Vector Calculus (H.1)
Curl & Div Operators

20160310

∇ $\boxed{\text{Scalar function}}$ grad(f)

$\nabla \cdot$ $\boxed{\text{Vector field}}$ div ($\vec{F}$)

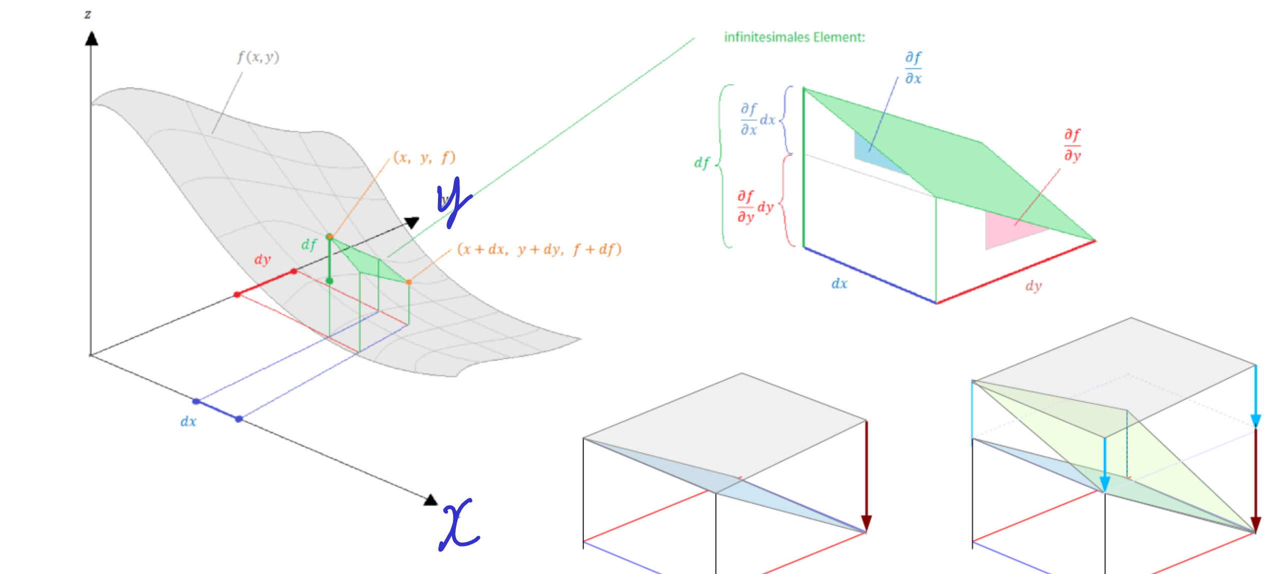
$\nabla \times$ $\boxed{\text{Vector field}}$ curl ($\vec{F}$)

Vector field $\vec{F} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$

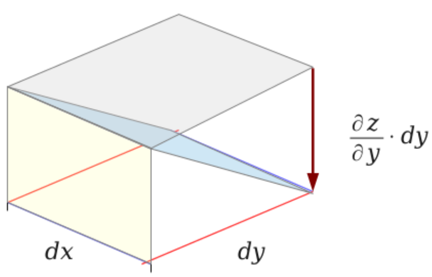
Gradient Vector field $\vec{F} = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

$= \nabla f$

Total Differential



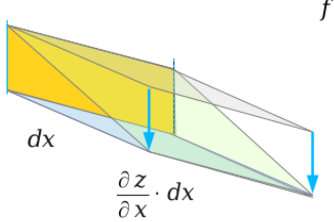
http://de.wikipedia.org/wiki/Totales_Differential
Muhammet Cakir

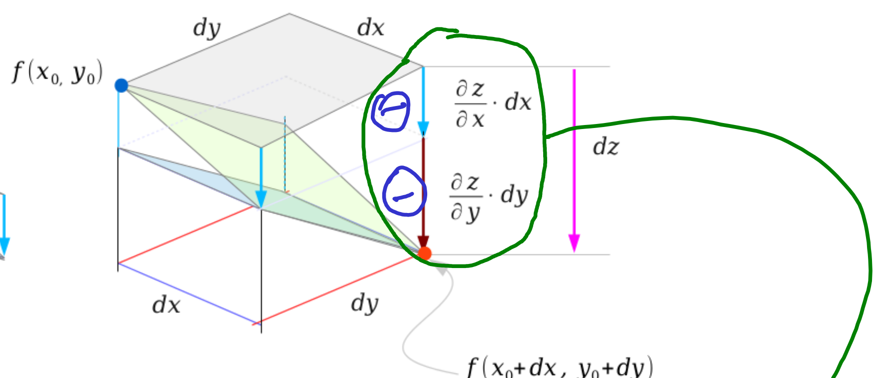


$$z = f(x, y)$$

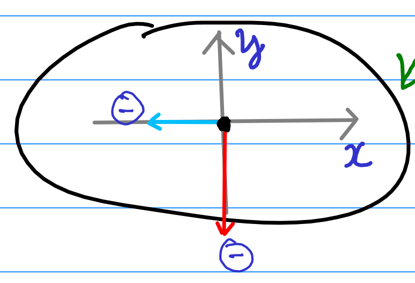
$$\boxed{dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy}$$

