# Group Delay and Phase Delay (1A)

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7/19/12

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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# Phase Shift and Time Shift

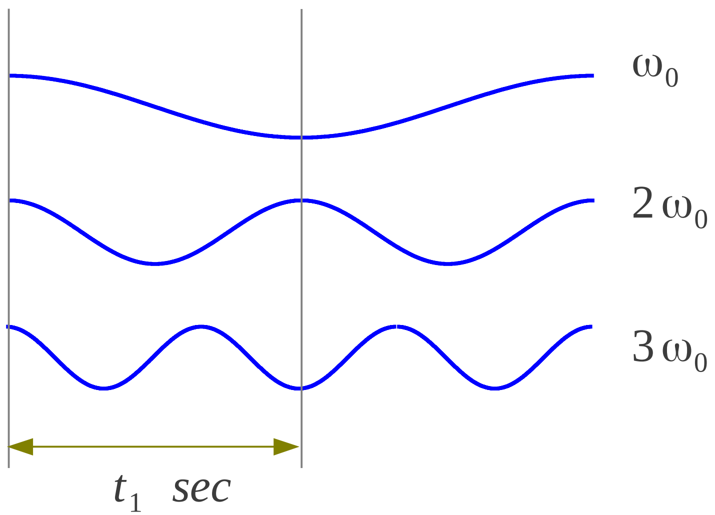
*measure phase shift not <u>in second</u>*
*but **<u>in portions</u>** of a cosine wave cycle*

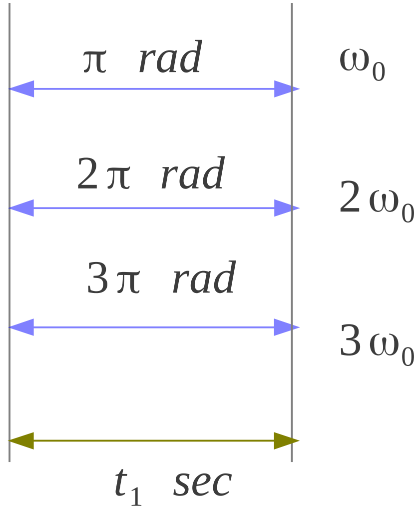
*within phase change in one cycle*

Phase Shift → in radians, degrees
Delay → in seconds (time)

Given time shift (delay) $t_1 \; sec$

The actual phase shift is different according to the frequency $\pi, 2\pi, 3\pi \; rad$



