Propositional Logic (3A)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using LibreOffice and Octave.

Propositional Logic

Propositional calculus (also called **propositional logic**, **statement logic**, **sentential calculus**, **sentential logic**, or sometimes **zeroth-order logic**) is the branch of logic concerned with the study of propositions (whether they are true or false) that are formed by other propositions with the use of logical connectives. First-order logic extends propositional logic by allowing a proposition to be expressed as constructs such as "for every", "exists", "equality" and "membership", whereas in proposition logic, propositions are thought of as atoms.

https://en.wikipedia.org/wiki/Propositional_calculus

Example (1)

Logical connectives are found in natural languages. In English for example, some examples are "and" (conjunction), "or" (disjunction), "not” (negation) and "if" (but only when used to denote material conditional).

The following is an example of a very simple inference within the scope of propositional logic:

Premise 1: If it's raining then it's cloudy.

Premise 2: It's raining.

Conclusion: It's cloudy.

Both premises and the conclusion are propositions. The premises are taken for granted and then with the application of modus ponens (an inference rule) the conclusion follows.

https://en.wikipedia.org/wiki/Propositional_calculus

Example (2)

As propositional logic is not concerned with the structure of propositions beyond the point where they can't be decomposed anymore by logical connectives, this inference can be restated replacing those *atomic* statements with statement letters, which are interpreted as variables representing statements:

Premise 1: $P \to Q$

Premise 2: P

Conclusion: Q

The same can be stated succinctly in the following way:

$$P \to Q, P \vdash Q$$

When P is interpreted as "It's raining" and Q as "it's cloudy" the above symbolic expressions can be seen to exactly correspond with the original expression in natural language. Not only that, but they will also correspond with any other inference of this *form*, which will be valid on the same basis that this inference is.

https://en.wikipedia.org/wiki/Propositional_calculus

Valid, Satisfiable, and Unsatisfiable Formulas

A **formula** is **valid** iff
Its truth value is **T** in all interpretations (**tautology**: ⊤)

A **formula** is **satisfiable** iff
Its truth value is **T** in at least one interpretation

A **formula** is **unsatisfiable** iff
Its truth value is **F** in all interpretations (**contradiction**: ⊥)

https://en.wikipedia.org/wiki/Propositional_calculus

Inference rules and axioms

Propositional logic may be studied through a formal system in which formulas of a formal language may be interpreted to represent propositions. A system of inference rules and axioms allows certain formulas to be derived. These derived formulas are called theorems and may be interpreted to be true propositions. A constructed sequence of such formulas is known as a *derivation* or *proof* and the last formula of the sequence is the theorem. The derivation may be interpreted as proof of the proposition represented by the theorem.