differential, or
total differential

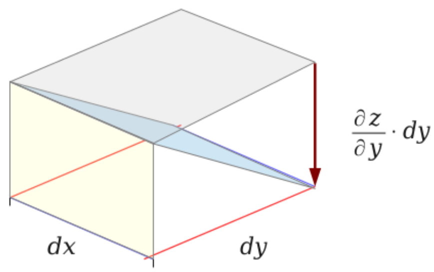




a vector representation



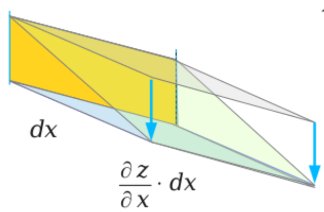
Gradient Vector Field

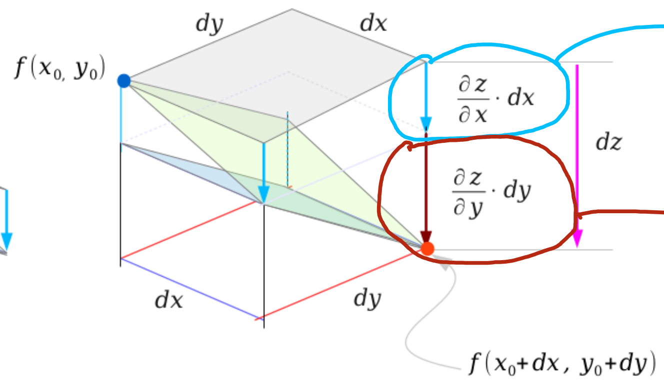


$z = f(x, y)$

$$\boxed{dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy}$$

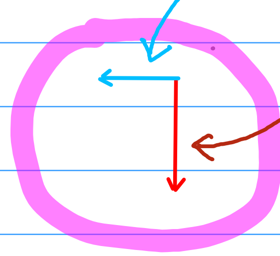
differential, or total differential



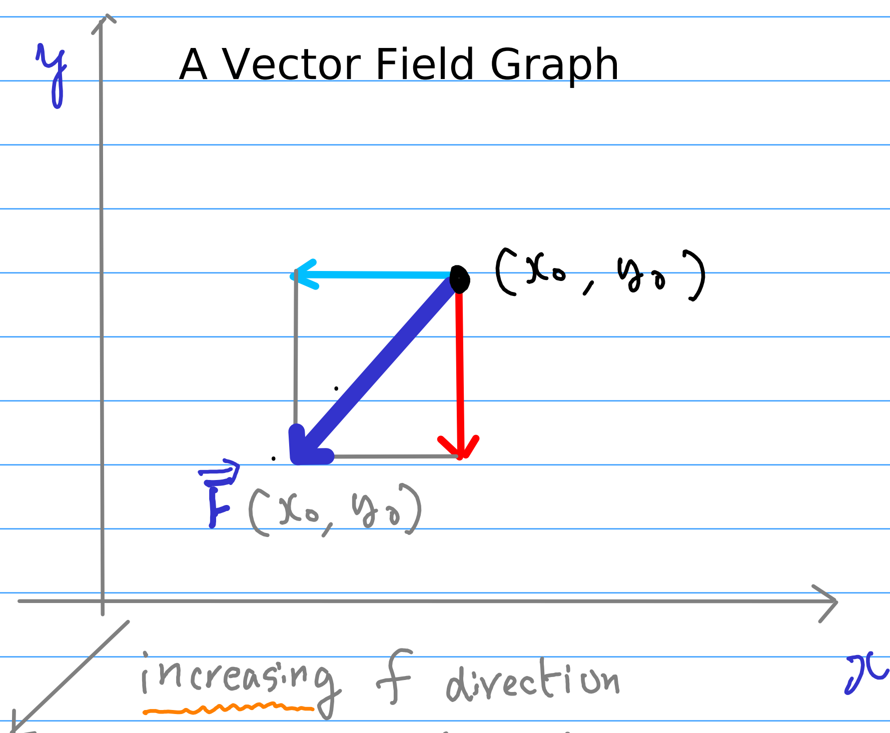


$\frac{\partial f}{\partial x} < 0 \Rightarrow$ a negative x direction vector

$\frac{\partial f}{\partial y} < 0 \Rightarrow$ a negative y direction vector



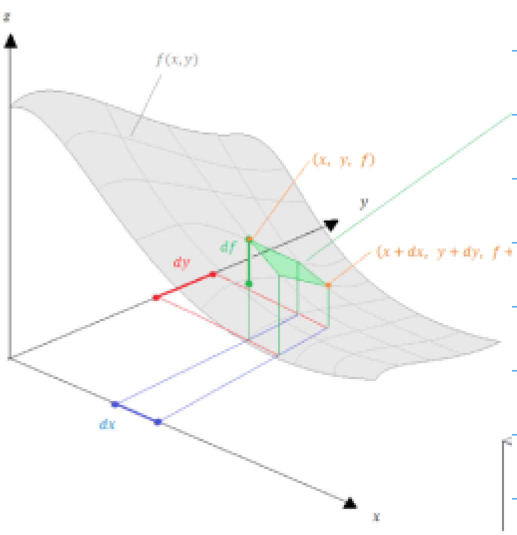
A Vector Field Graph

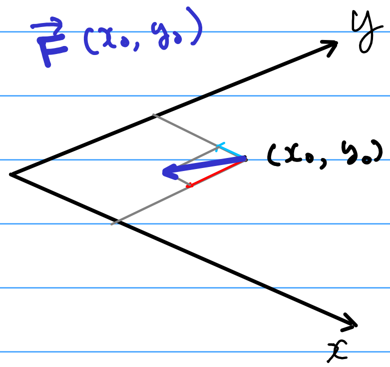


Gradient Vector Field

$f(x,y) \rightarrow$ 3D graph

$\vec{F}(x,y) \rightarrow$ 2D graph





mathematics, the **gradient** is a generalization of the usual concept of derivative of a function in one dimension to a function in several dimensions. If $f(x_1, \ldots, x_n)$ is a differentiable, scalar-valued function of standard Cartesian coordinates in Euclidean space, its gradient is the vector whose components are the *n* partial derivatives of *f*. It is thus a vector-valued function.

Similarly to the usual derivative, the gradient represents the slope of the tangent of the graph of the function. More precisely, the gradient points in the direction of the greatest rate of increase of the function and its magnitude is the slope of the graph in that direction. The components of the gradient in coordinates are the coefficients of the variables in the equation of the tangent space to the graph. This characterizing property of the gradient allows it to be defined independently of a choice of coordinate system, as a vector field whose components in a coordinate system will transform when going from one coordinate system to another.

wikipedia.org

Del Operator and a scalar function

	$f(x,y)$	$f(x,y,z)$
	$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}$	$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$
$\boxed{\text{Gradient vector}}$	$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$	$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

$$\boxed{f} \xrightarrow{\nabla f} \boxed{\vec{F}}$$

a scalar function

a gradient vector field

$$f(x,y) \xrightarrow{\nabla f} \vec{F}(x\ y) \quad R^2$$

$$f(x,y,z) \xrightarrow{\nabla f} \vec{F}(x,y,z) \quad R^3$$

a Gradient Vector Field

$$\boxed{f(x,y)} \xrightarrow{\nabla f} \boxed{\vec{F}(x,y)}$$

2 Variable function

2 component vector in R^2

: 2 component functions

each 2 variable function

This a gradient vector field

not just a vector field

in R^2

$$\boxed{\begin{aligned} P(x,y) &= \frac{\partial f}{\partial x} \\ Q(x,y) &= \frac{\partial f}{\partial y} \end{aligned}}$$

$$\boxed{f(x,y,z)} \xrightarrow{\nabla f} \boxed{\vec{F}(x,y,z)}$$

3 Variable function

3 component vector in R^3

: 3 component functions

each 3 variable functio

This a gradient vector field

not just a vector field

in R^3

$$\boxed{\begin{aligned} P(x,y,z) &= \frac{\partial f}{\partial x} \\ Q(x,y,z) &= \frac{\partial f}{\partial y} \\ R(x,y,z) &= \frac{\partial f}{\partial z} \end{aligned}}$$

a Vector Field

~~Gradient~~ Vector Field

$\cancel{f(x, y)}$ $\xrightarrow{\nabla f}$ $\boxed{\vec{F}(x, y)}$

2 Variable function

2 component vector in $\mathbb{R}^2$
: 2 component functions
each 2 variable function

even when we
can't find $f(x, y)$
such that

$$\begin{cases} P(x, y) = \frac{\partial f}{\partial x} \\ Q(x, y) = \frac{\partial f}{\partial y} \end{cases}$$

$P(x, y)$
$Q(x, y)$

$\longrightarrow$ Still a vector field $\vec{F}(x, y)$

$\cancel{f(x, y, z)}$ $\xrightarrow{\nabla f}$ $\boxed{\vec{F}(x, y, z)}$

3 Variable function

3 component vector in $\mathbb{R}^3$
: 3 component functions
each 3 variable functio

even when we
can't find $f(x, y)$
such that

$$P(x, y, z) = \frac{\partial f}{\partial x}$$
$$Q(x, y, z) = \frac{\partial f}{\partial y}$$
$$R(x, y, z) = \frac{\partial f}{\partial z}$$

$P(x, y, z)$
$Q(x, y, z)$
$R(x, y, z)$

$\longrightarrow$ Still a vector field $\vec{F}(x, y, z)$

2-d Gradient Vector Field

$$\boxed{f} \xrightarrow{\nabla f} \boxed{\vec{F}}$$

a scalar function

a gradient vector field

$f(x, y)$

$P\vec{i}$ $Q\vec{j}$

$\boxed{\text{2-variable scalar function } f(x,y)} \xrightarrow{\nabla f}$ vectors in $\mathbb{R}^2$

$\vec{F} = \langle$ x-directional derivatives , y-directional derivatives $\rangle$

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$P \qquad Q$

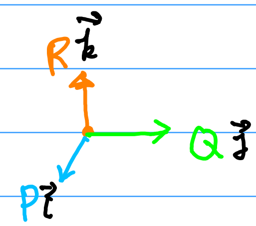
3-d Gradient Vector Field

$$\boxed{f} \xrightarrow{\nabla f} \boxed{\vec{F}}$$

a scalar function

a gradient vector field

$f(x, y, z)$



3-variable scalar function $f(x, y, z)$ $\xrightarrow{\nabla f}$ vectors in $\mathbb{R}^3$

$\vec{F} = \langle$ x-directional derivatives, y-directional derivatives, z-directional derivatives $\rangle$

$$= \left\langle \underset{P}{\frac{\partial f}{\partial x}}, \underset{Q}{\frac{\partial f}{\partial y}}, \underset{R}{\frac{\partial f}{\partial z}} \right\rangle$$

Vector Field

$\vec{F} = P\,\vec{i} + Q\,\vec{j}$	$\vec{F} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$

2D $\quad \vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$

3D $\quad \vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$

Vector in $\begin{pmatrix} 2D \\ 3D \end{pmatrix}$ $\quad \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ component functions with $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ variables