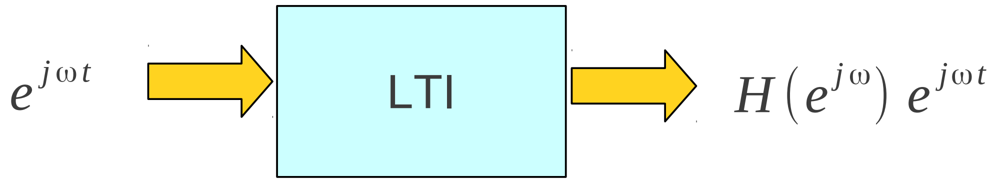




The same <u>delay</u>
applied to all frequencies

The different <u>phase shift</u>
to the different frequency

# Frequency Response

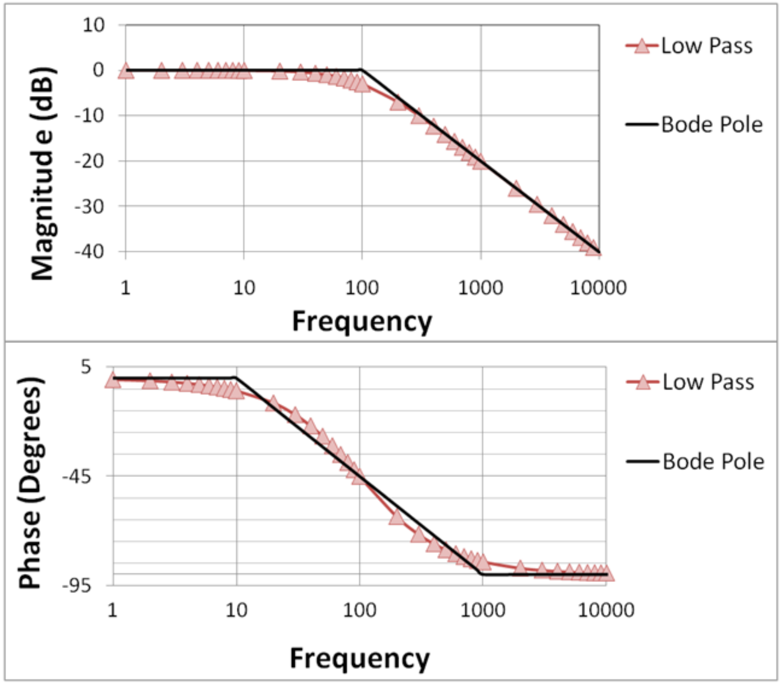
Frequency Response $H(e^{j\omega})$




LPF example

$|H(e^{j\omega})|$ Magnitude Response

$\angle H(e^{j\omega})$ Phase Response



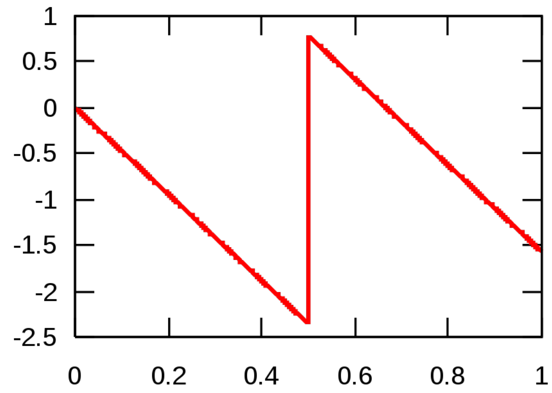
# Linear Phase System

## Linear Phase System

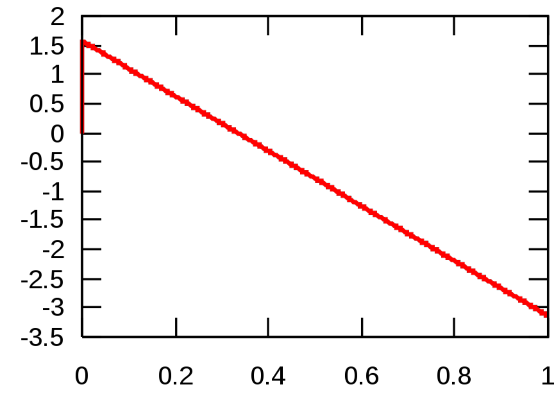
Phase Shift $\propto$ Frequency

$$\angle H\left(e^{j\omega}\right) \propto \omega$$

a) FIR Filter (Type II) having Linear Phase

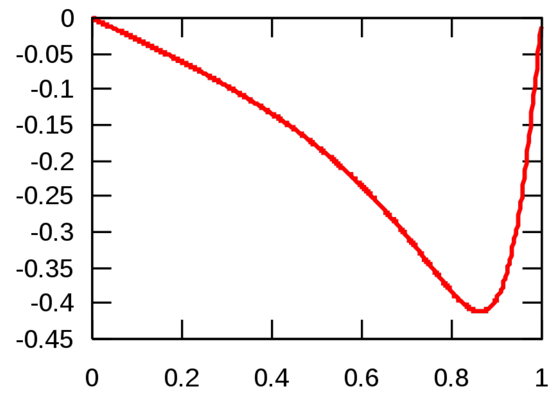



b) FIR Filter (Type IV) having Linear Phase

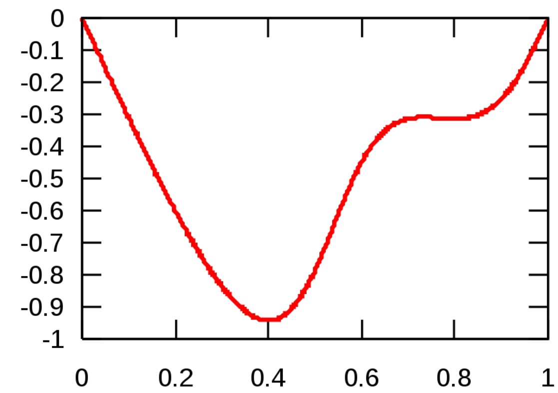



## Non-Linear Phase System

c) IIR Filter having Non-Linear Phase

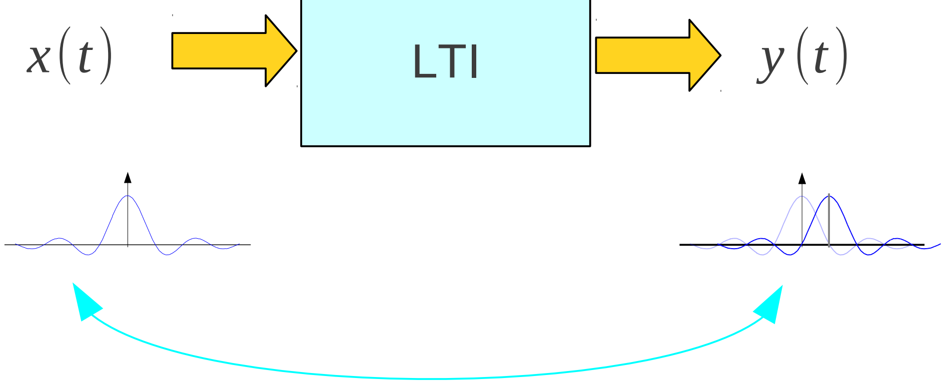



d) FIR Filter having Non-Linear Phase

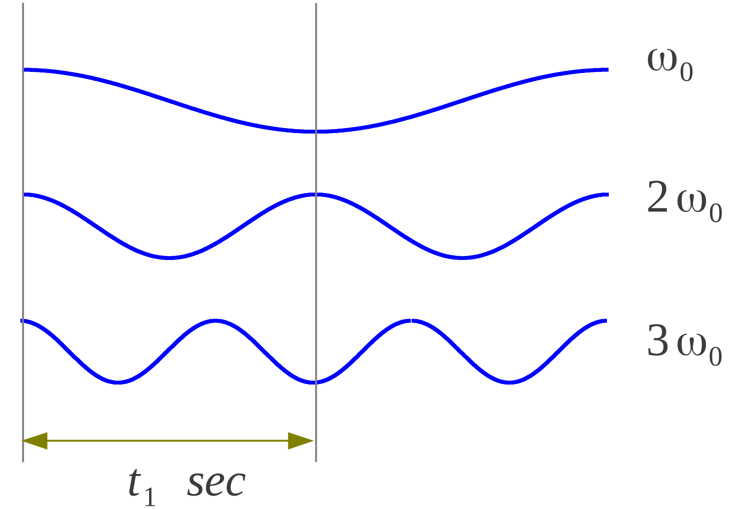


# Uniform Time Delay (1)

Frequency Response $H(e^{j\omega})$



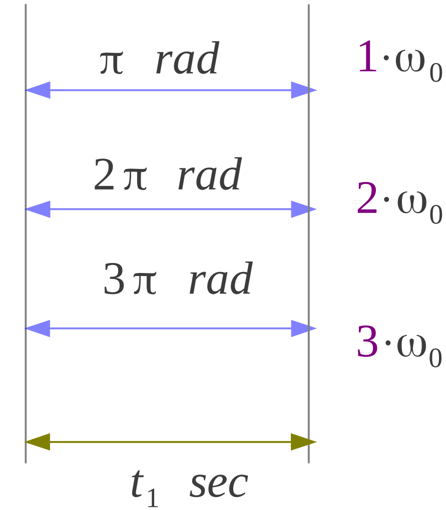

The waveform shape can be preserved.




- uniform magnitude $|H(e^{j\omega})| = c$
- uniform time delay

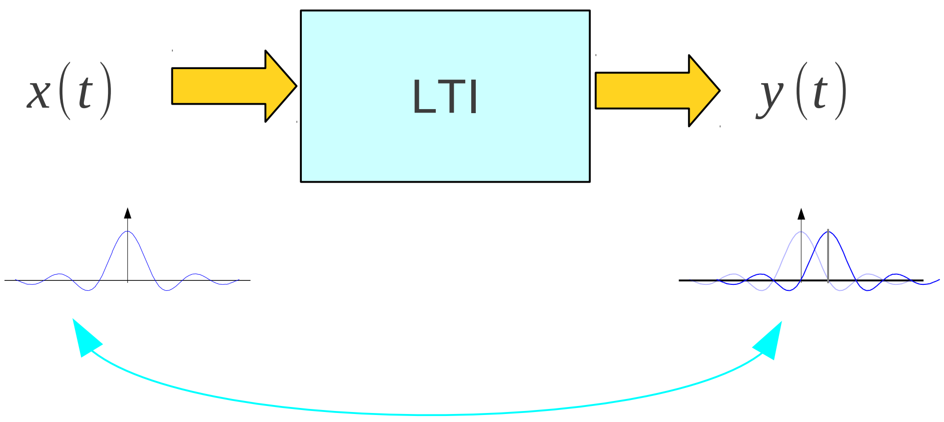
⇕

linear phase $\angle H(e^{j\omega}) = k\omega$



# Uniform Time Delay (2)

Frequency Response $H(e^{j\omega})$




$x(t)$ → LTI → $y(t)$

The waveform shape can be preserved.

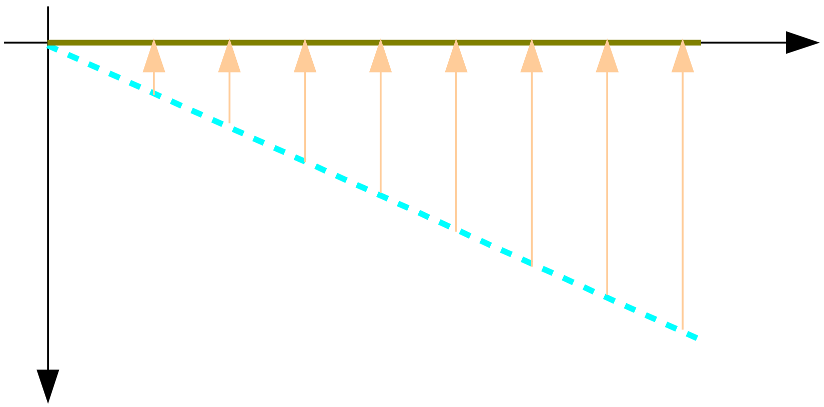
- uniform magnitude $\left|H(e^{j\omega})\right| = c$
- uniform time delay

↕

linear phase $\angle H(e^{j\omega}) = k\omega$

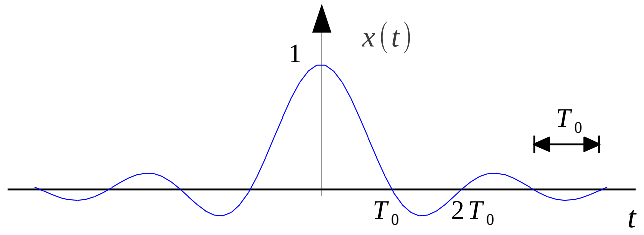
Uniform Time Delay

*Could remove delay*
*from the <u>phase response</u>*
*to achieve a horizontal line at*
***zero degree** (No delay)*



# CTFT of Sinc Function

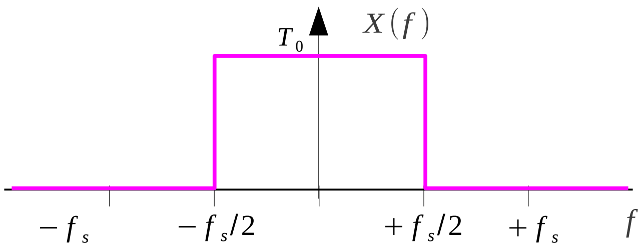
$t = \pm T_0, \pm 2T_0, \pm 3T_0, \cdots \;\Rightarrow\; x(t) = 0$