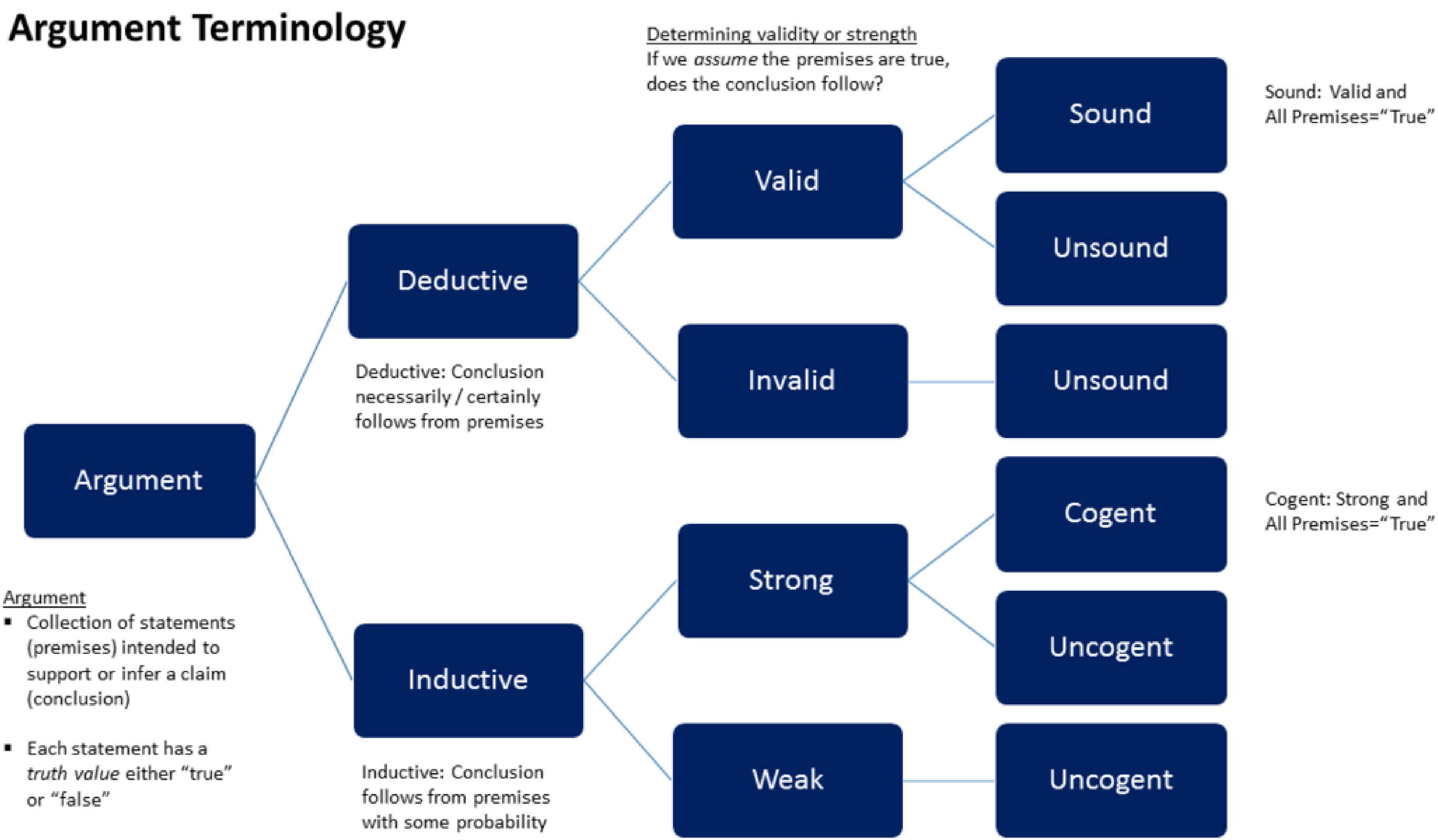
https://en.wikipedia.org/wiki/Propositional_calculus

Argument

In logic and philosophy, an **argument** is a series of statements typically used to persuade someone of something or to present reasons for accepting a conclusion.[1][2] The general form of an argument in a natural language is that of premises (variously propositions, statements or sentences) in support of a claim: the conclusion.[3][4][5] The structure of some arguments can also be set out in a formal language, and formally defined "arguments" can be made independently of natural language arguments, as in math, logic, and computer science.

https://en.wikipedia.org/wiki/Propositional_calculus

Terminology



https://en.wikipedia.org/wiki/Propositional_calculus

Source Information: Patrick J. Hurley "A Concise Introduction to Logic-12th Ed."

Deductive Arguments

- A deductive argument asserts that the truth of the conclusion is a logical consequence of the premises. Based on the premises, the conclusion follows necessarily (with certainty). For example, given premises that A=B and B=C, then the conclusion follows necessarily that A=C. Deductive arguments are sometimes referred to as "truth-preserving" arguments.
- A deductive argument is said to be valid or invalid. If one *assumes* the premises to be true (ignoring their actual truth values), would the conclusion follow with certainty? If yes, the argument is valid. Otherwise, it is invalid. In determining validity, the structure of the argument is essential to the determination, not the actual truth values. For example, consider the argument that because bats can fly (premise=true), and all flying creatures are birds (premise=false), therefore bats are birds (conclusion=false). If we assume the premises are true, the conclusion follows necessarily, and thus it is a valid argument.
- If a deductive argument is valid and its premises are all true, then it is also referred to as sound. Otherwise, it is unsound, as in the "bats are birds" example.