$\boxed{\text{Gradient}}$ Vector Field

$\vec{F} = \nabla f$	$\vec{F} = \nabla f$
$= \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$	$= \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$
2-D	3-D

Del Operator (vector-like, vector-making)

$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}$	$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$

Gradient Vector

$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$	$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

$$\nabla \blacksquare = \frac{\partial}{\partial x}\blacksquare\,\vec{i} + \frac{\partial}{\partial y}\blacksquare\,\vec{j}$$
$$= \frac{\partial \blacksquare}{\partial x}\vec{i} + \frac{\partial \blacksquare}{\partial y}\vec{j}$$

$$\nabla \bullet = \frac{\partial}{\partial x}\bullet\,\vec{i} + \frac{\partial}{\partial y}\bullet\,\vec{j} + \frac{\partial}{\partial z}\bullet\,\vec{k}$$
$$= \frac{\partial \bullet}{\partial x}\vec{i} + \frac{\partial \bullet}{\partial y}\vec{j} + \frac{\partial \bullet}{\partial z}\vec{k}$$

Curl & Div operators

$$\boxed{\begin{array}{l} \text{Curl } \vec{F} \\ \text{div } \vec{F} \end{array}}$$

Curl & div operates over a $\boxed{\text{Vector field } \vec{F}}$

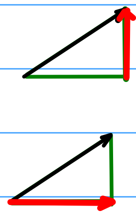
$\vec{F}$ must be a vector field

$$\begin{cases} P\vec{i} + Q\vec{j} \quad \text{or} & (2D) \\ P\vec{i} + Q\vec{j} + R\vec{k} & (3D) \end{cases}$$

Curl & Div operators

$\text{Curl } \vec{F}$: ∇ cross $\times$ product with $\vec{F}$

$\text{div } \vec{F}$: ∇ inner $\cdot$ product with $\vec{F}$



$$\nabla = \begin{cases} \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} & \text{(2D)} \\ \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} & \text{(3D)} \end{cases}$$

$$\vec{F} = \begin{cases} P\vec{i} + Q\vec{j} & \text{(2D)} \\ P\vec{i} + Q\vec{j} + R\vec{k} & \text{(3D)} \end{cases}$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F}$$

Curl & Div operators

For any vector field $\vec{F}$

$$\boxed{\begin{aligned} \vec{F} &= P\vec{i} + Q\vec{j} + R\vec{k} \\ \text{Curl } \vec{F} &= \nabla \times \vec{F} \qquad \text{cross product} \\ \text{div } \vec{F} &= \nabla \cdot \vec{F} \qquad \text{inner product} \end{aligned}}$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \nabla \times \vec{F} \qquad \text{a vector}$$

$$\text{div } \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \vec{F} \qquad \text{a scalar}$$

Curl & Div over a gradient field

6 $\vec{F}$ can be a gradient vector field

$$\vec{F} = \boxed{P}\,\vec{i} + \boxed{Q}\,\vec{j} + \boxed{R}\,\vec{k}$$
$$= \boxed{\frac{\partial f}{\partial x}}\,\vec{i} + \boxed{\frac{\partial f}{\partial y}}\,\vec{j} + \boxed{\frac{\partial f}{\partial z}}\,\vec{k}$$

$\exists f$ a scalar function f s.t.

$$\vec{F} = \nabla f$$

For a gradient vector field $\vec{F}$

$$\boxed{\begin{aligned} \vec{F} &= P\vec{i} + Q\vec{j} + R\vec{k} = \nabla f \\ \text{Curl } \vec{F} &= \nabla \times \vec{F} = \boxed{\nabla \times \nabla f} \\ \text{div } \vec{F} &= \nabla \cdot \vec{F} = \boxed{\nabla \cdot \nabla f} \end{aligned}}$$

3D Curl & Div

a Vector field $\vec{F}$

$$\vec{F} = \left(\boxed{P}\,\vec{i} + \boxed{Q}\,\vec{j} + \boxed{R}\,\vec{k}\right)$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \times \left(\boxed{P}\,\vec{i} + \boxed{Q}\,\vec{j} + \boxed{R}\,\vec{k}\right)$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(\boxed{P}\,\vec{i} + \boxed{Q}\,\vec{j} + \boxed{R}\,\vec{k}\right)$$

$$\vec{F} = \left(\boxed{P}\,\vec{i} + \boxed{Q}\,\vec{j} + \boxed{R}\,\vec{k}\right)$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)$$

a gradient Vector field $\vec{F}$ $\quad (\vec{F} = \nabla f)$

$$\vec{F} = \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right)$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\vec{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\vec{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{k}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right)$$

2D Curl & Div

a Vector field $\vec{F}$

$$\vec{F} = \left(\boxed{P}\,\vec{i} + \boxed{Q}\,\vec{j} + 0\,\vec{k}\right)$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + 0\,\vec{k}\right) \times \left(\boxed{P}\,\vec{i} + \boxed{Q}\,\vec{j} + 0\,\vec{k}\right)$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + 0\,\vec{k}\right) \cdot \left(\boxed{P}\,\vec{i} + \boxed{Q}\,\vec{j} + 0\,\vec{k}\right)$$

$$\vec{F} = \left(\boxed{P}\,\vec{i} + \boxed{Q}\,\vec{j}\right)$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)$$

a gradient Vector field $\vec{F}$

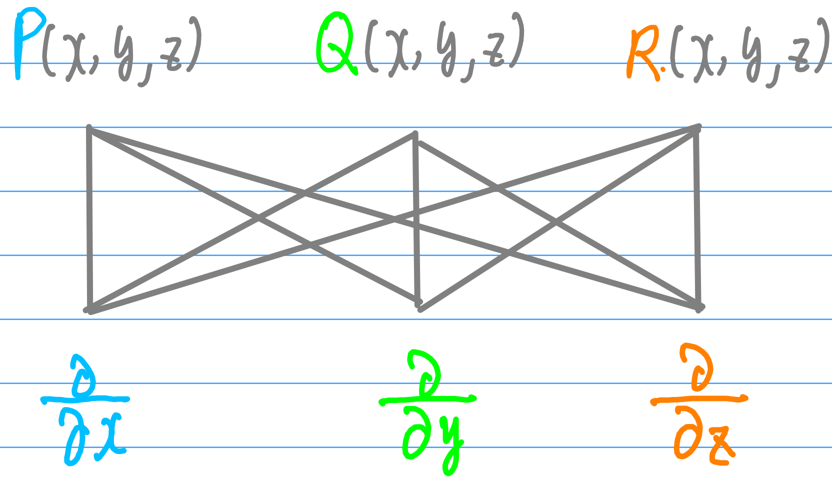
$(\vec{F} = \nabla f)$

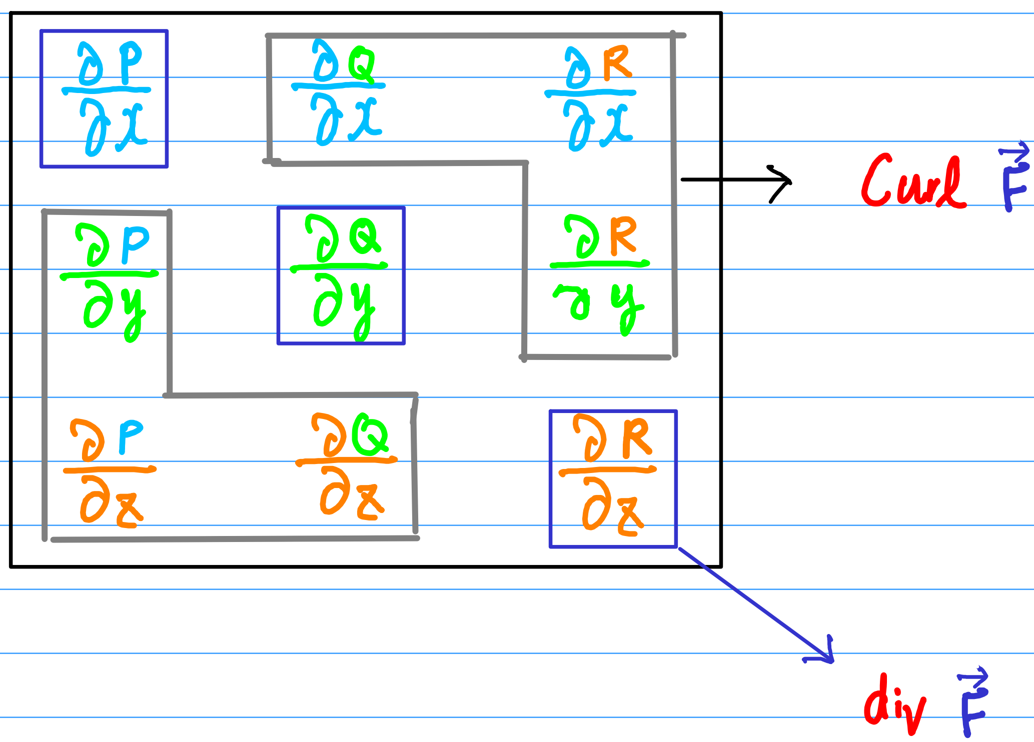
$$\vec{F} = \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}\right)$$

