Conformal Mapping (H.1)

20160609

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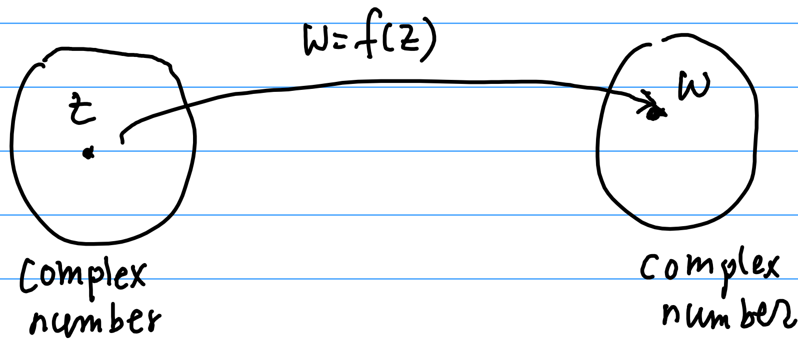
Mapping $\quad w = f(z)$



$$W = f(z) = u + iv$$

$$w = f(x + iy) = u(x, y) + i\,v(x, y)$$

$$\boxed{z = x + iy} \longrightarrow \boxed{w = u + iv}$$

image of z under f

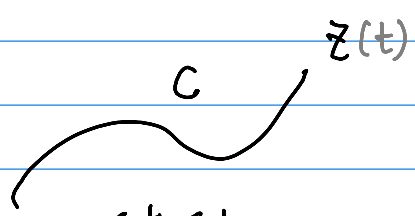
Images of Curves

$$\boxed{z = x + iy} \longrightarrow \boxed{w = u + iv}$$

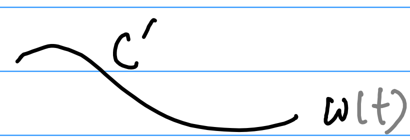
image of z
under f

$z(t) = x(t) + i\,y(t)$ $\qquad$ $w(t) = u(t) + i\,v(t)$

$$w = f(z(t))$$
$$= u(x(t), y(t)) + i\,v(x(t), y(t))$$



$a \le t \le b$



$a \le t \le b$

$f(z) = z + z_0$ translation

$g(z) = e^{i\theta_0} \cdot z$ rotation

$h(z) = \alpha z$ magnification

α : fixed positive real number

Real Power Function

$f(z) = z^{\alpha}$ $\qquad$ α: a fixed positive real number

z : $\qquad 0 \leq \mathrm{Arg}(z) \leq \theta_0$

$w = f(z)$: $\qquad 0 \leq \mathrm{Arg}(w) \leq \alpha\theta_0$

Angle Preserving Mappings

A complex mapping $w = f(z)$
defined on domain D

Conformal at $z = z_0$ in D

when f preserves the angle
between any 2 curves in D
that intersects at z_0

C_1 and C_2 intersect in D at z_0
the corresponding images C_1' and C_2'

the angle between C_1 and C_2
the angle between C_1' and C_2'

$$|z_1' - z_2'|^2 = |z_1'|^2 + |z_2'|^2 - 2|z_1'||z_2'|\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{|z_1'|^2 + |z_2'|^2 - |z_1' - z_2'|^2}{2|z_1'||z_2'|}\right)$$

$$\varphi = \cos^{-1}\left(\frac{|w_1'|^2 + |w_2'|^2 - |w_1' - w_2'|^2}{2|w_1'||w_2'|}\right)$$

$f(z)$: analytic in the domain D

$f'(z_0) \neq 0 \Rightarrow f$ is conformal at $z = z_0$

$z = z(t) \qquad w = f(z(t))$

$\qquad\qquad\quad w' = f'(z(t))\, z'(t)$

C_1 and C_2 intersect in D at z_0

$$w_1' = f'(z_0)\, z_1'$$
$$w_2' = f'(z_0)\, z_2'$$

$$\varphi = \cos^{-1}\left(\frac{|w_1'|^2 + |w_2'|^2 - |w_1' - w_2'|^2}{2\,|w_1'|\,|w_2'|} \right)$$

$$= \cos^{-1}\left(\frac{|f'(z_0) z_1'|^2 + |f'(z_0) z_2'|^2 - |f'(z_0) z_1' - f'(z_0) z_2'|^2}{2\,|f'(z_0) z_1'|\,|f'(z_0) z_2'|} \right)$$

$$= \cos^{-1}\left(\frac{|z_1'|^2 + |z_2'|^2 - |z_1' - z_2'|^2}{2\,|z_1'|\,|z_2'|} \right)$$

$$= \theta$$

Transform Theorem for Harmonic Functions

f : analytic function $D \to D'$

U : harmonic in D'

the real-valued function $u(x, y) = U(f(z))$
harmonic in D

Linear Fractional Transformations

$$T(z) = \frac{az+b}{cz+d}$$

$$T'(z) = \frac{ad-bc}{(cz+d)^2}$$

$$\lim_{z \to z_0} T(z) = \infty \qquad z_0 = -\frac{d}{c}$$

$$\lim_{|z| \to \infty} T(z) = \lim_{|z| \to \infty} \frac{a + b/z}{c + d/z} = \frac{a}{c}$$

Circle Preserving Property

$$T(z) = \frac{az+b}{cz+d}$$ Linear Fractional Trans.

$\Big\downarrow\ c \leftarrow 0$

$$T(z) = Az + B$$ Linear Trans

Linear Complex Mapping   Composite of $\begin{cases} \text{rotation} \\ \text{magnification} \\ \text{translation} \end{cases}$

circle in the z-plane $\longrightarrow$ circle in the w-plane

$$T(z) = \frac{az+b}{cz+d}$$ Linear Fractional Trans.