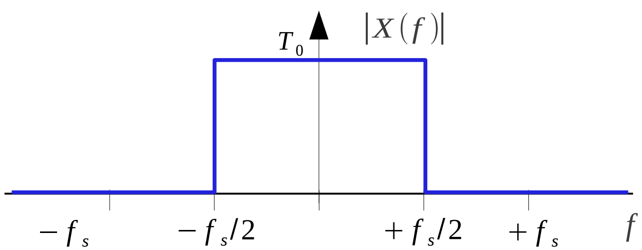

$$x(t) = \frac{\sin(\pi t/T_0)}{\pi t/T_0} = \frac{\sin(\pi f_s t)}{\pi f_s t}$$

$$\frac{1}{T_0} \equiv f_s$$

**CTFT**

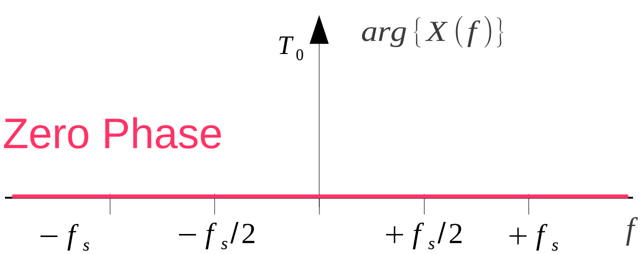



$$H(f) = \begin{cases} T_0, & |f| \le f_s/2 \\ 0, & otherwise \end{cases}$$

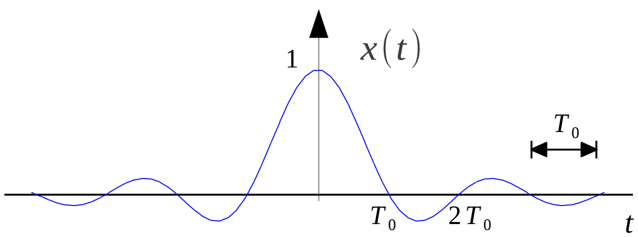



Real Symmetric Signal

Zero Phase

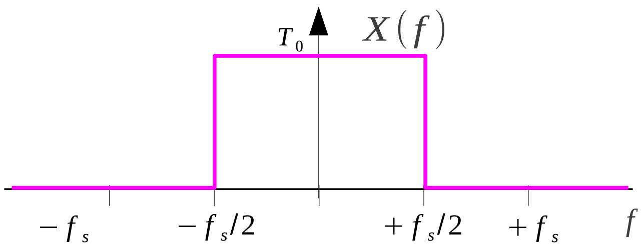


# CTFT Time Shifting Property




$$\frac{1}{T_0} \equiv f_s$$

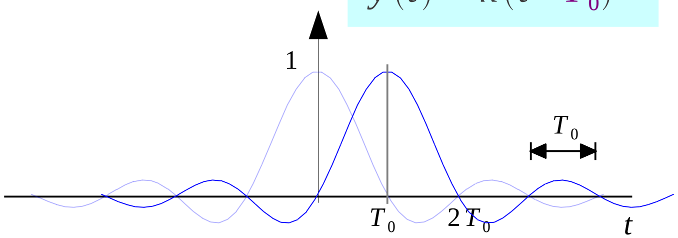
**CTFT**




$$x(t) = \frac{\sin(\pi t/T_0)}{\pi t/T_0} = \frac{\sin(\pi f_s t)}{\pi f_s t}$$

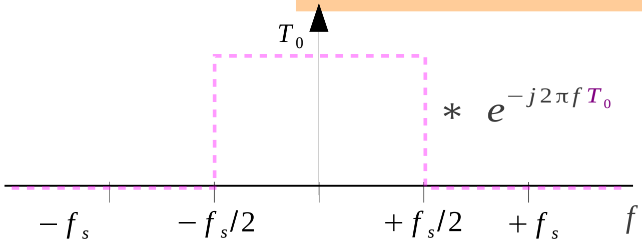
$$X(f) = \begin{cases} T_0, & |f| \le f_s/2 \\ 0, & otherwise \end{cases}$$

$$y(t) = x(t - T_0)$$



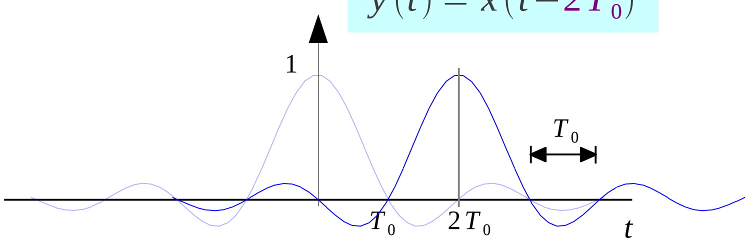

**CTFT**

$$Y(f) = X(f) \cdot e^{-j2\pi f T_0}$$



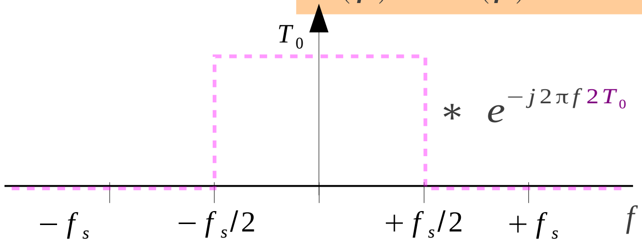

$$* \quad e^{-j2\pi f T_0}$$

$$y(t) = x(t - 2T_0)$$



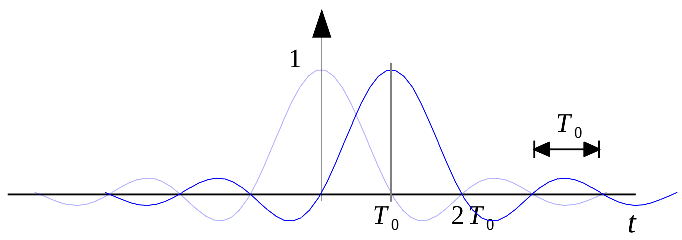

**CTFT**

$$Y(f) = X(f) \cdot e^{-j2\pi f 2T_0}$$



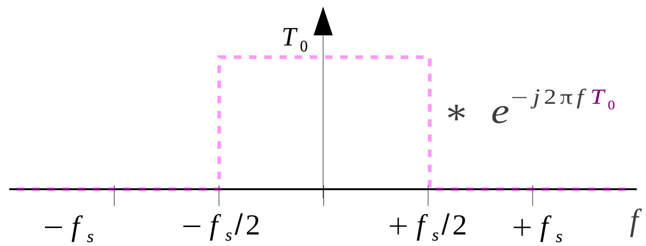

$$* \quad e^{-j2\pi f 2T_0}$$

# CTFT of Sinc Function Shifted by $T_0$




$$\frac{1}{T_0} \equiv f_s$$

**CTFT**



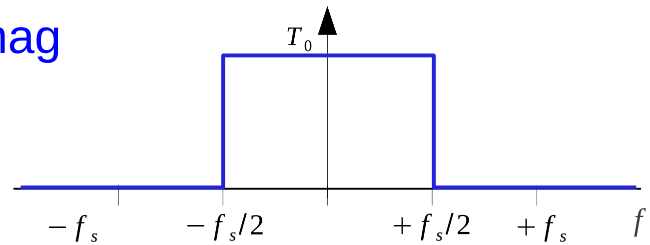

$$x(t) = \frac{\sin(\pi t/T_0)}{\pi t/T_0} = \frac{\sin(\pi f_s t)}{\pi f_s t}$$

$$X(f) = \begin{cases} T_0, & |f| \le f_s/2 \\ 0, & otherwise \end{cases}$$

$$y(t) = x(t - T_0)$$

$$Y(f) = X(f) \cdot e^{-j2\pi f T_0}$$

mag

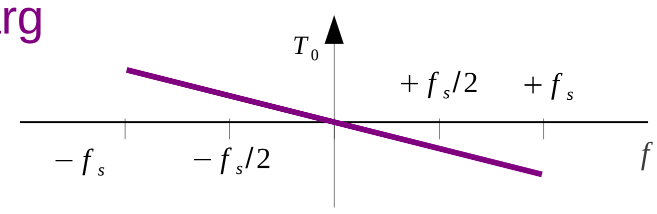



$$Arg \Rightarrow \Phi(f)$$

$$slope = \frac{d\Phi}{df} = -2\pi T_0 \Rightarrow \frac{d\Phi}{d\omega} = -T_0$$