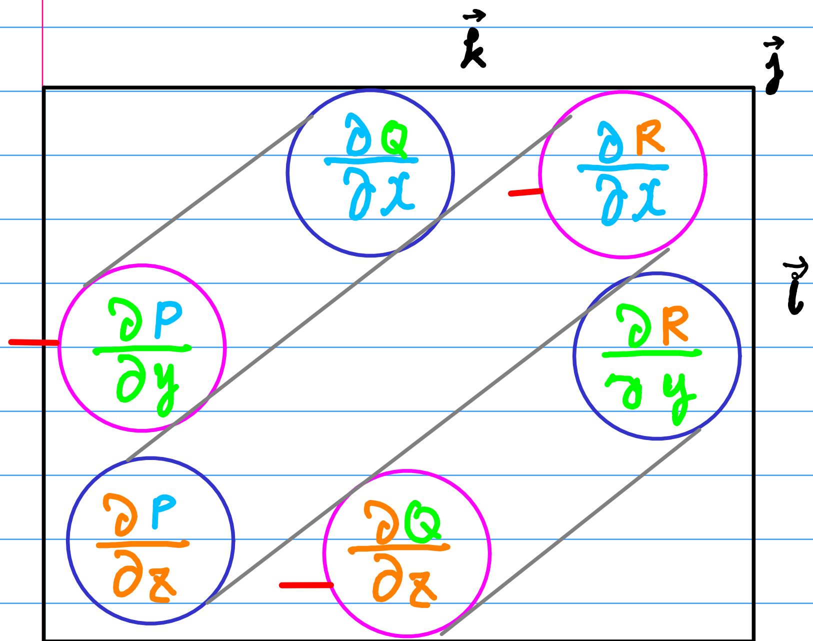
$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{k}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$$

