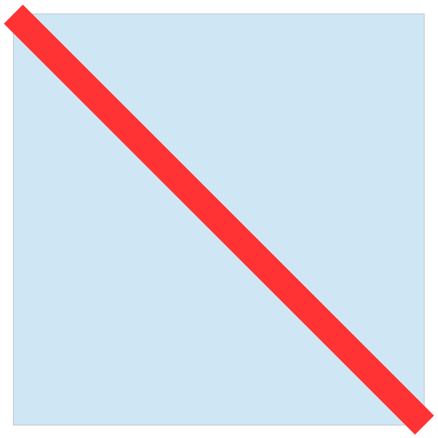


Reflexive, Symmetric, Anti-symmetric

$$\forall x \quad (x, x) \in R$$

$$\forall x, \forall y \ \left[\ (x, y) \in R\ \right] \rightarrow \left[\ (y, x) \in R\ \right]$$

$$\forall x, \forall y \ \left[\ (x, y) \in R \ \wedge \ (y, x) \in R\ \right] \rightarrow \left[\ x = y\ \right]$$



Reflexive
Also, symmetric (no relation for (x, y) where x≠y)
Also, anti-symmetric (no relation for (x, y) where x≠y)

not symmetric $\neq$ anti-symmetric
not anti-symmetric $\neq$ symmetric

Matrix of a Relation

$R_1 \in \{(1,a), (2,b), (3,a), (3,b)\}$

$$A_1 = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

$R_2 \in \{(a,x), (a,y), (b,y), (b,z)\}$

$$A_2 = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \end{array}$$

Sufficient Part

$$A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_1 = \begin{array}{c|c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

$$A_2 = \begin{array}{c|c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{c|cc} & a & b \\ i & s & t \end{array} \qquad \begin{array}{c|c} & j \\ a & u \\ b & v \end{array}$$

i, j: $su + tv$

$i \in \{1, 2, 3\}$

$j \in \{x, y, z\}$

$s \in \{0, 1\}$

$t \in \{0, 1\}$

$u \in \{0, 1\}$

$v \in \{0, 1\}$

$su + tv \neq 0$

$su = 1$	$tv = 1$
$(s = 1)$ $(u = 1)$	$(t = 1)$ $(v = 1)$

$$\frac{(i, a) \in R_1 \quad (a, j) \in R_2}{(i, j) \in R_2 \circ R_1} \qquad \frac{(i, b) \in R_1 \quad (b, j) \in R_2}{(i, j) \in R_2 \circ R_1}$$

Necessary Part

$$A_1 A_2 = \begin{bmatrix} 1 & 0 \\ \boxed{0} & \boxed{1} \\ 1 & 1 \end{bmatrix} \begin{array}{c} \\ \begin{bmatrix} 1 & \boxed{1} & 0 \\ 0 & \boxed{1} & 1 \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & \boxed{1} & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_1 = \begin{array}{c|cc} & a & b \\ \hline 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 1 \end{array}$$

$$A_2 = \begin{array}{c|ccc} & x & y & z \\ \hline a & 1 & 1 & 0 \\ b & 0 & 1 & 1 \end{array}$$

$$i \rightarrow \begin{array}{cc} a & b \\ \boxed{s} & \boxed{t} \end{array} \qquad \begin{array}{cc} & j \\ a & u \\ b & v \end{array}$$

$$i \rightarrow \boxed{} \rightarrow su + tv$$

$i \in \{1, 2, 3\}$

$j \in \{x, y, z\}$

$s \in \{0, 1\}$

$t \in \{0, 1\}$

$u \in \{0, 1\}$

$v \in \{0, 1\}$

$(i, j) \in R_2 \circ R_1 \Rightarrow$

$(i, a) \in R_1$	$(i, b) \in R_1$
$(a, j) \in R_2$	$(b, j) \in R_2$
$(s = 1)$	$(t = 1)$
$(u = 1)$	$(v = 1)$
$su = 1$	$tv = 1$

$su + tv \neq 0$

Necessary Part

$$A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

$$A_2 = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$i \rightarrow \begin{array}{cc} a & b \\ s & t \end{array} \qquad \begin{array}{cc} & j \\ a & u \\ b & v \end{array}$$

$$i \rightarrow \begin{array}{c} j \\ \boxed{} \end{array} \rightarrow su+tv$$

$i \in \{1, 2, 3\}$

$j \in \{x, y, z\}$

$s \in \{0, 1\}$

$t \in \{0, 1\}$

$u \in \{0, 1\}$

$v \in \{0, 1\}$

$su+tv \neq 0$ *nonzero* $(i, j)^{th}$ *element of* $A_1 A_2$

$\Leftrightarrow (i, j) \in R_2 \circ R_1$

Transitivity Test

$$A = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & * & * & * & * \\ 2 & * & \boxed{*} & * & * \\ 3 & * & * & * & * \\ 4 & * & * & * & * \end{array}$$

$$A^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \\ \begin{bmatrix} * & * & * & * \\ a & b & c & d \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \\ \begin{bmatrix} * & e & * & * \\ * & f & * & * \\ * & g & * & * \\ * & h & * & * \end{bmatrix} \end{array} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & * & * & * & * \\ 2 & * & \boxed{*} & * & * \\ 3 & * & * & * & * \\ 4 & * & * & * & * \end{array}$$

$$nonzero\ (i,j)^{th}\ element\ of\ A^2 \quad \Rightarrow \quad nonzero\ (i,j)^{th}\ element\ of\ A$$