Group Delay $\quad -\dfrac{d\Phi}{d\omega} = T_0$

arg




Pure Delay (No Dispersion) → Linear Phase Change

$slope = -2\pi T_0$



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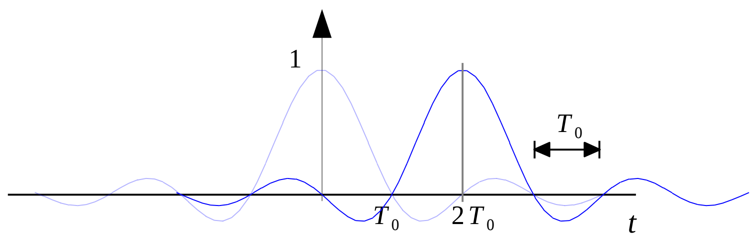
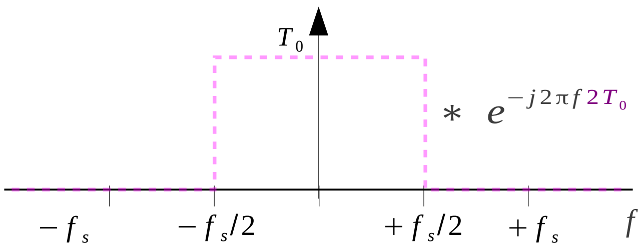
# CTFT of Sinc Function Shifted by $2T_0$




$$\frac{1}{T_0} \equiv f_s$$

**CTFT**



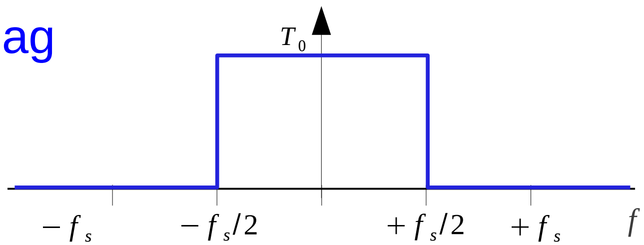

$$x(t) = \frac{\sin(\pi t/T_0)}{\pi t/T_0} = \frac{\sin(\pi f_s t)}{\pi f_s t}$$

$$X(f) = \begin{cases} T_0, & |f| \le f_s/2 \\ 0, & otherwise \end{cases}$$

$$y(t) = x(t - 2T_0)$$

$$Y(f) = X(f) \cdot e^{-j2\pi f 2T_0}$$

mag

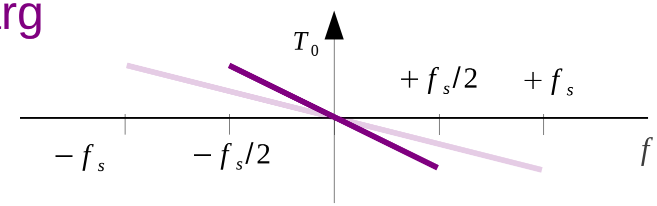



$Arg$ → $\Phi(f)$

$$slope = \frac{d\Phi}{df} = -2\pi 2T_0 \rightarrow \frac{d\Phi}{d\omega} = -2T_0$$

Group Delay $$-\frac{d\Phi}{d\omega} = 2T_0$$

arg




Pure Delay (No Dispersion) → Linear Phase Change

$$slope = -2\pi 2T_0$$

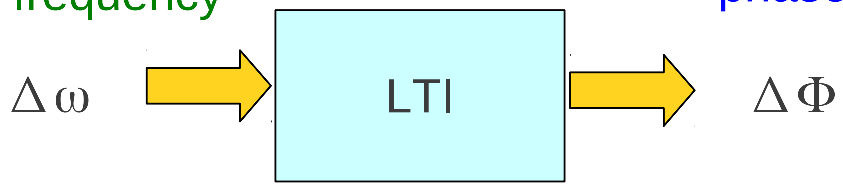
# Group Delay (1)

Consider the cosine components
at *closely spaced frequencies* and
*their phase shifts* in relation to each other

**Group Delay**:
The phase shift changes
for small changes in frequency

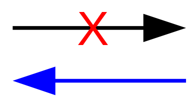
small changes in frequency

phase shift changes




A uniform, waveform preserving
phase response → linear

Constant Group Delay




Uniform Time Delay (linear phase)