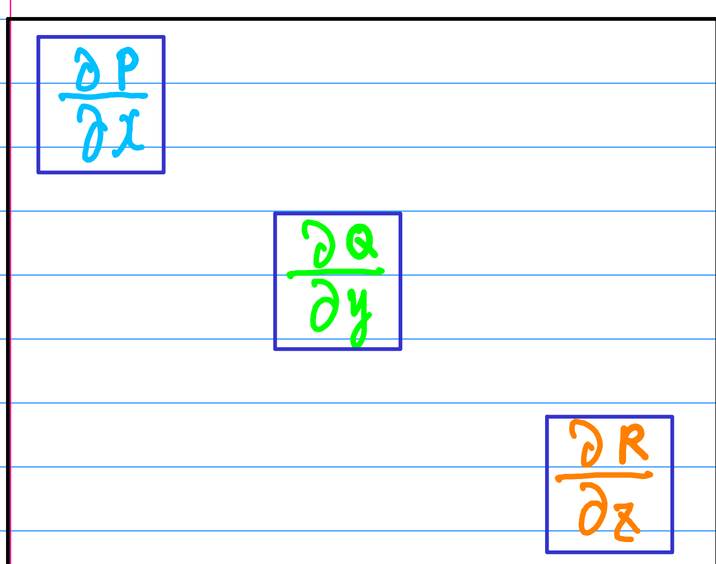
Combination of Partial Derivatives





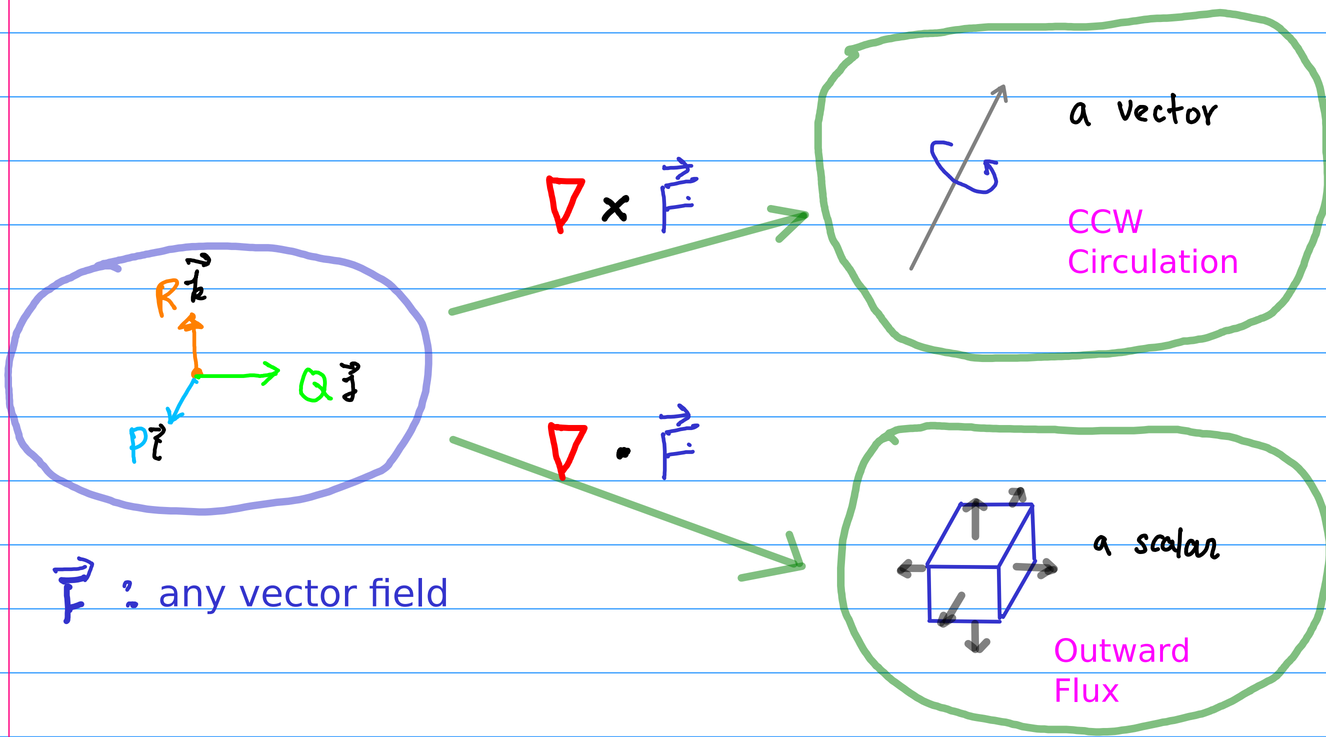


$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$



$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)$$

Physical Interpretations



Component Functions of a Vector Field

$$\boxed{\vec{F} = P\,\vec{i} + Q\,\vec{j}}$$

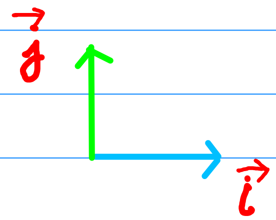
component functions

$\boxed{P(x,y)}$ $\boxed{Q(x,y)}$

partial derivatives

$\dfrac{\partial P}{\partial x}$ $\dfrac{\partial Q}{\partial y}$

$\dfrac{\partial P}{\partial y}$ $\dfrac{\partial Q}{\partial x}$



$\vec{i}$ component $\vec{j}$ component

$\boxed{P(x,y)}$ $\boxed{Q(x,y)}$

$\boxed{\dfrac{\partial}{\partial x}}$ $\boxed{\dfrac{\partial}{\partial y}}$

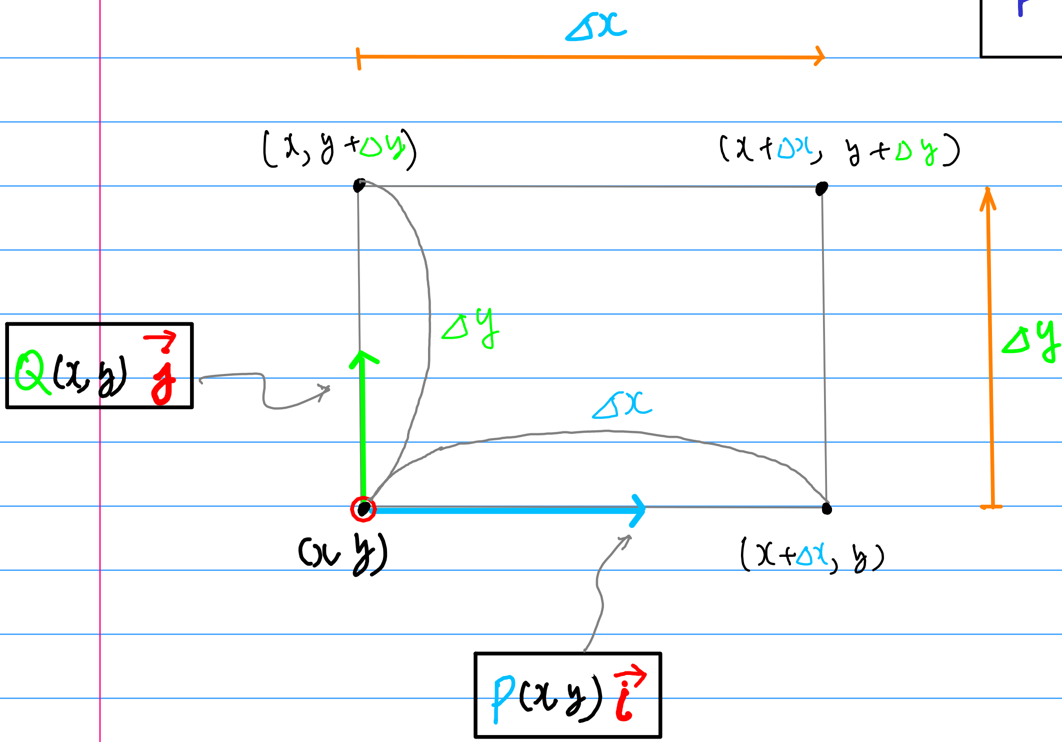
$\vec{i}$ component $\vec{j}$ component

$\boxed{P(x,y)}$ $\boxed{Q(x,y)}$

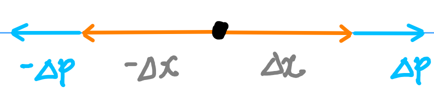
$\boxed{\dfrac{\partial}{\partial x}}$ $\boxed{\dfrac{\partial}{\partial y}}$

Vector Field & Partial Derivatives

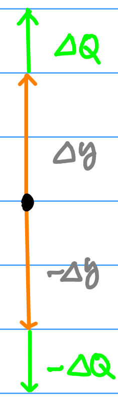
$$\vec{F} = P\vec{i} + Q\vec{j}$$



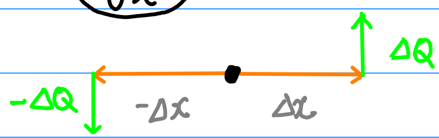
$$\frac{\partial P}{\partial x}$$



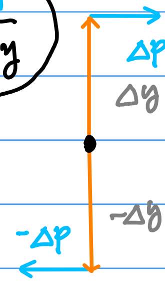
$$\frac{\partial Q}{\partial y}$$



$$\frac{\partial Q}{\partial x}$$



$$\frac{\partial P}{\partial y}$$

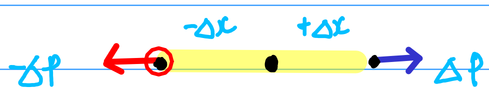


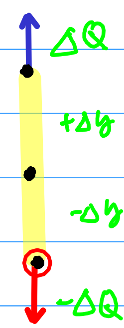
Differentials $\boxed{dP \ \& \ dQ}$

$\Delta P \ \vec{i}$ $\qquad\qquad$ $\Delta Q \ \vec{j}$

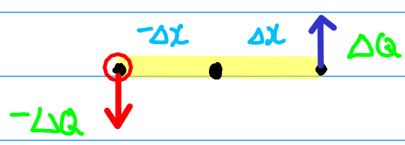
$\Delta x \ \vec{i}$ $\qquad\qquad$ $\Delta y \ \vec{j}$

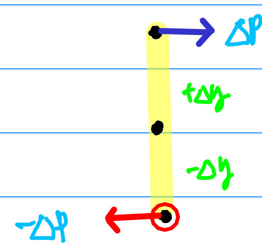
$$\boxed{dP = \frac{\partial P}{\partial x}\, dx} \qquad \boxed{dQ = \frac{\partial Q}{\partial y}\, dy}$$





$$\boxed{dQ = \frac{\partial Q}{\partial x}\, dx} \qquad \boxed{dP = \frac{\partial P}{\partial y}\, dy}$$





Interpretations of Partial Derivatives

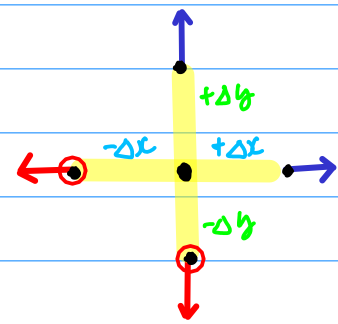
$$\boxed{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}}$$

outward flux

magnitude only

$\boxed{\Delta p\ \vec{i}}$ $\boxed{\Delta Q\ \vec{j}}$

$\boxed{\Delta x\ \vec{i}}$ $\boxed{\Delta y\ \vec{j}}$



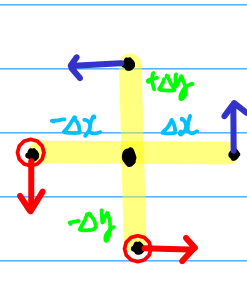
$$\boxed{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}$$

c.c.w circulation

magnitude & direction

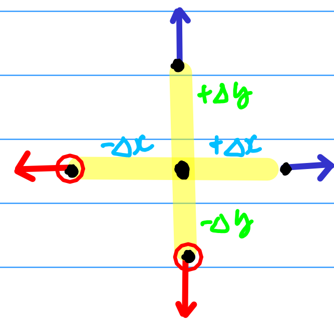
$-$ $\boxed{\Delta p\ \vec{i}}$ $\boxed{\Delta Q\ \vec{j}}$

$\boxed{\Delta x\ \vec{i}}$ $\boxed{\Delta y\ \vec{j}}$



$$\begin{array}{ccccc} \vec{i} & \leftarrow & \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} & \rightarrow & \vec{j} \\ \vec{i} & \leftarrow & & \rightarrow & \vec{j} \end{array}$$