$ae = 1$	$bf = 1$	$cg = 1$	$dh = 1$	
$(2,1) \in R$	$(2,2) \in R$	$(2,3) \in R$	$(2,4) \in R$	$\Rightarrow (2,2) \in R$
$(1,2) \in R$	$(2,2) \in R$	$(3,2) \in R$	$(4,2) \in R$	

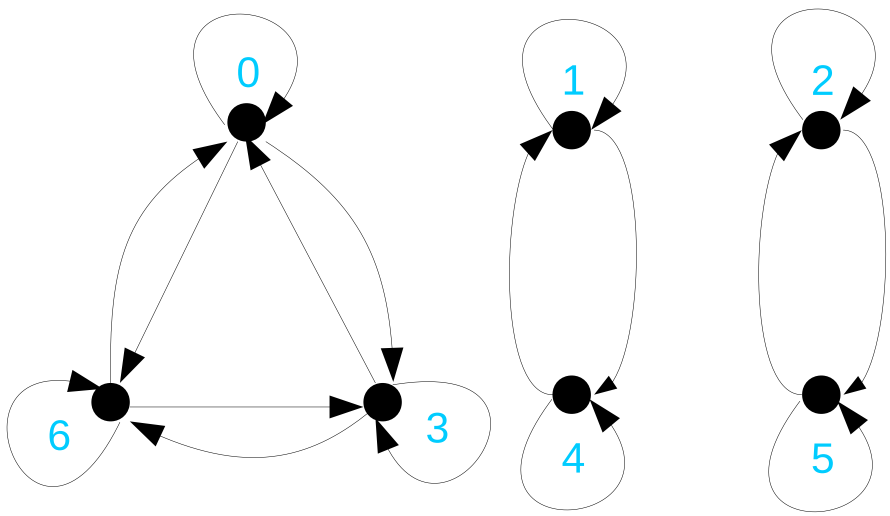
Binary Relations and Digraphs

$$A = \{0,1,2,3,4,5,6\}$$

$$R \subset A \times A$$

$$R = \{(a,b) \mid a \equiv b \pmod 3\}$$

$$R = \begin{array}{c|c} & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



http://www.math.fsu.edu/~pkirby/mad2104/SlideShow/s7_1.pdf

Young Won Lim
5/10/17

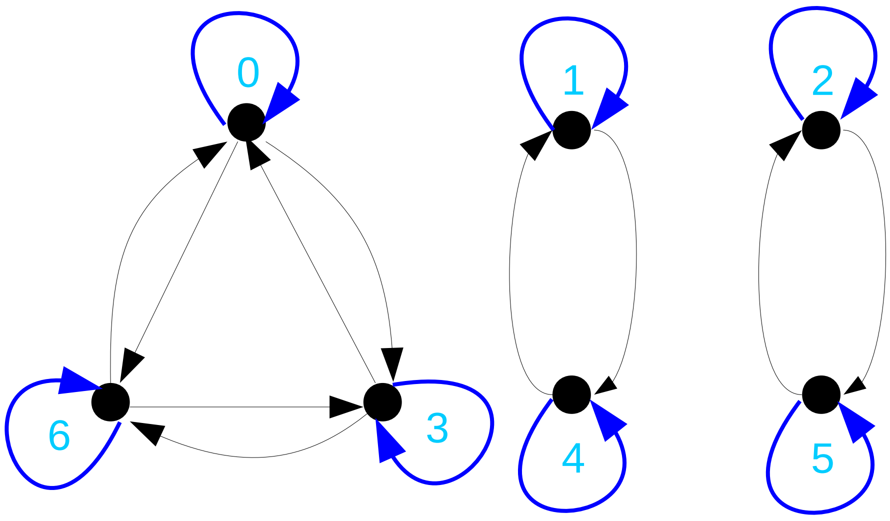
Reflexive Relation

$$A = \{0,1,2,3,4,5,6\}$$

$$R \subset A \times A$$

$$R = \{(a,b) \mid a \equiv b \ (mod\ 3)\}$$

$$R = \begin{array}{c|c} & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



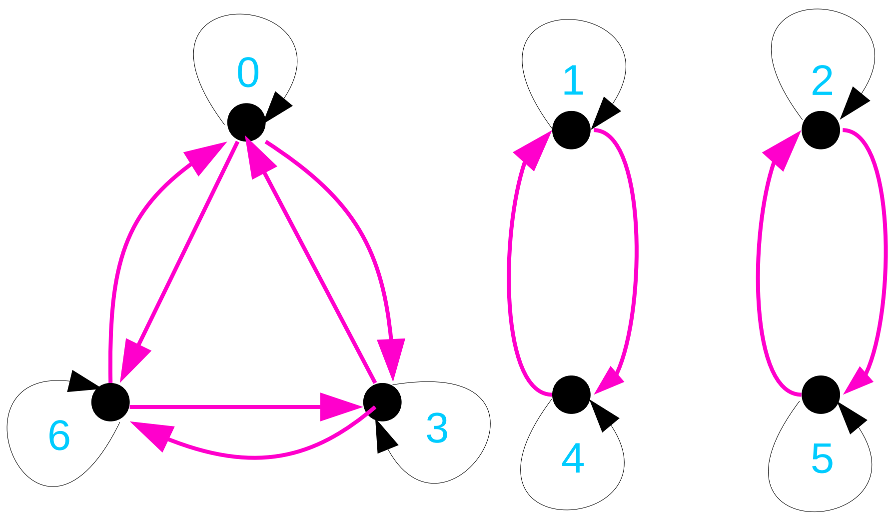
Symmetric Relation

$$A = \{0,1,2,3,4,5,6\}$$

$$R \subset A \times A$$

$$R = \{(a,b) \mid a \equiv b \ (mod\ 3)\}$$

$$\begin{array}{cc} & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ R = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



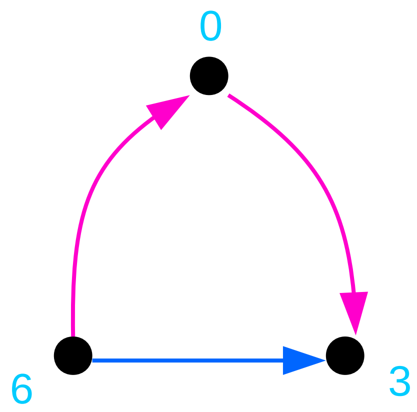
Transitive Relation