https://en.wikipedia.org/wiki/Propositional_calculus

Inductive Arguments

- An inductive argument, on the other hand, asserts that the truth of the conclusion is supported to some degree of probability by the premises. For example, given that the U.S. military budget is the largest in the world (premise=true), then it is probable that it will remain so for the next 10 years (conclusion=true). Arguments that involve predictions are inductive, as the future is uncertain.
- An inductive argument is said to be strong or weak. If the premises of an inductive argument are *assumed* true, is it probable the conclusion is also true? If so, the argument is strong. Otherwise, it is weak.
- A strong argument is said to be cogent if it has all true premises. Otherwise, the argument is uncogent. The military budget argument example above is a strong, cogent argument.

https://en.wikipedia.org/wiki/Propositional_calculus

Validity

A *deductive argument* is one that, if valid, has a conclusion that is entailed by its premises. In other words, the truth of the conclusion is a logical consequence of the premises—if the premises are true, then the conclusion must be true. It would be self-contradictory to assert the premises and deny the conclusion, because the negation of the conclusion is contradictory to the truth of the premises.

https://en.wikipedia.org/wiki/Propositional_calculus

Soundness

An argument is **sound** if and only if

1. The argument is valid, and
2. All of its premises are true.

For instance,

All men are mortal.
Socrates is a man.
Therefore, Socrates is mortal.

The argument is valid (because the conclusion is true based on the premises, that is, that the conclusion follows the premises) and since the premises are in fact true, the argument is sound.

The following argument is valid but not sound:

All organisms with wings can fly.
Penguins have wings.
Therefore, penguins can fly.

Since the first premise is actually false, the argument, though valid (the premises of an argument do not have to be true in order for the argument to be valid), is not sound.

https://en.wikipedia.org/wiki/Soundness

Valid and Sound Arguments

a valid argument

$H_1 = \boldsymbol{T}$	$\boldsymbol{F}$	$\boldsymbol{T}$	$\boldsymbol{F}$	$\boldsymbol{T}$	$\boldsymbol{F}$	$\boldsymbol{T}$	$\boldsymbol{F}$
$H_2 = \boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{F}$	$\boldsymbol{F}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{F}$	$\boldsymbol{F}$
$H_2 = \boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{F}$	$\boldsymbol{F}$	$\boldsymbol{F}$	$\boldsymbol{F}$
$C = \boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$

a sound argument

a invvalid argument

$H_1 = \boldsymbol{T}$
$H_2 = \boldsymbol{T}$
$H_2 = \boldsymbol{T}$
$C = \boldsymbol{F}$

https://en.wikipedia.org/wiki/Soundness

Logical equivalence and bi-conditionals

Modus ponens (conditional elimination)

From p and $(p \to q)$, infer q.

That is, $\{p, p \to q\} \vdash q$.

Conditional proof (conditional introduction)

From [accepting p allows a proof of q], infer $(p \to q)$.

That is, $(p \vdash q) \vdash (p \to q)$.

https://en.wikipedia.org/wiki/Propositional_calculus

Argument Rules (1)

Negation introduction

From $(p \to q)$ and $(p \to \neg q)$, infer $\neg p$.

That is, $\{(p \to q), (p \to \neg q)\} \vdash \neg p$.

Negation elimination

From $\neg p$, infer $(p \to r)$.

That is, $\{\neg p\} \vdash (p \to r)$.

Double negative elimination

From $\neg\neg p$, infer p.

That is, $\neg\neg p \vdash p$.

Conjunction introduction

From p and q, infer $(p \land q)$.

That is, $\{p, q\} \vdash (p \land q)$.

Conjunction elimination

From $(p \land q)$, infer p.

From $(p \land q)$, infer q.

That is, $(p \land q) \vdash p$ and $(p \land q) \vdash q$.

https://en.wikipedia.org/wiki/Propositional_calculus

Argument Rules (2)

Disjunction introduction

From p, infer $(p \lor q)$.