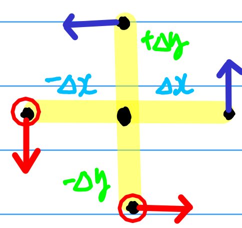
$$\boxed{\vec{i} \cdot \vec{i} = 1}$$



$$\boxed{\vec{j} \cdot \vec{j} = 1}$$

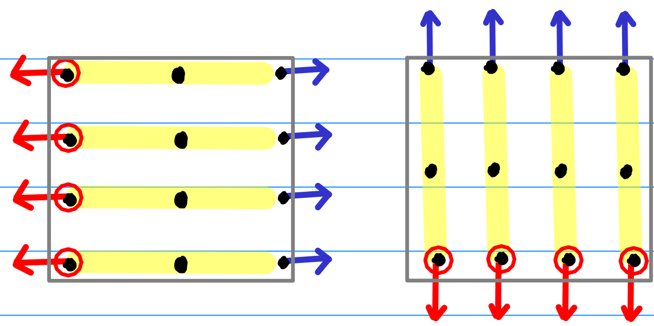
$$\begin{array}{ccccc} \vec{j} & \leftarrow & \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} & \rightarrow & \vec{i} \\ \vec{i} & \leftarrow & & \rightarrow & \vec{j} \end{array}$$

$$\boxed{\vec{i} \times \vec{j} = \vec{k}}$$

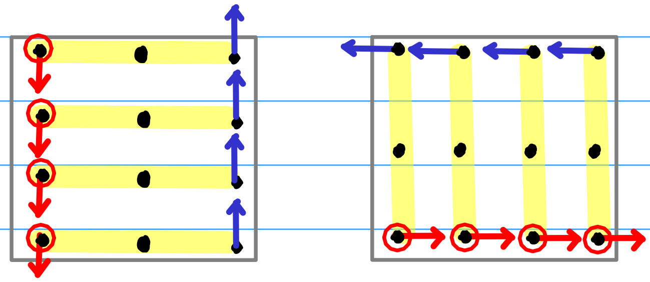


$$- \boxed{\vec{j} \times \vec{i} = -\vec{k}}$$

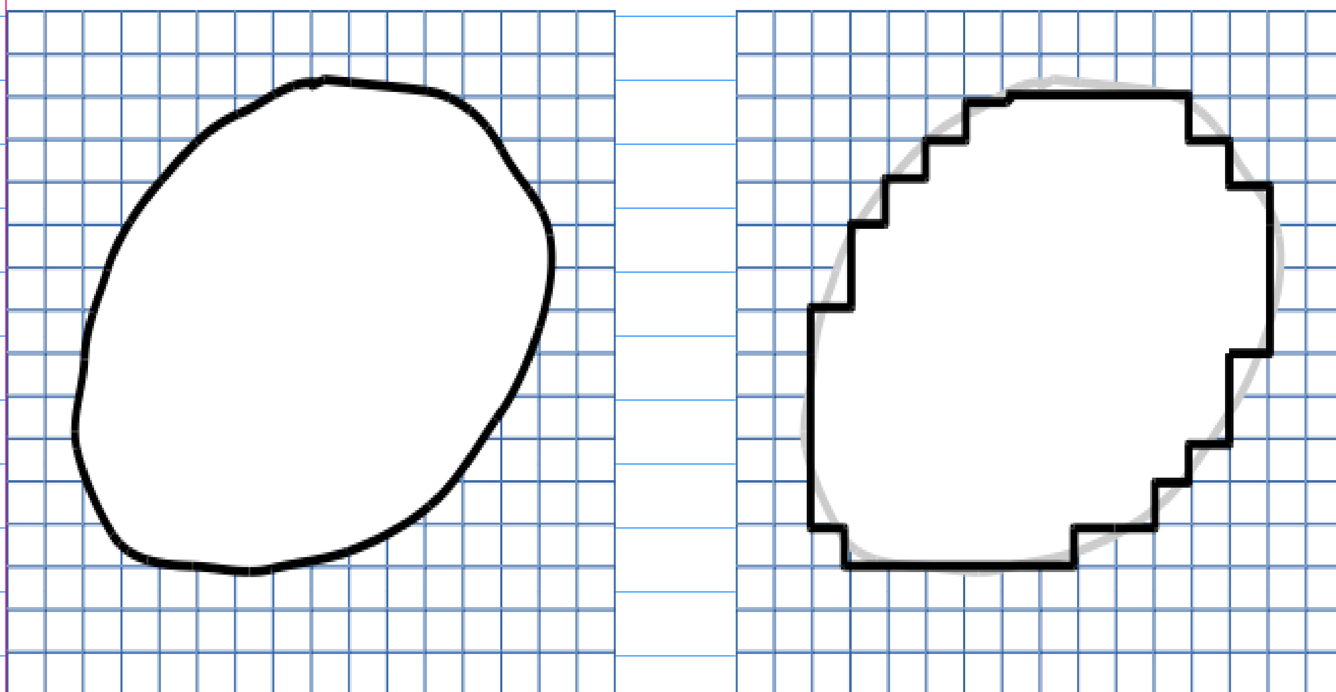
$$\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\Delta x\,\Delta y$$



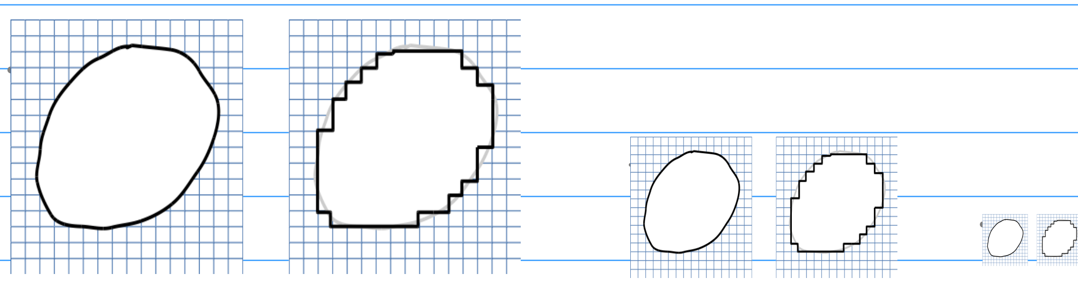
$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\Delta x\,\Delta y$$



Approximation (Digitization)



approximated by a collection of rectangles



Some Identity Equations

Gradient

Curl of the gradient

$$\nabla \times (\nabla f) = \vec{0}$$ zero vector

Divergence of the gradient

$$\nabla \cdot (\nabla f) = \boxed{\nabla^2 f}$$ Laplacian

Curl

Curl of the curl

$$\nabla \times (\nabla \times \vec{F}) = \nabla (\nabla \cdot \vec{F}) - \boxed{\nabla^2 \vec{F}}$$

Vector Laplacian

Divergence of the curl

$$\nabla \cdot (\nabla \times \vec{F}) = 0$$ zero scalar

Div

Curl of the divergence $\neq$ Curl (a vector)

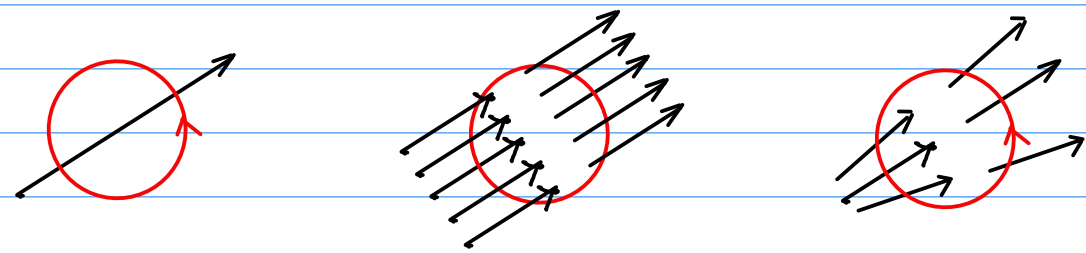
$$\cancel{\nabla \times (\nabla \cdot \vec{F})}$$ must be $\nabla \times$ (Vector)

Divergence of the divergence $\neq$ Div (a vector)

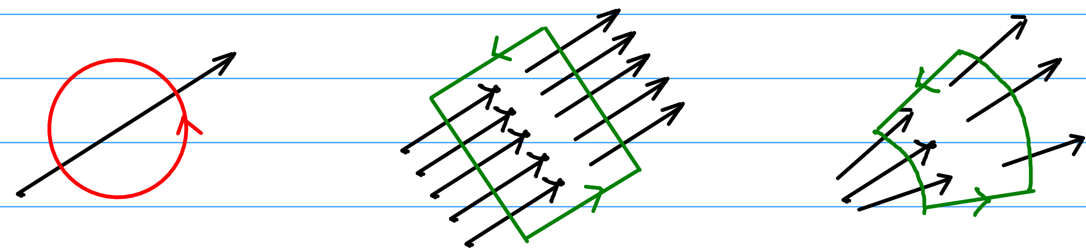
$$\cancel{\nabla \cdot (\nabla \cdot \vec{F})}$$ must be $\nabla \cdot$ (Vector)