# Group Delay (2)

Constant slope ➡ Constant Group Delay

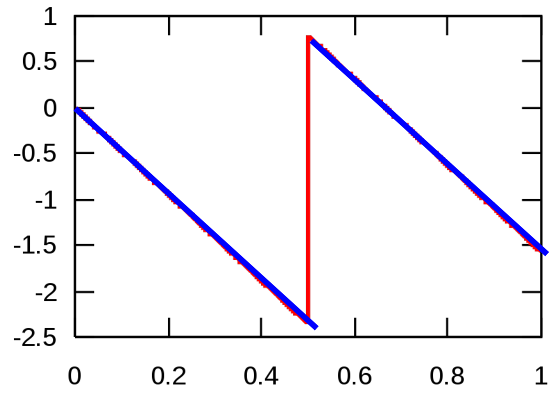
**Linear Phase System**

Phase Shift $\propto$ Frequency

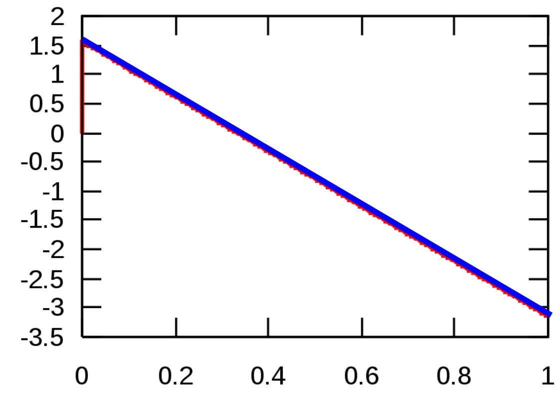
$\angle H(e^{j\omega}) \propto \omega$

No dispersion

a) FIR Filter (Type II) having Linear Phase



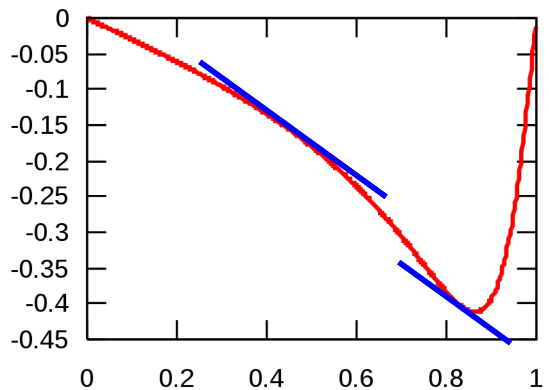

b) FIR Filter (Type IV) having Linear Phase



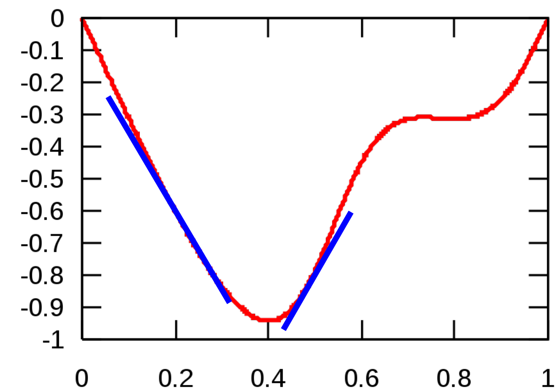

**Non-Linear Phase System**

Dispersion

c) IIR Filter having Non-Linear Phase

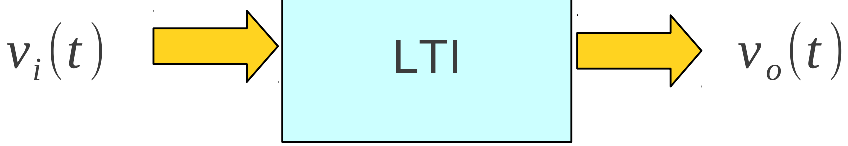


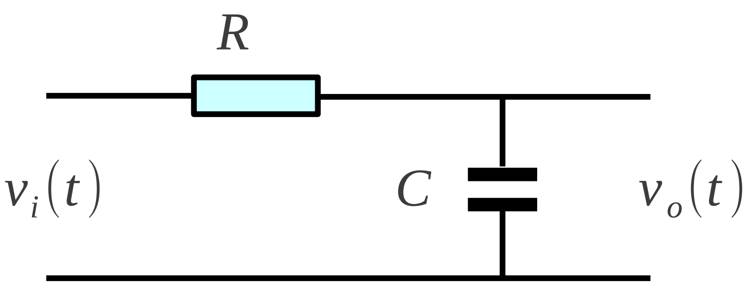

d) FIR Filter having Non-Linear Phase




Varying slope ➡ Varying Group Delay

## Frequency Response



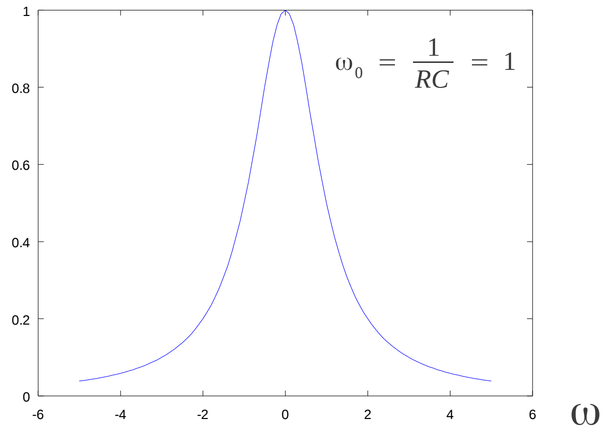

$$H(j\omega) = \frac{1}{1 + j\omega/\omega_0} \qquad \omega_0 = \frac{1}{RC}$$

$$A(j\omega) = |H(j\omega)| = \frac{1}{\sqrt{1+\omega^2/\omega_0^2}}$$

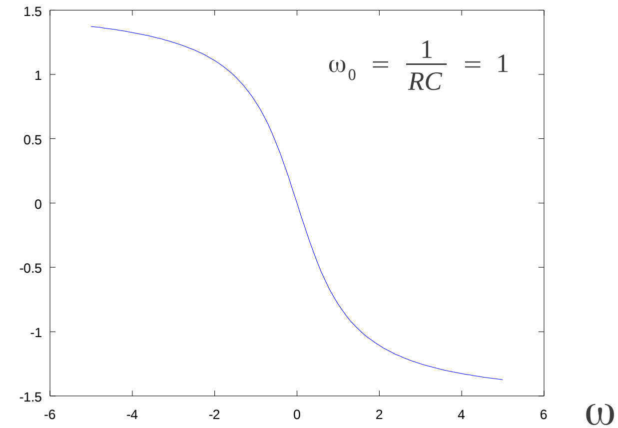
$$H(j\omega) = A(j\omega) e^{j\phi(j\omega)}$$

$$\phi(j\omega) = arg\{H(j\omega)\} = \tan^{-1}(-\omega/\omega_0)$$

Magnitude Response $|H(j\omega)|$

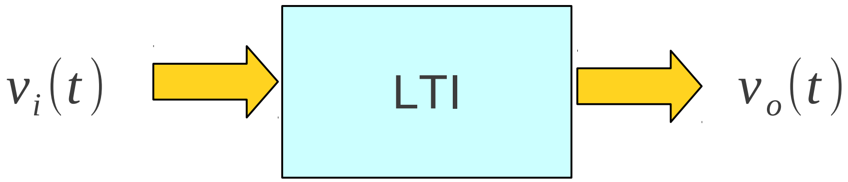


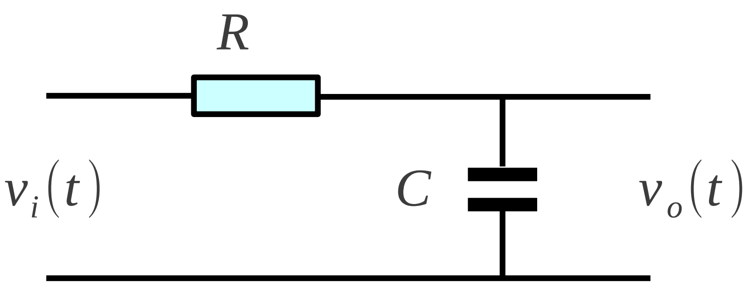

Phase Response $arg\{H(j\omega)\}$



# Simple LPF – Group Delay

## Frequency Response



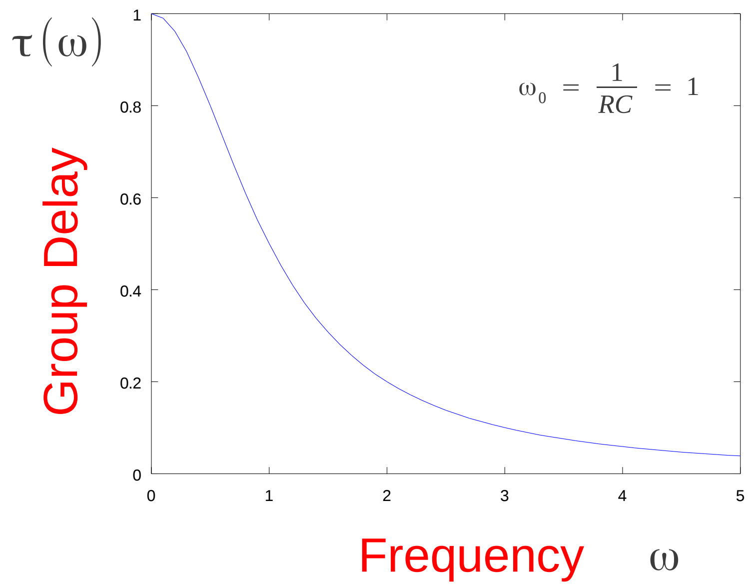

$$H(j\omega) = \frac{1}{1 + j\omega/\omega_0} \qquad \omega_0 = \frac{1}{RC}$$

$$H(j\omega) = A(j\omega) e^{j\phi(j\omega)}$$

$$A(j\omega) = \frac{1}{\sqrt{1+\omega^2/\omega_0^2}}$$

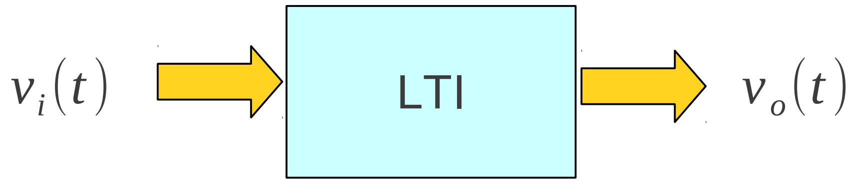
$$\phi(j\omega) = \tan^{-1}(-\omega/\omega_0)$$

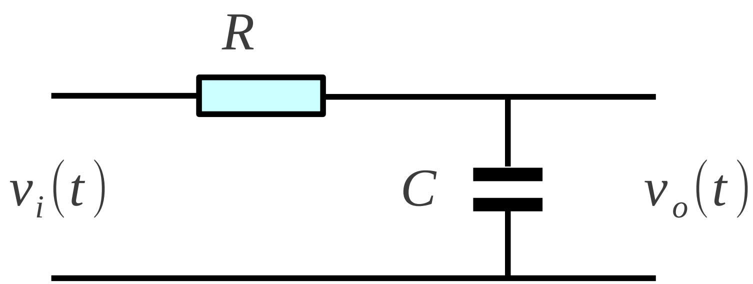
$$\tau(\omega) = -\frac{d\phi}{d\omega} = \frac{1}{1+\omega^2/\omega_0^2}$$



# Simple LPF – Step Response

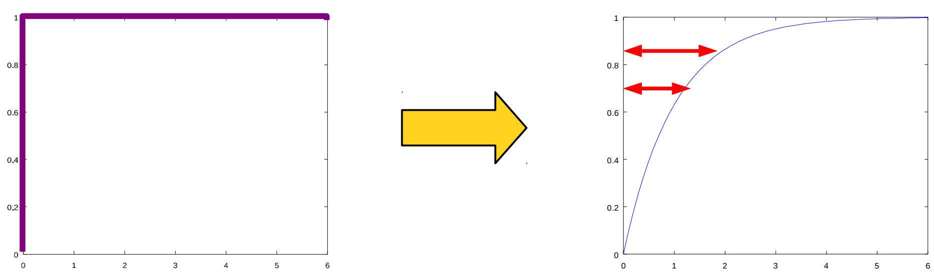
Frequency Response

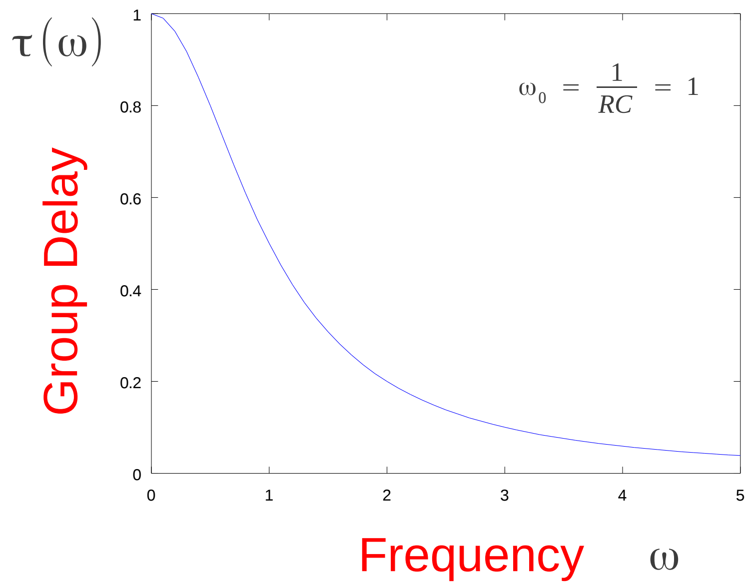



$$v_o(t) = 1 - e^{-\frac{t}{\tau}} \qquad \omega_0 = \frac{1}{RC} = \frac{1}{\tau}$$