$\Big\downarrow\ c \neq 0$

$$T(z) = \frac{bd-ad}{c}\,\frac{1}{cz+d} + \frac{a}{c}$$

$$= A\,\frac{1}{cz+d} + B$$

$z_1 = cz+d$

$$= A\,\frac{1}{z_1} + B$$

$z_2 = \frac{1}{z_1}$

$$= A\,z_2 + B$$

General Linear Fractional Trans.

composite of two linear functions
the inversion $w = 1/z$

$|z - z_1| = r \qquad w = \frac{1}{z}$

$$\left|\frac{1}{w} - \frac{1}{w_1}\right| = \frac{|w - w_1|}{|w||w_1|} = r$$

$$|w - w_1| = r|w_1|\,|w - 0|$$

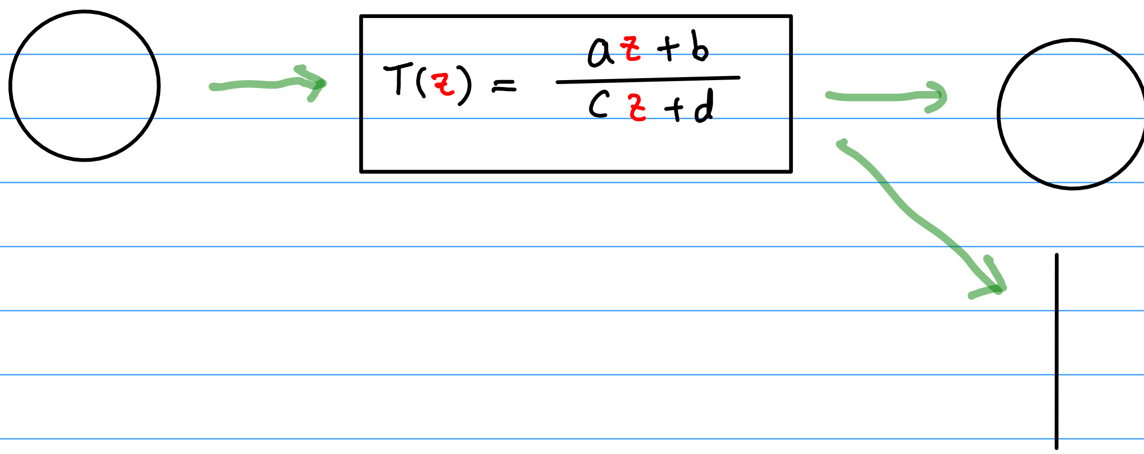
$$|w - w_1| = \lambda\,|w - w_2|$$

$\lambda = 1$ line
$\lambda \neq 1,\ \lambda > 0$ circle

the image of the circle $|z - z_1| = r$
under the inversion $w = 1/z$
a circle except when $r = 1/|w_1| = |z_1|$

Circle - Preserving Property

A Linear Fractional Transformation



$$\boxed{T(z) = \frac{az+b}{cz+d}}$$

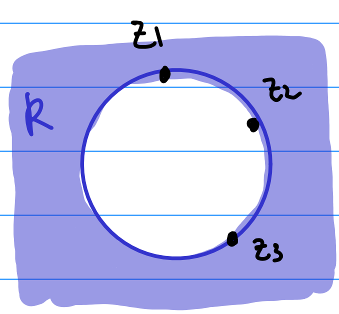
To solve Dirichelet problems
construct special functions
that map a given circular region R
to a target region R'
where the corresponding Dirichlet problem
is solvable.

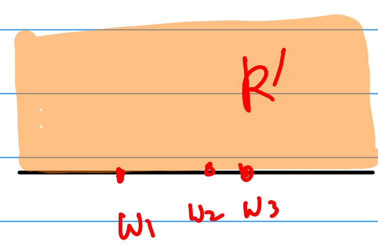
A Circular Boundary – by 3 points

Find a linear frac trans $w = T(z)$

z_1, z_2, z_3 in $R \longrightarrow w_1, w_2, w_3$ in R'

interior of $R \longrightarrow$ interior of R'





$$T(z) = \frac{az+b}{cz+d} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$T_1(z) = \frac{a_1 z+b_1}{c_1 z+d_1} \qquad A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

$$T_2(z) = \frac{a_2 z+b_2}{c_2 z+d_2} \qquad A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

$$w = T(z) = T_2(T_1(z)) \qquad A = A_2 A_1$$

$$z = T^{-1}(w) \qquad \mathrm{adj}(A)$$

Triples of Triples

$$T(z) = \frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1} \qquad \begin{cases} T(z_1) = 0 \\ T(z_2) = 1 \\ T(z_3) = \infty \end{cases}$$

cross ratio of the complex numbers z, z_1, z_2, z_3

$$S(w) = \frac{w - w_1}{w - w_3} \frac{w_2 - w_3}{w_2 - w_1} \qquad \begin{cases} S(w_1) = 0 \\ S(w_2) = 1 \\ S(w_3) = \infty \end{cases}$$

$$\begin{cases} S^{-1}(0) = w_1 \\ S^{-1}(1) = w_2 \\ S^{-1}(\infty) = w_3 \end{cases}$$

$$\begin{cases} T(z_1) = 0 \\ T(z_2) = 1 \\ T(z_3) = \infty \end{cases} \qquad \begin{cases} S^{-1}(0) = w_1 \\ S^{-1}(1) = w_2 \\ S^{-1}(\infty) = w_3 \end{cases}$$

$w = S^{-1}(T(z))$ maps the triple z_1, z_2, and z_3
to the triple w_1, w_2, and w_3