$\nabla \cdot \vec{F}$: scalar

curl (grad)



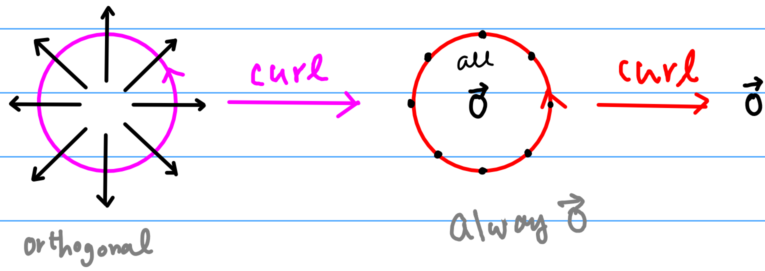
orthogonal



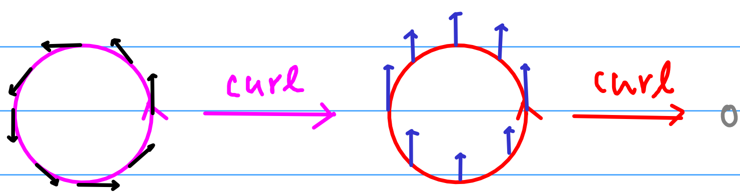
- can always find orthogonal contours
- path independence

curl (curl)

conservative field (irrotational)



non-conservative field (rotational)



in this case
curl becomes
conservative

Generally

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

$$\nabla \times (\nabla \times \vec{F}) = (\nabla \cdot \vec{F})\nabla - (\nabla \cdot \nabla)\vec{F}$$

$$= \nabla(\nabla \cdot \vec{F}) - (\nabla \cdot \nabla)\vec{F}$$

Curl(grad) = $\vec{0}$

$$\boxed{\begin{array}{l}\mathrm{Curl}(\nabla f) = \nabla \times \nabla f = \vec{0}\\ \mathrm{Curl}(\vec{F}) = \nabla \times \nabla f = \vec{0}\\ \\ \mathrm{div}(\nabla f) = \nabla \cdot \nabla f = \nabla^2 f\\ \mathrm{div}(\vec{F}) = \nabla \cdot \nabla f = \nabla^2 f\end{array}}$$

$\vec{F} = \nabla f$

$$\mathrm{Curl}\,\vec{F} = \nabla \times \vec{F}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$\vec{F} = \nabla f \Rightarrow \boxed{P = \frac{\partial f}{\partial x} \qquad Q = \frac{\partial f}{\partial y} \qquad R = \frac{\partial f}{\partial z}}$$

$$\boxed{\begin{array}{l}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = 0\\ \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \Rightarrow \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = 0\\ \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 0\end{array}}$$

Schwarz Theorem continuous 2nd order partial derivatives : symmetric

$$\operatorname{div}(\text{grad}) = \nabla^2 f$$

$$\text{Curl}(\nabla f) = \nabla \times \nabla f = \vec{0}$$

$$\text{Curl}(\vec{F}) = \nabla \times \nabla f = \vec{0}$$

$$\vec{F} = \nabla f$$

$$\boxed{\operatorname{div}(\nabla f) = \nabla \cdot \nabla f = \nabla^2 f}$$

$$\boxed{\operatorname{div}(\vec{F}) = \nabla \cdot \nabla f = \nabla^2 f}$$

$$\vec{F} = \nabla f = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) f(x, y, z)$$

$$= \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right)$$

$\boxed{\text{Laplace Operator}}$ $\quad \nabla \cdot \nabla = \nabla^2$

$$\operatorname{div}(\nabla f) = \nabla \cdot \nabla f = \nabla^2 f$$

$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right)$$

$$= \boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}}$$

$$\neq \left(\frac{\partial^2}{\partial x^2}\vec{i} + \frac{\partial^2}{\partial y^2}\vec{j} + \frac{\partial^2}{\partial z^2}\vec{k}\right) f(x, y, z)$$

$$\operatorname{div}(\text{curl}) = 0$$

$$\text{Curl}\ \vec{F} = \nabla \times \vec{F}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$\operatorname{div}\left(\text{Curl}\ \vec{F}\right) = \nabla \cdot \left(\nabla \times \vec{F}\right)$$

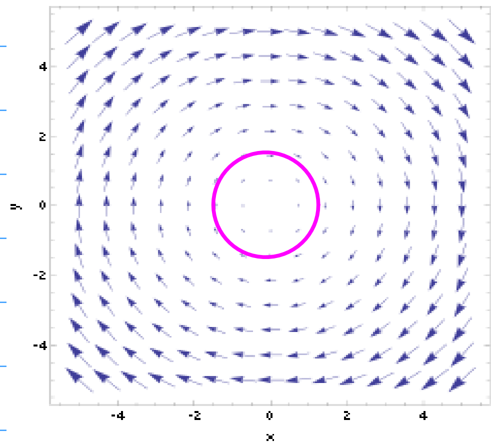
$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= 0$$

$\vec{F} = y\vec{i} - x\vec{j}$

$\mathrm{Curl}(\vec{F}) \neq 0 \qquad \vec{F} \neq \nabla f$



non-gradient vector field

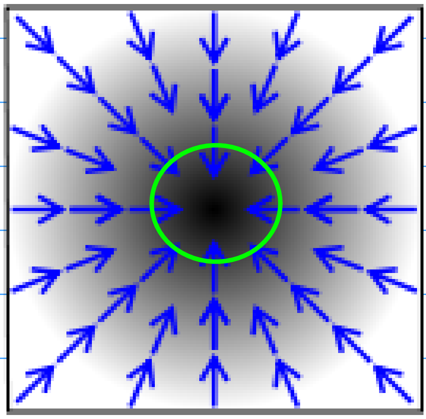
Can be rotating

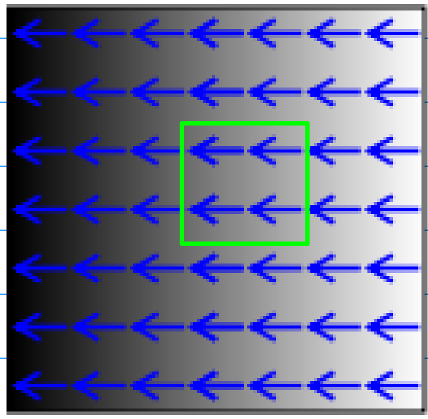
https://en.wikipedia.org/wiki/Curl_%28mathematics%29
https://en.wikipedia.org/wiki/Gradient

* Gradient Vector Field

always $\mathrm{Curl}(\nabla f) = 0$ always $\mathrm{Curl}(\nabla f) = 0$

conservative → irrotational





$\nabla^2 f = \mathrm{div}(\nabla f) < 0$ $\nabla^2 f = \mathrm{div}(\nabla f) = 0$

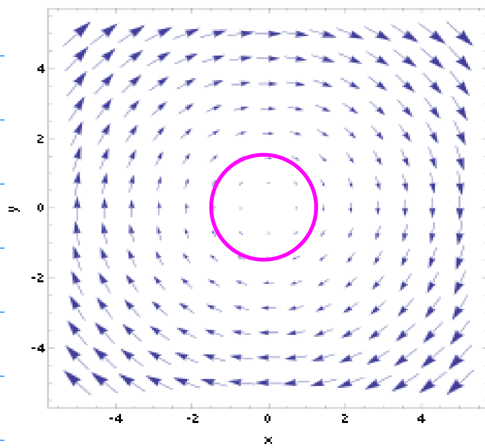
Sink   source X
sink X

$\vec{F} = y\vec{i} - x\vec{j}$

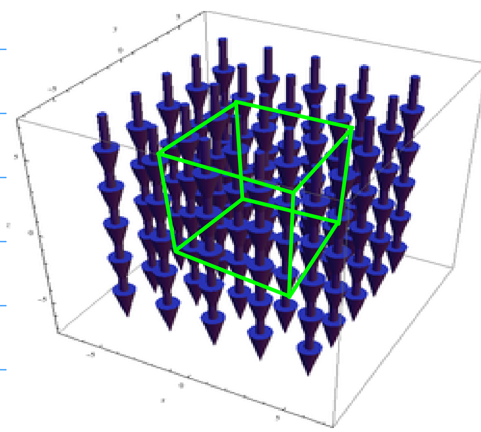
$\nabla \times \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = (-1 - 1)\vec{k}$