*Which is the group delay ?*



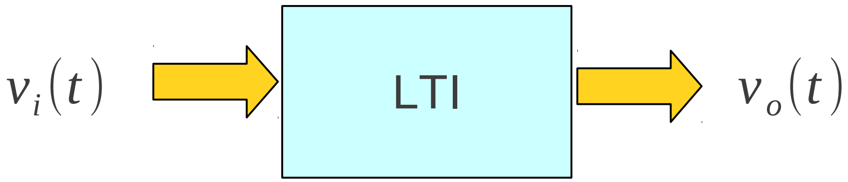


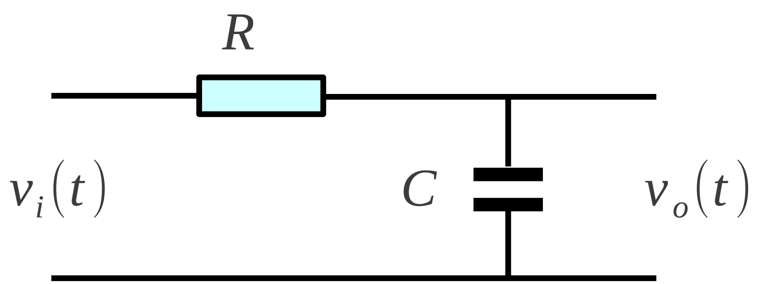

Group delay is not constant

Dispersion

# Simple LPF – Narrow Band Signal

Frequency Response




$$v_o(t) = 1 - e^{-\frac{t}{\tau}} \qquad \omega_0 = \frac{1}{RC} = \frac{1}{\tau}$$