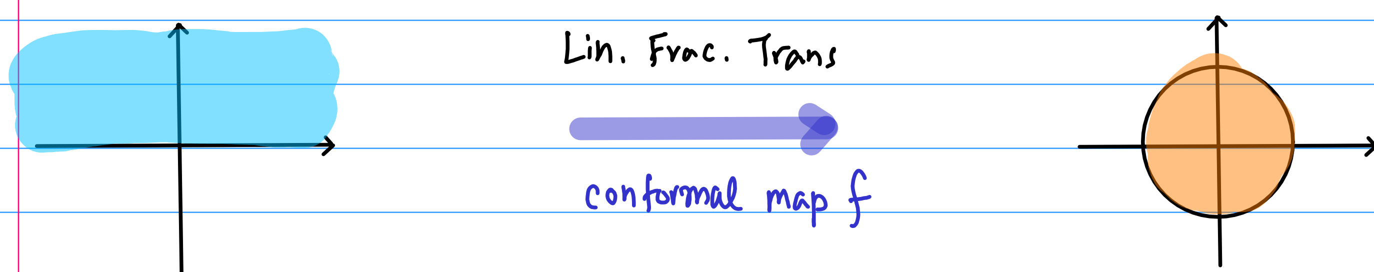
$$S(w) = T(z)$$

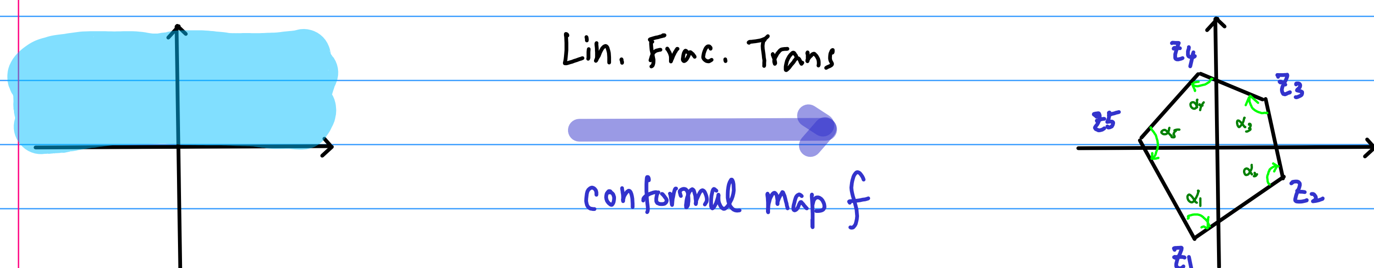
$$\frac{w - w_1}{w - w_3} \frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1}$$

Riemann Mapping Theorem

D' a simply connected domain
with at least one boundary point

the existence of an analytic function g
which conformaly maps
the unit open disk $|z| < 1$
onto D'





Schwarz - Christoffel Transformation

Unlike the Riemann mapping theorem

can specify
a form of for the derivative $f'(z)$
of a conformal mapping $f(z)$
from the upper half plane
to a bounded or unbounded polygonal region

mapping

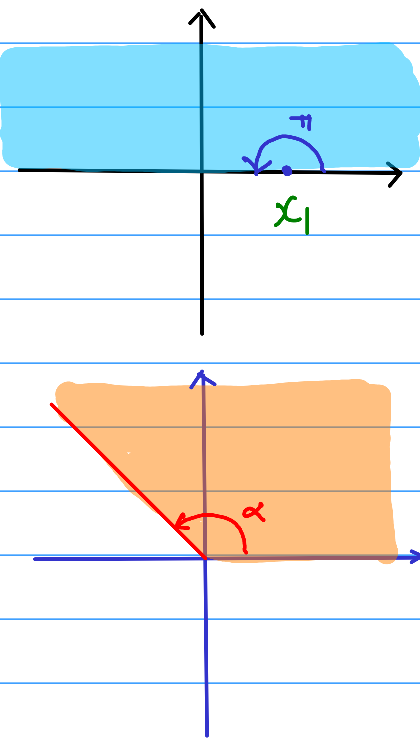
$f(z) = (z - x_1)^{\alpha/\pi}$, $0 \leq \alpha < 2\pi$

on the upper half plane $y \geq 0$

$\zeta = (z - x_1)$

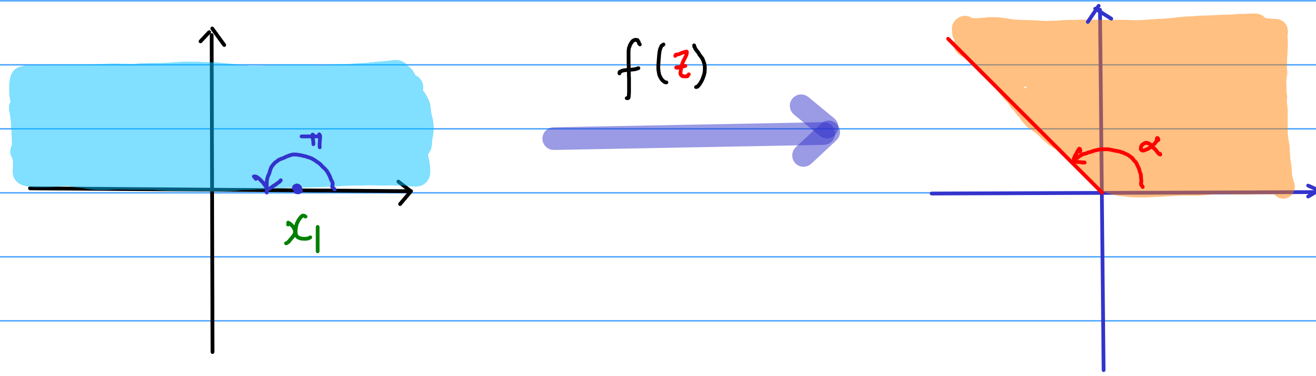
$w = \zeta^{\alpha/\pi}$

$[0, \pi]$
$\downarrow \quad \downarrow$
$[0, \alpha]$



$f'(z) = \frac{\alpha}{\pi}(z - x_1)^{\alpha/\pi - 1}$

$= A(z - x_1)^{\alpha/\pi - 1}$



$$f(z) = (z - x_1)^{\alpha/\pi} \qquad 0 \le \alpha < 2\pi$$

$$\begin{tikzcd}
z \arrow[r] \arrow[rr, bend left] & \zeta \arrow[r] & w
\end{tikzcd}$$

$$\zeta = (z - x_1) \qquad w = \zeta^{\alpha/\pi}$$

Arg $[0, \pi] \to [0, \alpha]$

$$f(z) = (z - x_1)^{\alpha/\pi} \qquad 0 \le \alpha < 2\pi$$

$$f'(z) = \frac{\alpha}{\pi}(z - x_1)^{\alpha/1}$$

$$= A(z - x_1)^{\alpha/\pi - 1}$$

Assum $f(z)$ analytic in the upper half plane $(y \geq 0)$

$$f'(z) = A\,(z - x_1)^{\frac{\alpha_1}{\pi}-1}\,(z - x_2)^{\frac{\alpha_2}{\pi}-1}$$