$= -2\vec{k}$

Vector field $\vec{F}$



curl$(\vec{F})$



non-gradient vector field

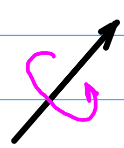
curl$(\cancel{\text{grad}}) = 0$

can't be applied

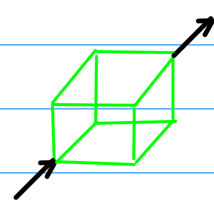
div(curl) = 0

always

curl $\neq 0$



div(curl) = 0



https://en.wikipedia.org/wiki/Curl_%28mathematics%29
https://en.wikipedia.org/wiki/Gradient

https://en.wikipedia.org/wiki/Vector_calculus_identities

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\begin{aligned}
\nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \\
&= (\nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla)\mathbf{A} - (\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla)\mathbf{B} \\
&= \nabla \cdot (\mathbf{B}\mathbf{A}^{\mathrm{T}}) - \nabla \cdot (\mathbf{A}\mathbf{B}^{\mathrm{T}}) \\
&= \nabla \cdot (\mathbf{B}\mathbf{A}^{\mathrm{T}} - \mathbf{A}\mathbf{B}^{\mathrm{T}})
\end{aligned}$$

- The scalar triple product is invariant under a circular shift of its three operands (**a**, **b**, **c**):

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

- Swapping the positions of the operators without re-ordering the operands leaves the triple product unchanged. This follows from the preceding property and the commutative property of the dot product.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

- Swapping any two of the three operands negates the triple product. This follows from the circular-shift property and the anticommutativity of the cross product.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$$
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$$
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$$

- The scalar triple product can also be understood as the determinant of the 3 × 3 matrix (thus also its inverse) having the three vectors either as its rows or its columns (a matrix has the same determinant as its transpose):

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

https://en.wikipedia.org/wiki/Triple_product

- If the scalar triple product is equal to zero, then the three vectors **a**, **b**, and **c** are coplanar, since the "parallelepiped" defined by them would be flat and have no volume.
- If any two vectors in the triple scalar product are equal, then its value is zero:

$$\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{a})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{b})=\mathbf{a}\cdot(\mathbf{a}\times\mathbf{a})=0$$

- Moreover,

$$[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\mathbf{a}=(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c})$$

- The simple product of two triple products (or the square of a triple product), may be expanded in terms of dot products:[1]

$$((\mathbf{a}\times\mathbf{b})\cdot\mathbf{c})\;((\mathbf{d}\times\mathbf{e})\cdot\mathbf{f})=\det\left[\begin{pmatrix}\mathbf{a}\\\mathbf{b}\\\mathbf{c}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{d}&\mathbf{e}&\mathbf{f}\end{pmatrix}\right]=\det\Bigg[$$

https://en.wikipedia.org/wiki/Triple_product

The **vector triple product** is defined as the cross product of one vector with the cross product of the other two. The following relationship holds:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

This is known as **triple product expansion**, or **Lagrange's formula**,[2][3] although the latter name is also used for several other formulae. Its right hand side can be remembered by using the mnemonic "BAC − CAB", provided one keeps in mind which vectors are dotted together. A proof is provided below.

Since the cross product is anticommutative, this formula may also be written (up to permutation of the letters) as:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -(\mathbf{c} \cdot \mathbf{b})\mathbf{a} + (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$$

From Lagrange's formula it follows that the vector triple product satisfies:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$$

which is the Jacobi identity for the cross product. Another useful formula follows:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{b} \times (\mathbf{a} \times \mathbf{c})$$

These formulas are very useful in simplifying vector calculations in physics. A related identity regarding gradients and useful in vector calculus is Lagrange's formula of vector cross-product identity:[4]

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{f}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{f}) - (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{f}$$

Laplacian Operator

$$\boxed{\begin{array}{ll} \nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} = & \text{vector} \\ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla \cdot \nabla & \text{scalar} \end{array}}$$

$$\nabla \square = \frac{\partial \square}{\partial x}\vec{i} + \frac{\partial \square}{\partial y}\vec{j} + \frac{\partial \square}{\partial z}\vec{k} \qquad \text{vetor}$$

$$\nabla^2 \square = \frac{\partial^2 \square}{\partial x^2} + \frac{\partial^2 \square}{\partial y^2} + \frac{\partial^2 \square}{\partial z^2} \qquad \text{scalar}$$

$$\boxed{\nabla^2 f = \nabla \cdot \nabla f}$$

Laplacian Operator

$$\boxed{\begin{aligned}\nabla &= \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \\ \nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\end{aligned}}$$

$$\nabla \cdot \nabla = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$(\nabla \cdot \nabla) f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$\nabla \cdot (\nabla f) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\boxed{\nabla^2 f = (\nabla \cdot \nabla) f = \nabla \cdot (\nabla f)}$$

Vector Laplacian Operator

$$\nabla \times (\nabla \times \vec{F})$$

$$= \nabla \times (\vec{B})$$

$$\vec{B} = \nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$\nabla \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) & \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) & \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\right]\vec{i}$$

$$- \left[\frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\right]\vec{j}$$

$$\left[\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\right]\vec{k}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial x}\right) - \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{\partial R}{\partial x}\right)\right]\vec{i}$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right) - \frac{\partial^2 Q}{\partial x^2} - \frac{\partial^2 Q}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{\partial R}{\partial y}\right)\right]\vec{j}$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial z}\right) - \frac{\partial^2 R}{\partial x^2} - \frac{\partial^2 R}{\partial y^2} + \frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial z}\right)\right]\vec{k}$$

$$\nabla \cdot \vec{F} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right)$$

$$\nabla \left(\nabla \cdot \vec{F} \right) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \right) \cdot \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right)\vec{i} +$$

$$\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right)\vec{j} +$$

$$\frac{\partial}{\partial z}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right)\vec{k}$$

$$= \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial z}\right) \right)\vec{i} +$$

$$\left(\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x}\right) + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial}{\partial y}\left(\frac{\partial R}{\partial z}\right) \right)\vec{j} +$$

$$\left(\frac{\partial}{\partial z}\left(\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial y}\right) + \frac{\partial^2 R}{\partial z^2} \right)\vec{k}$$

$$
\begin{aligned}
& \nabla \times (\nabla \times \vec{F}) - \nabla(\nabla \cdot \vec{F}) \\
&= \left[\frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial x}\right) - \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{\partial R}{\partial x}\right) \right] \vec{i} \\
&\quad + \left[\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right) - \frac{\partial^2 Q}{\partial x^2} - \frac{\partial^2 Q}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{\partial R}{\partial y}\right) \right] \vec{j} \\
&\quad + \left[\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial z}\right) - \frac{\partial^2 R}{\partial x^2} - \frac{\partial^2 R}{\partial y^2} + \frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial z}\right) \right] \vec{k} \\
&\quad - \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial z}\right) \right) \vec{i} \\
&\quad - \left(\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x}\right) + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial}{\partial y}\left(\frac{\partial R}{\partial z}\right) \right) \vec{j} \\
&\quad - \left(\frac{\partial}{\partial z}\left(\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial y}\right) + \frac{\partial^2 R}{\partial z^2} \right) \vec{k} \\
&= -\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} \right) \vec{i} \\
&\quad - \left(\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 Q}{\partial z^2} \right) \vec{j} \\
&\quad - \left(\frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2} + \frac{\partial^2 R}{\partial z^2} \right) \vec{k} \\
&= -\nabla^2 \vec{F}
\end{aligned}
$$