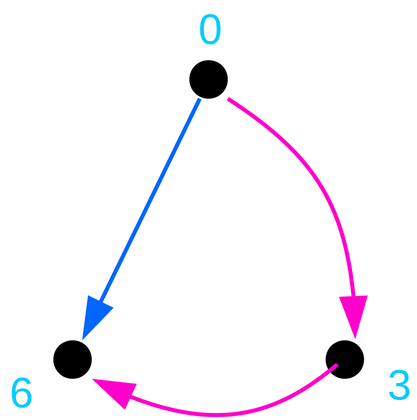
$$A = \{0,1,2,3,4,5,6\}$$

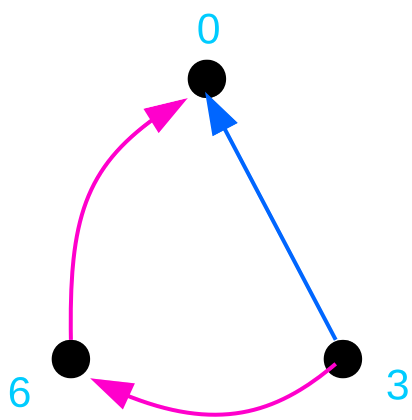
$$R \subset A \times A$$

$$R = \{(a,b) \mid a \equiv b \ (mod\ 3)\}$$

$$RR = \begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 0 & 3 & 0 & 0 & 3 & 0 & 0 & 3 \\ 1 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 \\ 3 & 3 & 0 & 0 & 3 & 0 & 0 & 3 \\ 4 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 5 & 0 & 0 & 2 & 0 & 0 & 2 & 0 \\ 6 & 3 & 0 & 0 & 3 & 0 & 0 & 3 \end{array}$$







Equivalence Relation

$$A = \{0, 1, 2, 3, 4, 5, 6\}$$

$$R \subset A \times A$$

$$R = \{(a, b) \mid a \equiv b \ (mod\ 3)\}$$

Equivalence Relation $\Leftrightarrow$ Reflexive Relation &
Symmetric Relation &
Transitive Relation

Equivalence Class

$A = \boldsymbol{Z}^{+} = \{0, 1, 2, 3, 4, 5, 6, \cdots\}$

$R \subset A \times A$

$R = \{(a, b) \mid a \equiv b \ (mod\ 3)\}$

$\{0,\ 3,\ 6,\ 9,\ \cdots\}$	$[0]$	$[33]$
$\{1,\ 4,\ 7,\ 10,\ \cdots\}$	$[1]$	$[331]$
$\{2,\ 5,\ 8,\ 11,\ \cdots\}$	$[2]$	$[3332]$

https://www.cse.iitb.ac.in/~nutan/courses/cs207-12/notes/lec7.pdf

References

[1] http://en.wikipedia.org/
[2]