$x_1 < x_2$

$\arg(w'(t)) = \text{const}$ $\Rightarrow$

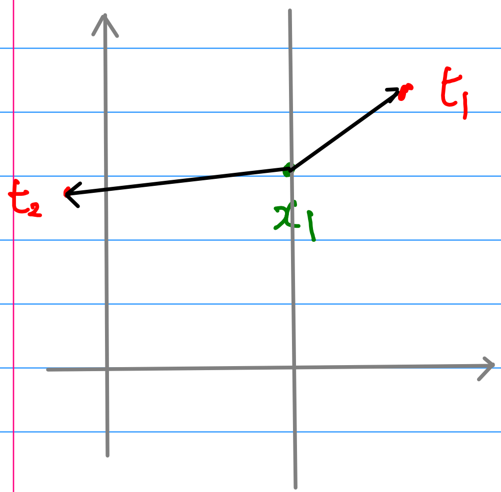
a curve $w = w(t)$: a line

in the w-plane

$$f'(t) = A\,(t - x_1)^{\frac{\alpha_1}{\pi}-1}\,(t - x_2)^{\frac{\alpha_2}{\pi}-1}$$

$$\begin{aligned}\arg\{f'(t)\} &= \operatorname{Arg}(A) \\ &\quad + \left(\frac{\alpha_1}{\pi}-1\right)\operatorname{Arg}(t - x_1) \\ &\quad + \left(\frac{\alpha_2}{\pi}-1\right)\operatorname{Arg}(t - x_2)\end{aligned}$$

(X)



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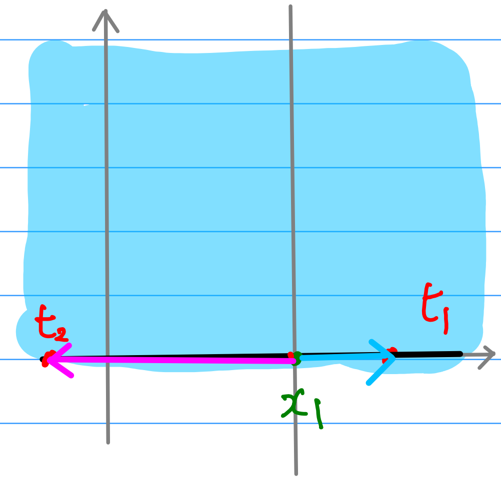


$$\begin{cases} t < x_1 & \operatorname{Arg}(t - x_1) = \pi \\ t > x_1 & \operatorname{Arg}(t - x_1) = 0 \end{cases}$$

$$\begin{cases} t < x_2 & \operatorname{Arg}(t - x_2) = \pi \\ t > x_2 & \operatorname{Arg}(t - x_2) = 0 \end{cases}$$

①	②	③
$t < x_1$	$x_1 < t < x_2$	$t > x_2$
$\mathrm{Arg}(t - x_1) = \pi$	$\mathrm{Arg}(t - x_1) = 0$	$\mathrm{Arg}(t - x_1) = 0$
$\mathrm{Arg}(t - x_2) = \pi$	$\mathrm{Arg}(t - x_2) = \pi$	$\mathrm{Arg}(t - x_2) = 0$
$\boxed{\begin{array}{l} \mathrm{Arg}(A) \\ + (\alpha_1 - \pi) \\ + (\alpha_2 - \pi) \end{array}}$	$\boxed{\begin{array}{l} \mathrm{Arg}(A) \\ + 0 \\ + (\alpha_2 - \pi) \end{array}}$	$\boxed{\begin{array}{l} \mathrm{Arg}(A) \\ + 0 \\ + 0 \end{array}}$
const	const	const

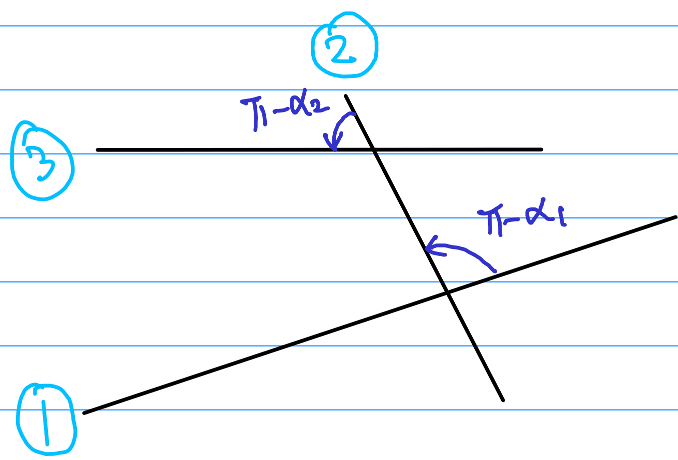
$\arg\{f'(t)\}$

$+ (\pi - \alpha_1)$ $\qquad$ $+ (\pi - \alpha_2)$

①	x_1	②	x_2	③
$t < x_1$		$x_1 < t < x_2$		$t > x_2$
$\mathrm{Arg}(A) + (\alpha_1 - \pi) + (\alpha_2 - \pi)$		$\mathrm{Arg}(A) + (\alpha_2 - \pi)$		$\mathrm{Arg}(A)$