From q, infer $(p \lor q)$.

That is, $p \vdash (p \lor q)$ and $q \vdash (p \lor q)$.

Disjunction elimination

From $(p \lor q)$ and $(p \to r)$ and $(q \to r)$, infer r.

That is, $\{p \lor q, p \to r, q \to r\} \vdash r$.

Biconditional introduction

From $(p \to q)$ and $(q \to p)$, infer $(p \leftrightarrow q)$.

That is, $\{p \to q, q \to p\} \vdash (p \leftrightarrow q)$.

Biconditional elimination

From $(p \leftrightarrow q)$, infer $(p \to q)$.

From $(p \leftrightarrow q)$, infer $(q \to p)$.

That is, $(p \leftrightarrow q) \vdash (p \to q)$ and $(p \leftrightarrow q) \vdash (q \to p)$.

https://en.wikipedia.org/wiki/Propositional_calculus

Argument Rules (3)

Modus ponens (conditional elimination)

From p and $(p \to q)$, infer q.

That is, $\{p, p \to q\} \vdash q$.

Conditional proof (conditional introduction)

From [accepting p allows a proof of q], infer $(p \to q)$.

That is, $(p \vdash q) \vdash (p \to q)$.

https://en.wikipedia.org/wiki/Propositional_calculus

Modus Ponens

The *modus ponens* rule is written as the statement of a truth-functional tautology or theorem of propositional logic:

$$((P \to Q) \land P) \to Q$$

where P, and Q are propositions expressed in some formal system.

Or in sequent notation:

$$P \to Q,\ P \ \vdash \ Q$$

where $\vdash$ is a metalogical symbol meaning that Q is a syntactic consequence of $P \to Q$ and P in some logical system.

https://en.wikipedia.org/wiki/Modus_ponens

Modus Ponens

The validity of *modus ponens* in classical two-valued logic can be clearly demonstrated by use of a truth table.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

In instances of *modus ponens* we assume as premises that $p \to q$ is true and p is true. Only one line of the truth table—the first—satisfies these two conditions (p and $p \to q$). On this line, q is also true. Therefore, whenever $p \to q$ is true and p is true, q must also be true.

https://en.wikipedia.org/wiki/Modus_ponens

Modus Tollens

The *modus tollens* rule may be written in sequent notation:

$$P \to Q, \neg Q \vdash \neg P$$

where $\vdash$ is a metalogical symbol meaning that $\neg P$ is a syntactic consequence of $P \to Q$ and $\neg Q$ in some logical system;

or as the statement of a functional tautology or theorem of propositional logic:

$$((P \to Q) \land \neg Q) \to \neg P$$

where P and Q are propositions expressed in some formal system;

or including assumptions:

$$\frac{\Gamma \vdash P \to Q \quad \Gamma \vdash \neg Q}{\Gamma \vdash \neg P}$$

though since the rule does not change the set of assumptions, this is not strictly necessary.

https://en.wikipedia.org/wiki/Modus_tollens

Modus Tollens

The validity of *modus tollens* can be clearly demonstrated through a truth table.

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

In instances of *modus tollens* we assume as premises that p → q is true and q is false. There is only one line of the truth table—the fourth line—which satisfies these two conditions. In this line, p is false. Therefore, in every instance in which p → q is true and q is false, p must also be false.

https://en.wikipedia.org/wiki/Modus_tollens

Modus Ponens

https://en.wikipedia.org/wiki/Modus_ponens

Modus Ponens

https://en.wikipedia.org/wiki/Modus_ponens

Modus Ponens

https://en.wikipedia.org/wiki/Modus_ponens

Modus Ponens

https://en.wikipedia.org/wiki/Modus_ponens

Modus Ponens

https://en.wikipedia.org/wiki/Modus_ponens

Modus Ponens

https://en.wikipedia.org/wiki/Modus_ponens

Modus Ponens

https://en.wikipedia.org/wiki/Modus_ponens

References

[1] http://en.wikipedia.org/
[2]