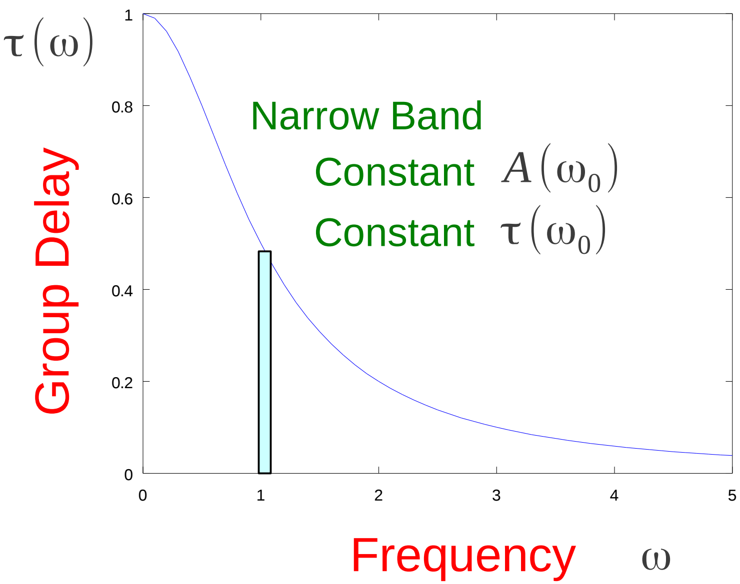
When focusing a narrow band signal

Output will be

Time delayed by $\tau(\omega_0)$

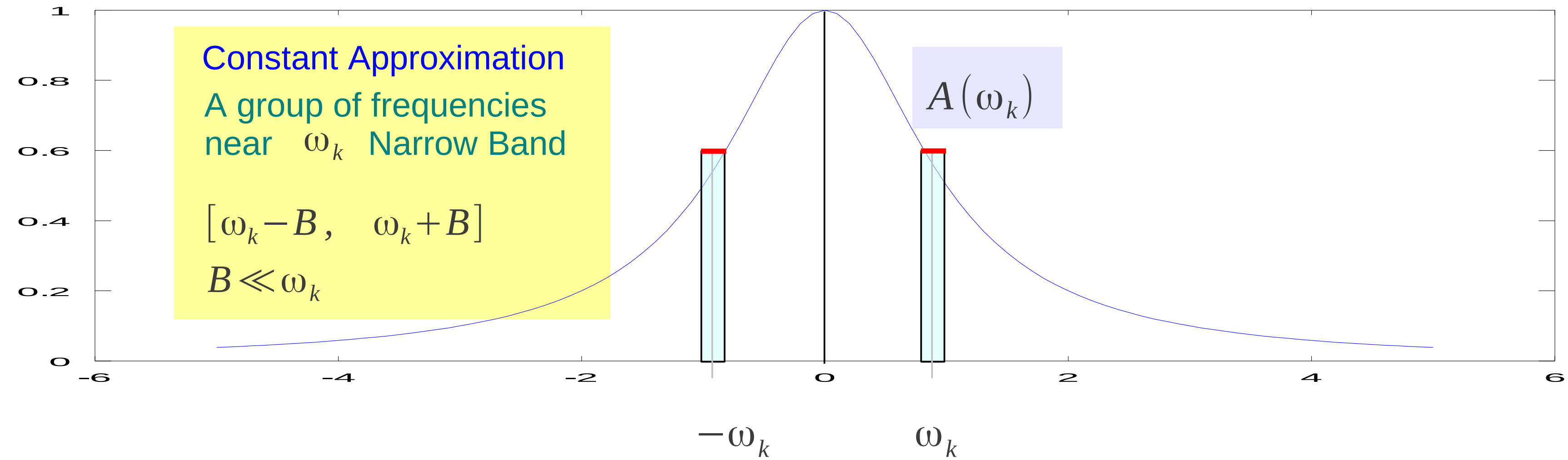
Amplitude scaled by $A(\omega_0)$

Phase shifted by $\phi(\omega_0)$

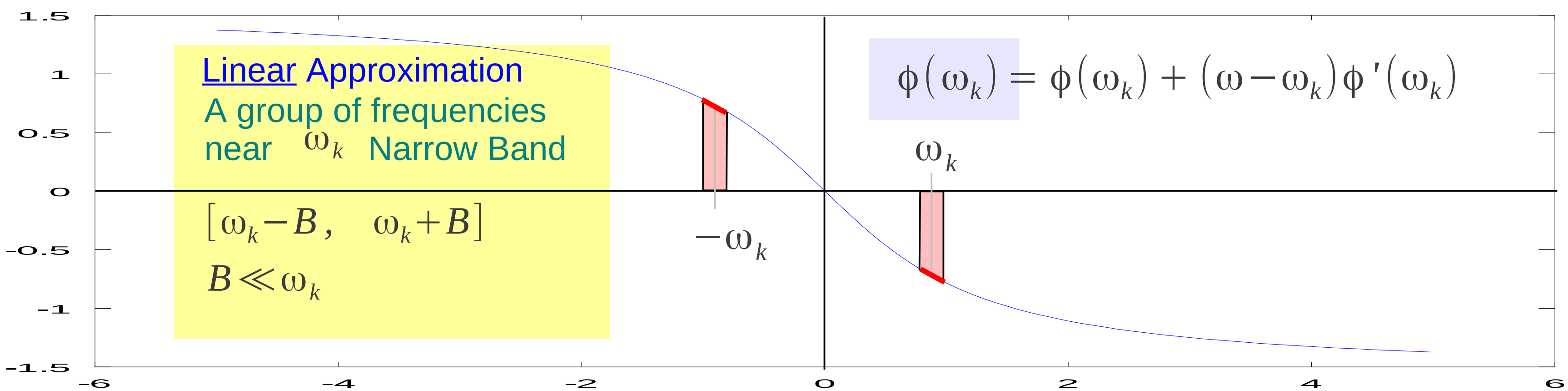


# Simple LPF – Approximation (1)

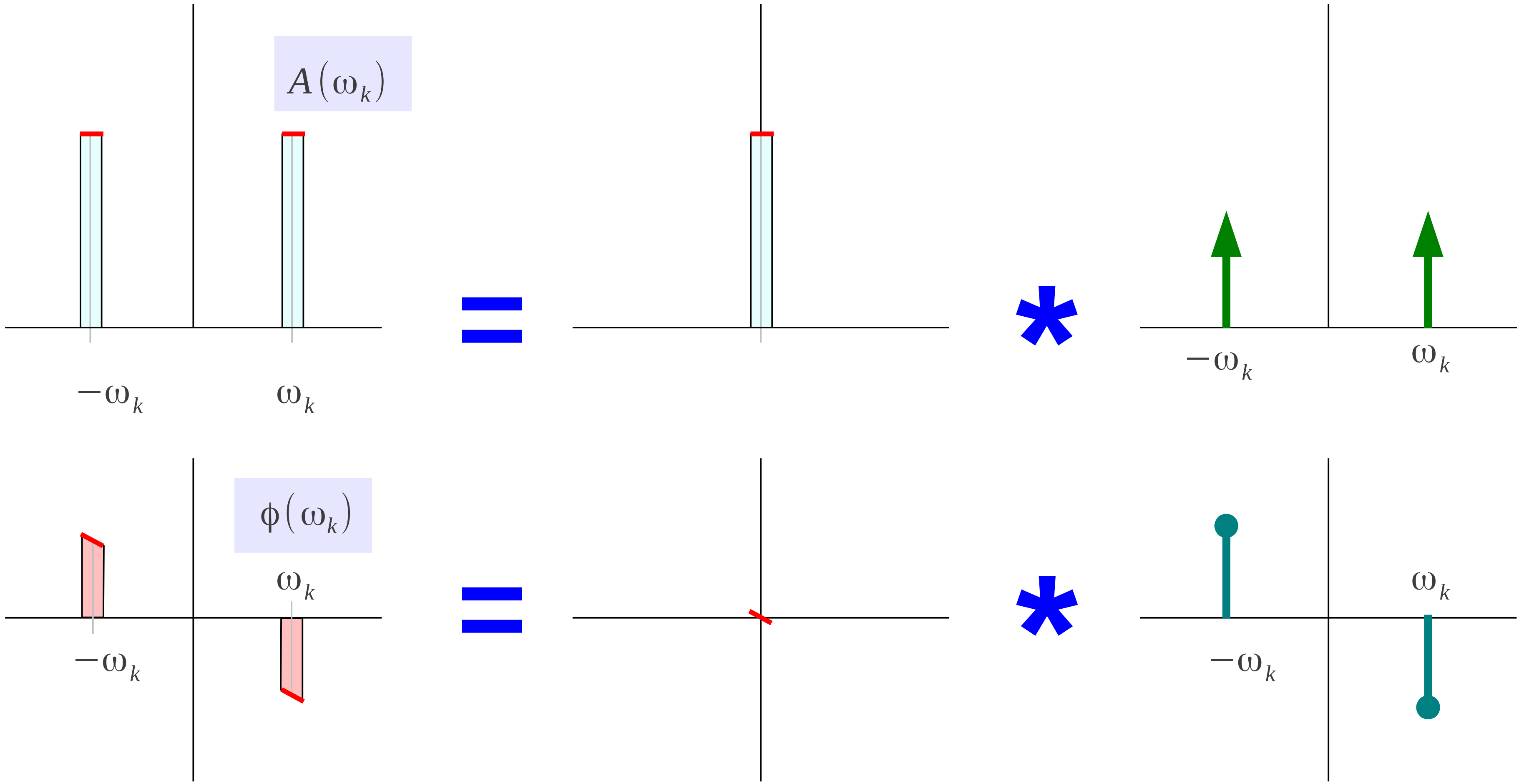
Magnitude Response $A(j\omega) = |H(j\omega)| = 1 \ / \ \sqrt{1+\omega^2/\omega_0^2}$ ➡ $\approx \sum_k A(\omega_k)$




Phase Response $\phi(j\omega) = arg\{H(j\omega)\} = \tan^{-1}(-\omega/\omega_0)$ ➡ $\approx \sum_k \phi(\omega_k)$



# Simple LPF – Approximation (2)

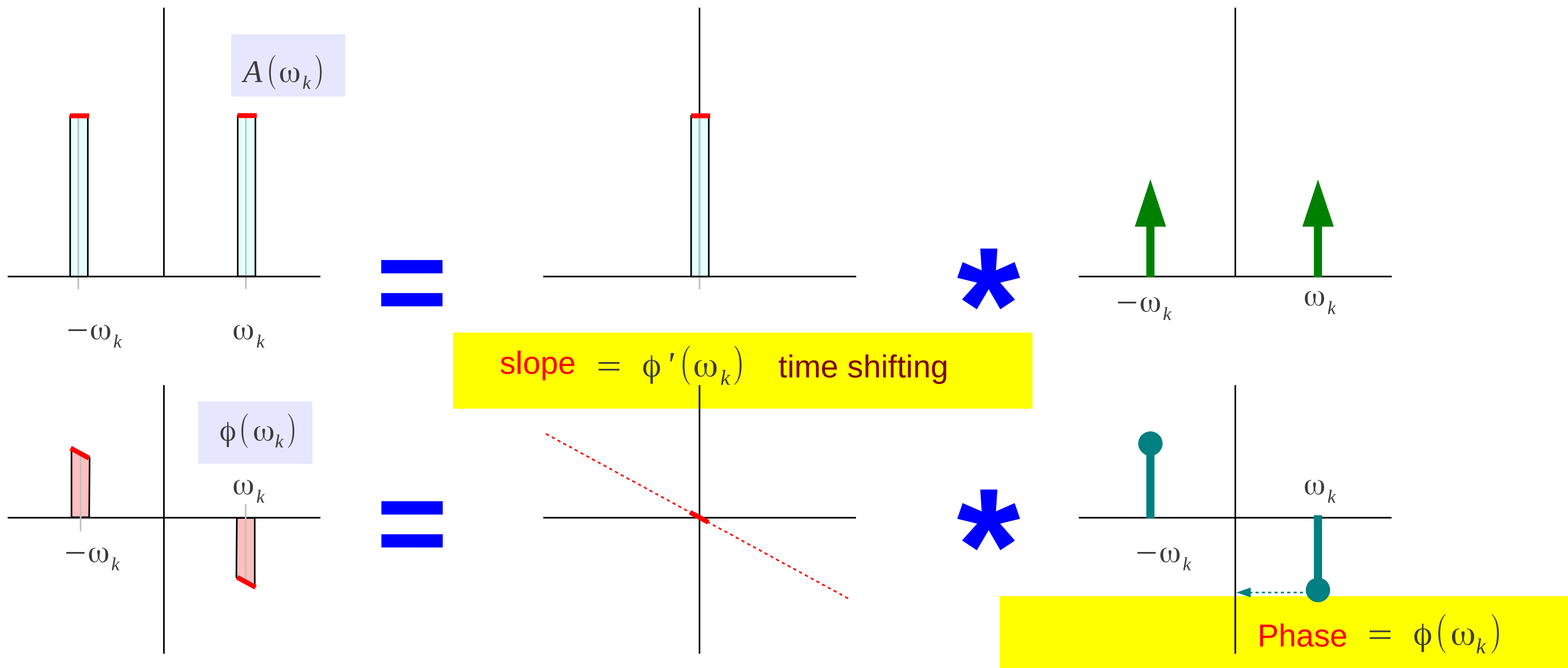



$$2A(\omega_k)B\frac{\sin[B(t+\phi'(\omega_k))]}{\pi B(t+\phi'(\omega_k))}$$

$$\begin{cases} A(\omega_k) & (-B \le \omega \le +B) \\ 0 & else \end{cases}$$

$$\pi[e^{+j\phi(\omega_k)}\delta(\omega-\omega_k) + e^{-j\phi(\omega_k)}\delta(\omega+\omega_k)]$$