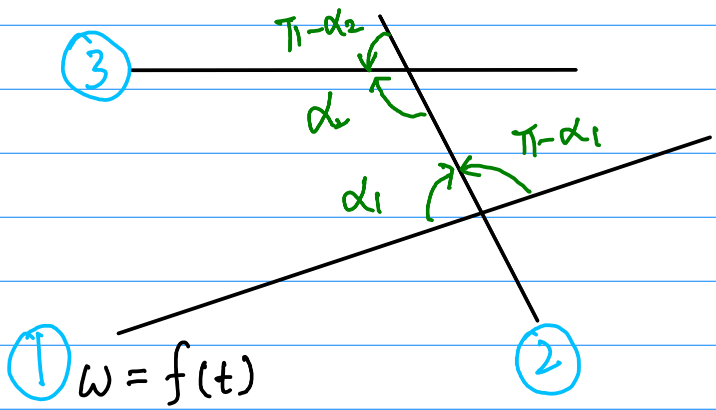
$$\arg\{f'(t)\} = \begin{cases} \mathrm{Arg}(A) + (\alpha_1 - \pi) + (\alpha_2 - \pi) & [t < x_1] \\ \mathrm{Arg}(A) + (\alpha_2 - \pi) & [x_1 < t < x_2] \\ \mathrm{Arg}(A) & [t > x_2] \end{cases}$$

$\boxed{w = f(t)}$



$w = f(t) \quad x_1 < t < x_2$

$w = f(t)$
$t > x_2$



$w = f(t)$

$t < x_1$

Schwarz – Christoffel Formula

$f(z)$ analytic in the upper half-plane $y > 0$

$$f'(z) = A(z - x_1)^{\alpha_1/\pi - 1}(z - x_2)^{\alpha_2/\pi - 1} \cdots (z - x_n)^{\alpha_n/\pi - 1}$$

$$x_1 < x_2 < \cdots < x_n$$

$$0 < \alpha_i < 2\pi \quad i = 1, \cdots, n$$

$f(z)$ maps the upper half-plane $y > 0$
to a polygonal region
with interior angles $\alpha_1, \alpha_2, \cdots, \alpha_n$

① can select the location of 3 points
x_k on the x-axis

careful selection simplifies the computation of $f(z)$

other remaining points depends on the shape of
the target polygon

② A general formula for $f(z)$

$$f'(z) = A(z-x_1)^{\alpha_1/\pi - 1}(z-x_2)^{\alpha_2/\pi - 1} \cdots (z-x_n)^{\alpha_n/\pi - 1}$$

$$f(z) = A\int \boxed{(z-x_1)^{\alpha_1/\pi - 1}(z-x_2)^{\alpha_2/\pi - 1} \cdots (z-x_n)^{\alpha_n/\pi - 1}}\, dz + B$$

$$= A\ g(z) + B$$

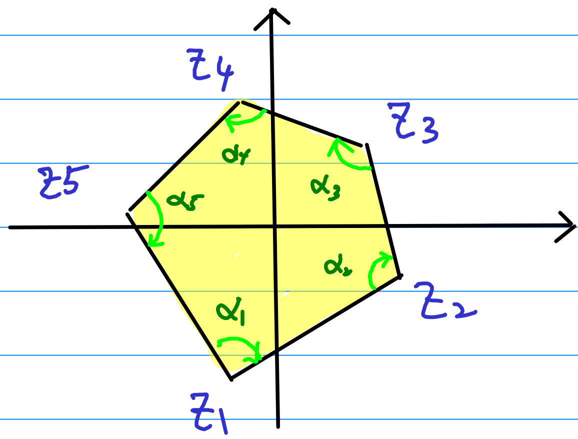
$$g(z) = \int \boxed{(z-x_1)^{\alpha_1/\pi - 1}(z-x_2)^{\alpha_2/\pi - 1} \cdots (z-x_n)^{\alpha_n/\pi - 1}}\, dz$$

$$h(z) = Az + B$$

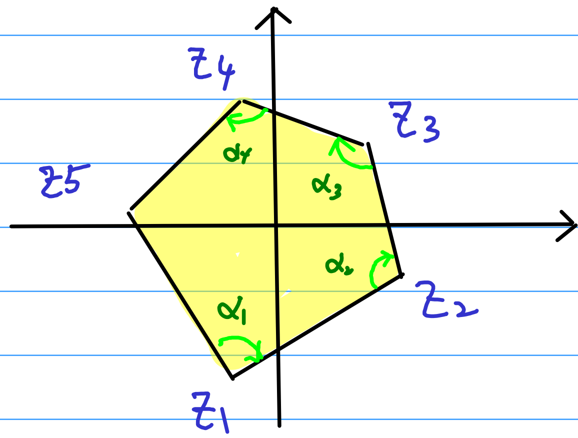
$\left\{ \begin{array}{l} \text{magnify} \\ \text{rotate} \\ \text{translate} \end{array} \right.$ the image polygon produced by $g(z)$

③ If the polygon regions are bounded

only n-1 of the n interior angles
should be included in the Swartz-Christoffel formula



Bounded



Bounded

α_1, α_2, α_3, α_4
are sufficient to
determine Schwartz-
Christoffel formula

Classical BVP's

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad k > 0$$ one-dim heat eq

$$a\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$ one-dim wave eq

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ two-dim Laplace's eq

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Initial Conditions

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad k > 0$$ one-dim heat eq

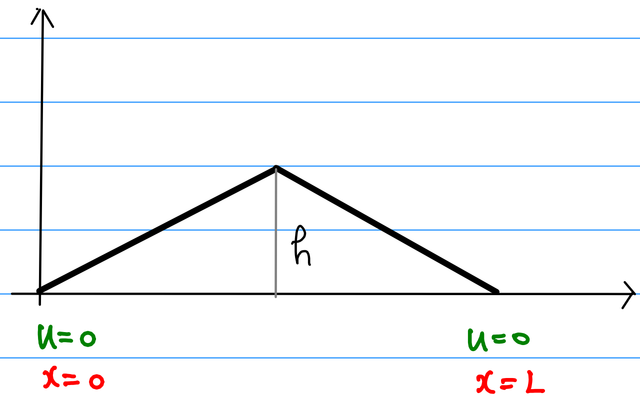
$$a\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$ one-dim wave eq

$u(x,t) \longrightarrow u(x,0)$ IC (Initial Conditions)

$$\begin{cases} u(x,0) = f(x) & 0 < x < L \\ \frac{\partial}{\partial t}u(x,0) = g(x) \end{cases}$$

$$\begin{cases} u(x,t)\Big|_{t=0} = f(x) & 0 < x < L \\ \frac{\partial}{\partial t}u(x,t)\Big|_{t=0} = g(x) \end{cases}$$

Boundary Conditions



plucked string

$u(0, t) = 0$ $\quad u(L, t) = 0$ $\quad t > 0$

Dirichelet Condition

Dirichlet problem for a disk

Solve Laplace's equation

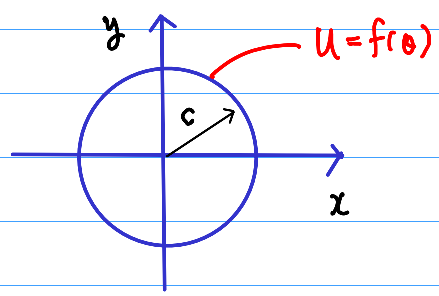
for the steady-state temperature $u(r, \theta)$

in a circular disk or plate of radius c

when the temperature of the circumference

is $u(c, \theta) = f(\theta)$, $0 < \theta < 2\pi$

two faces of the plate are insulated



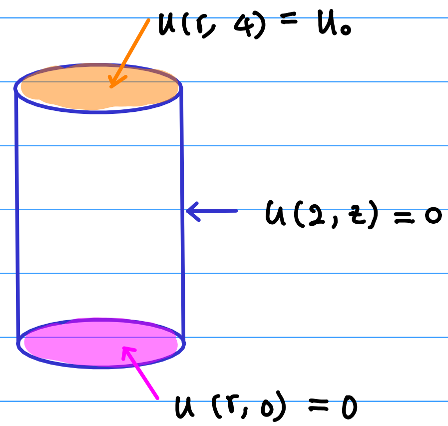
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Steady Temperature in a Circular Cylinder

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad 0<r<2, \qquad 0<z<4$$

$u(2, z) = 0 \qquad 0 < z < 4$

$u(r, 0) = 0 \qquad u(r, 4) = u_0 \qquad 0 < r < 2$



Laplace Equations

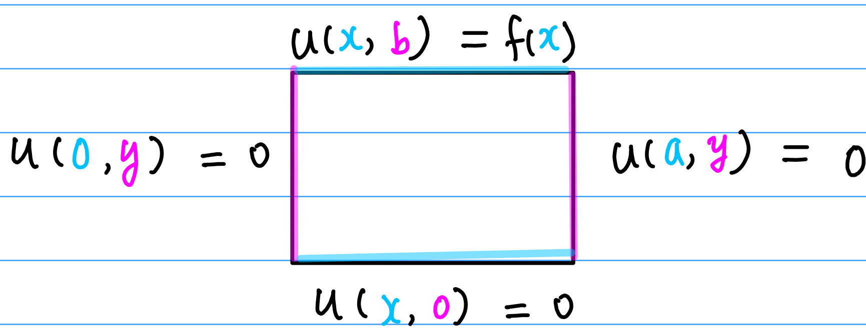
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < a \qquad 0 < y < b$$

Neuman condition (BC) $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0 \qquad 0 < y < b$

$\boxed{\text{Dirichlet Condition}}$ (BC) $u(x, 0) = 0, \quad u(x, b) = f(x) \qquad 0 < x < a$

$\boxed{\text{Dirichlet Condition}}$ (BC) $u(0, y) = 0, \quad u(a, y) = 0 \qquad 0 < y < b$

$\boxed{\text{Dirichlet Condition}}$ (BC) $u(x, 0) = 0, \quad u(x, b) = f(x) \qquad 0 < x < a$



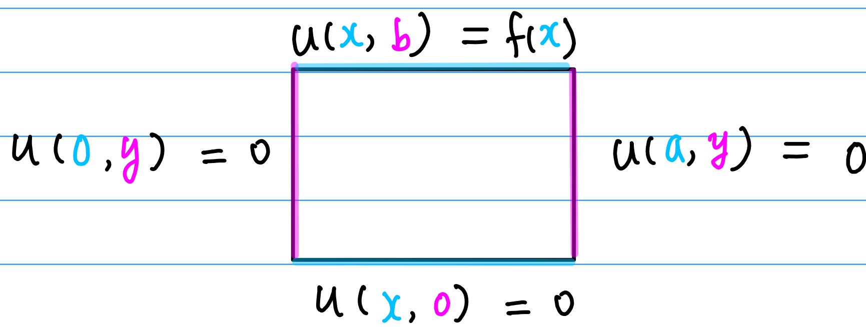
Dirichlet Problem

Elliptic partial differential equation

$\nabla^2 u = 0$ Laplace Eq

within a region R (in the plane or 3-space)

u takes a prescribed values
on the entire boundary of the region



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < a \qquad 0 < y < b$$

$\boxed{\text{Dirichlet Condition}}$ (BC) $u(0, y) = 0, \quad u(a, y) = 0 \qquad 0 < y < b$

$\boxed{\text{Dirichlet Condition}}$ (BC) $u(x, 0) = 0, \quad u(x, b) = f(x) \qquad 0 < x < a$

Dirichlet Problem

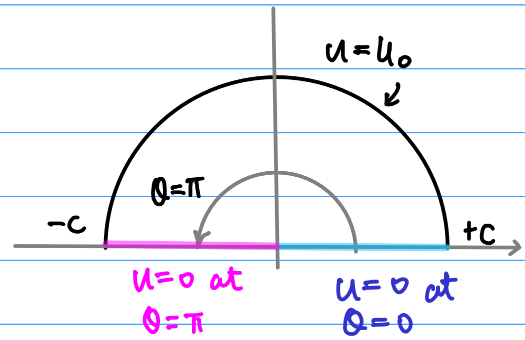
Steady-state temperature in a semi-circle plate

$u(r, \theta)$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad 0 < \theta < \pi, \quad 0 < r < c$$

$$u(c, \theta) = u_0 \qquad 0 < \theta < \pi$$

$$u(r, \theta) = 0 \qquad u(r, \pi) = 0 \qquad 0 < r < c$$

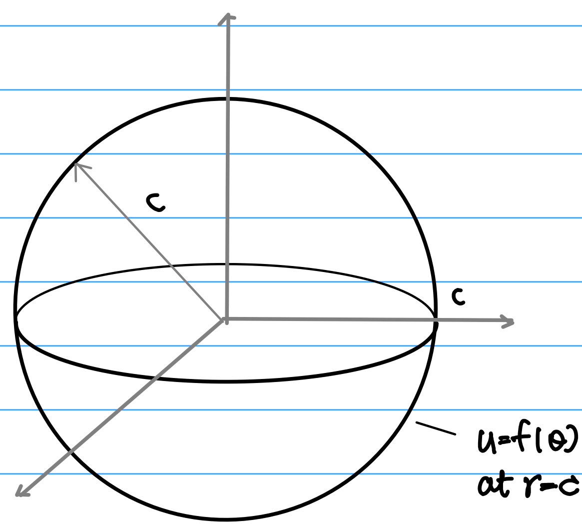


Dirichlet Problem

Steady-state temperature in a sphere

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0 \qquad 0<r<c, \qquad 0<\theta<\pi$$

$$u(c,\theta) = f(\theta) \qquad 0<\theta<\pi$$



Dirichlet Problem: Superposition

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0<x<a, \quad 0<y<b$$

$$u(0,y) = F(y) \qquad u(a,y) = G(y) \qquad 0<y<b$$

$$u(x,0) = f(x) \qquad u(x,b) = g(x) \qquad 0<x<a$$

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0 \qquad 0<x<a, \quad 0<y<b$$

$$u_1(0,y) = 0 \qquad u_1(a,y) = 0 \qquad 0<y<b$$

$$u_1(x,0) = f(x) \qquad u_1(x,b) = g(x) \qquad 0<x<a$$

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0 \qquad 0<x<a, \quad 0<y<b$$

$$u_2(0,y) = F(y) \qquad u_2(a,y) = G(y) \qquad 0<y<b$$

$$u_2(x,0) = 0 \qquad u_2(x,b) = 0 \qquad 0<x<a$$

Dirichlet Problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < 2, \quad 0 < y < 2$$

$$u(0, y) = 0 \qquad u(2, y) = y(2-y) \qquad 0 < y < b$$

$$u(x, 0) = 0 \qquad u(x, 2) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 < x < 2 \end{cases}$$

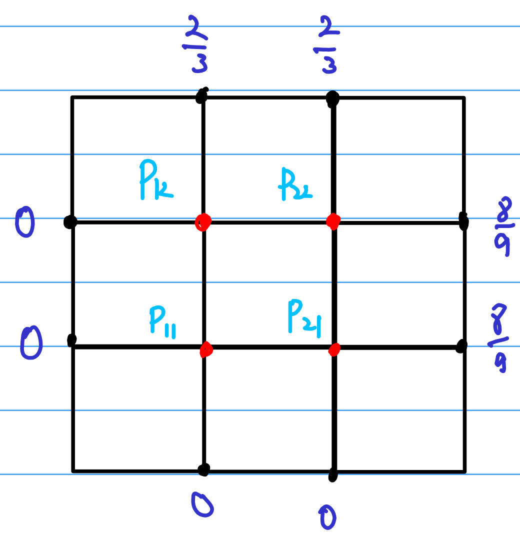
the Dirichlet Problem for Laplace Eq $\nabla^2 u = 0$

$u(x, y)$ - given on the boundary C of a region R

the approximate solution by using numerical method

interior mesh points

u_{ij}



Harmonic Functions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

the real & imaginary parts of an analytic function
cannot be chosen arbitrarily

both u and v must satisfy Laplace eq

the real-valued function $\phi(x, y)$
has continuous 2nd order partial derivatives
in a domain D
satisfies Laplace eq

$\Rightarrow$ harmonic in D

Criterion for Analyticity

Suppose the real-valued functions

$u(x,y)$, $v(x,y)$: continuous

the 1st order partial derivatives

$\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$: continuous

in a domain D

And if $u(x,y)$ & $v(x,y)$ meets the Cauchy – Rieman eq

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}} \qquad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y}}$$

Then the complex function $f(z) = u(x,y) + i\,v(x,y)$

is analytic in D

$$\Rightarrow \boxed{\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \end{aligned}} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Harmonic Functions

A real-valued function $\phi(x, y)$

- the second order partial derivatives : continous in a domain D

$$\frac{\partial^2 \phi}{\partial x^2}, \quad \frac{\partial^2 \phi}{\partial y^2}, \quad \frac{\partial^2 \phi}{\partial x \partial y}, \quad \frac{\partial^2 \phi}{\partial y \partial x}$$ continous

- satisfy Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$\Leftrightarrow$ $\phi(x, y)$ is harmonic in D

The source of Harmonic Functions

$f(z) = u(x, y) + i\, v(x, y)$ analytic in domain D → the function $u(x, y)$ & $v(x, y)$ are harmonic functions

u & v : continuous 2nd order partial derivatives

f : analytic → Cauchy - Rieman eq

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \qquad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

$$\frac{\partial^2 u}{\partial x^2} = \boxed{\frac{\partial^2 v}{\partial x \partial y}} \qquad \frac{\partial^2 u}{\partial y^2} = -\boxed{\frac{\partial^2 v}{\partial x \partial y}}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

$u(x, y)$: harmonic

$$\boxed{\frac{\partial^2 u}{\partial y \partial x}} = \frac{\partial^2 v}{\partial y^2} \qquad \boxed{\frac{\partial^2 u}{\partial x \partial y}} = -\frac{\partial^2 v}{\partial x^2}$$

$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

$v(x, y)$: harmonic

$f(z) = u(x, y) + i\, v(x, y)$ $\Longleftrightarrow$

$\boxed{\text{analytic}}$ in domain D

the function $u(x, y)$ & $v(x, y)$ are $\boxed{\text{harmonic}}$ functions

Harmonic Conjugate Functions

$f(z) = u(x,y) + i v(x,y)$ analytic in D

$\Rightarrow$ u, v : harmonic in D

given $u(x,y)$: harmonic in D

find $v(x,y)$: harmonic in D

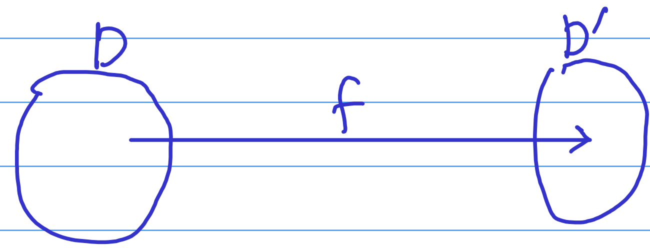
such that $u(x,y) + i v(x,y)$: analytic in D

u, v : harmonic conjugate function

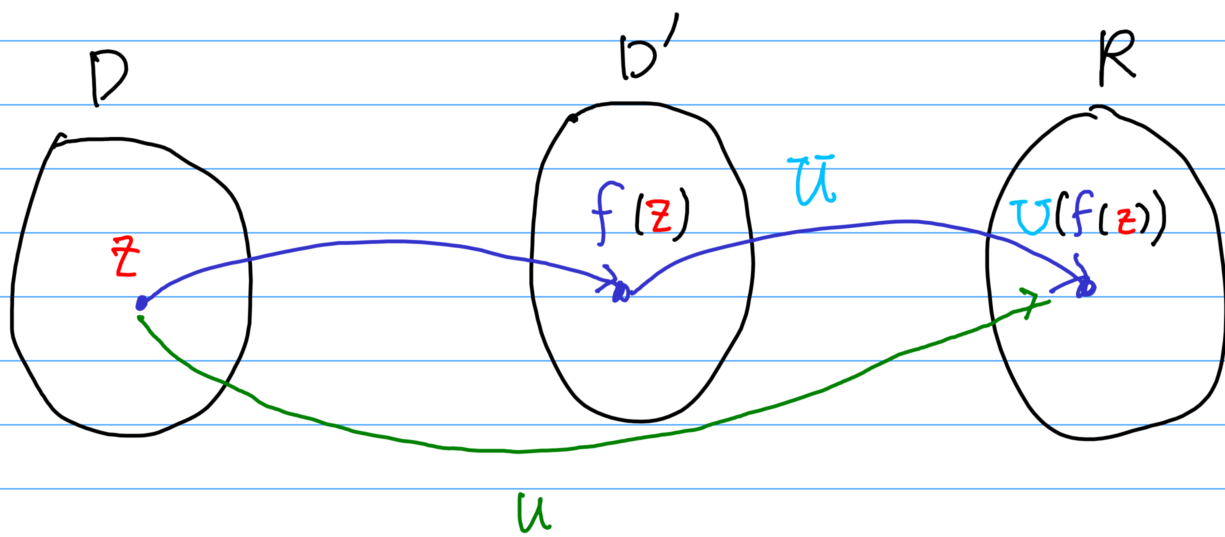
Harmonic Functions & the Dirichlet Problem

Transformation Theorem for Harmonic Functions

f : analytic function
that maps a domain D onto a domain D'



real valued function
$u(x,y) = U(f(z))$ ← U is harmonic in D'
harmonic in D



$$\boxed{\begin{array}{ll} U(z) \text{ harmonic in } D' & \rightarrow \\ U(f(z)) \text{ harmonic in } D & f: D \to D' \end{array}}$$

a special case where D is simply connected

U, V harmonic conjugate in D'

$\Rightarrow H = U + iV$ analytic in D'

$\Rightarrow H(f(z)) = U(f(z)) + i\,V(f(z))$ is analytic in D

$$\boxed{\begin{array}{l} f(z) = u(x, y) + i\,v(x, y) \\ \quad \text{analytic in domain } D \\ \\ \text{the function } u(x, y) \text{ \& } v(x, y) \text{ are harmonic functions} \end{array}}$$

$\rightarrow u(x, y) = U(f(z))$ harmonic in D

$U(z)$ has a harmonic conjugate $V(z)$

let $g(z) = \dfrac{\partial U}{\partial x} - i\dfrac{\partial U}{\partial y}$

To show $g(z)$: analytic

$Re\{g(z)\}$ and $Im\{g(z)\}$

1. are continuous
2. has continuous 1st order partial derivatives
3. satisfy the Cauchy-Rieman equations

$\dfrac{\partial U}{\partial x}, \quad \dfrac{\partial U}{\partial y}$ continuous

$\dfrac{\partial^2 U}{\partial x^2}, \quad \dfrac{\partial^2 U}{\partial x \partial y},$

$\dfrac{\partial^2 U}{\partial y \partial x}, \quad \dfrac{\partial^2 U}{\partial y^2}$ continuous

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{\partial U}{\partial x}\right) = \dfrac{\partial}{\partial y}\left(-\dfrac{\partial U}{\partial y}\right) \\[2ex] \dfrac{\partial}{\partial y}\left(\dfrac{\partial U}{\partial x}\right) = -\dfrac{\partial}{\partial x}\left(-\dfrac{\partial U}{\partial y}\right) \end{cases}$$

g : analytic $\to$ Cauchy - Rieman eq

$$\boxed{\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial U}{\partial y}\right)} \qquad \boxed{\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\right) = -\frac{\partial}{\partial x}\left(-\frac{\partial U}{\partial y}\right)}$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \qquad \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 U}{\partial x \partial y}$$

U : harmonic in D'

equal 2nd order mixed partial derivatives

$\Rightarrow$ U : analytic in D'

$\Rightarrow$ $g(z) = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$: analytic in D'

Fundamental Theorem for Contour Integrals

f continuous in a domain D

F antiderivative of f in D

C any contour in D

initial pt z_0, final pt z_1

$$\int_C f(z)\,dz = F(z_1) - F(z_0)$$