# Simple LPF – Approximation (3)




**Frequency Domain**

$$\begin{cases} A(\omega_k) & (-B \le \omega \le +B) \\ 0 & else \end{cases} \quad * \quad \pi[e^{+j\phi(\omega_k)}\delta(\omega-\omega_k) + e^{-j\phi(\omega_k)}\delta(\omega+\omega_k)]$$

Group Delay

Phase Delay

**Time Domain**

$$2A(\omega_k)B\frac{\sin[B(t+\phi'(\omega_k))]}{\pi B(t+\phi'(\omega_k))} \quad \times \quad \cos(\omega_k t+\phi(\omega_k))$$

# Beat Signal

**Very similar frequency signals**

1.1 Hz $\cos(2\pi * 1.1 * t)$

0.9 Hz $\cos(2\pi * 0.9 * t)$

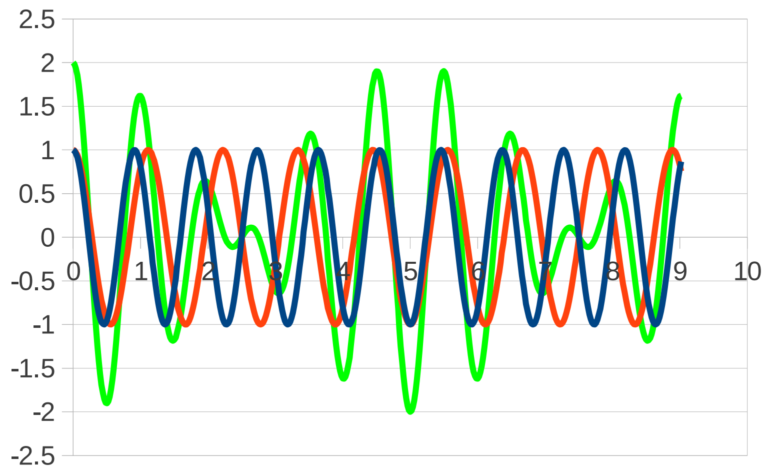
$$\cos(2\pi * 1.1 * t) + \cos(2\pi * 0.9 * t)$$

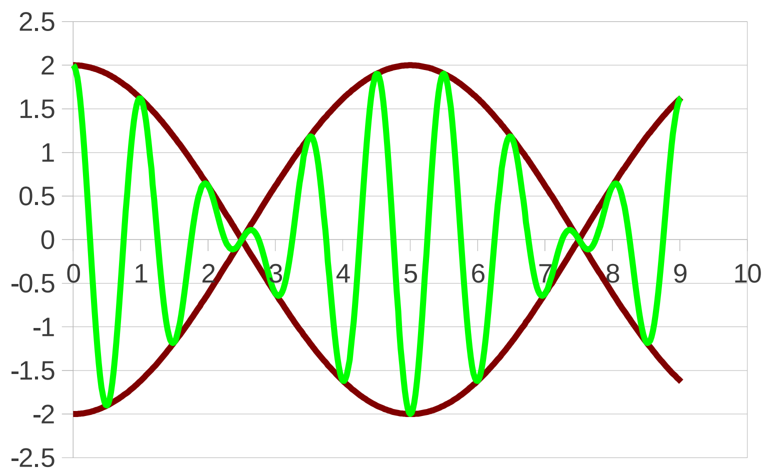
$$= \cos\left(2\pi * \frac{(1.1-0.9)}{2} * t\right) \cdot \cos\left(2\pi * \frac{(1.1+0.9)}{2} * t\right)$$

$$= \cos(2\pi * 0.1 * t) \cdot \cos(2\pi * 1.0 * t)$$

Slow moving envelop

Fast moving carrier

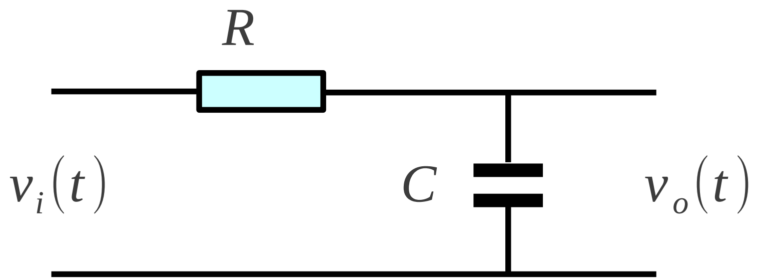



➡ **Bandlimited Narrowband Signal**

# Example – RC Circuit & Frequency Response

## Frequency Response

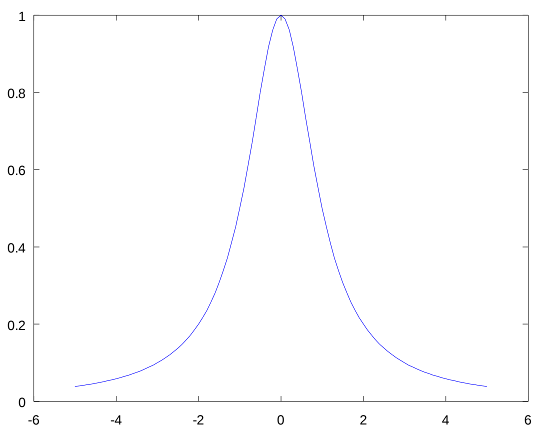
$$H(j\omega) = \frac{1}{1 + j\omega/\omega_0} \qquad \omega_0 = 1$$

$$|H(j\omega)| = \frac{1}{\sqrt{1+\omega^2}}$$

$$arg\{H(j\omega)\} = \tan^{-1}(-\omega)$$

$$arg\{H(j\omega)\} = \tan^{-1}(-\omega)$$

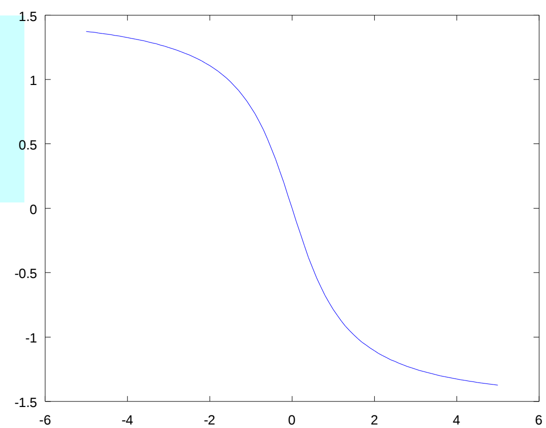
Magnitude Response $|H(j\omega)|$




$$|H(j\,0.9)| = 0.743$$
$$|H(j\,1.0)| = 0.707$$
$$|H(j\,1.1)| = 0.672$$

$\omega$

Phase Response $arg\{H(j\omega)\}$




$$\angle H(j\,0.9) = -0.73$$
$$\angle H(j\,1.0) = -0.79$$
$$\angle H(j\,1.1) = -0.83$$

$\omega$

# Example – Some Signal Responses

$\cos(0.9t) \Rightarrow$ LTI $\Rightarrow 0.743\cos(0.9t - 0.73)$

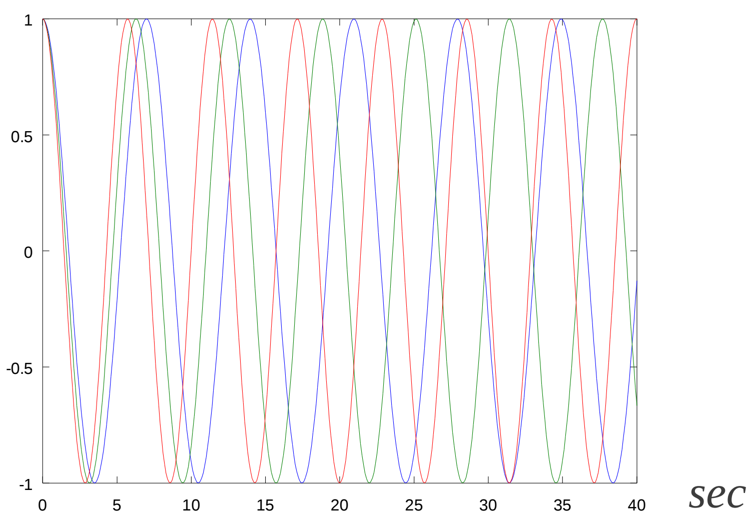
$\cos(1.0t) \Rightarrow$ LTI $\Rightarrow 0.707\cos(1.0t - 0.79)$

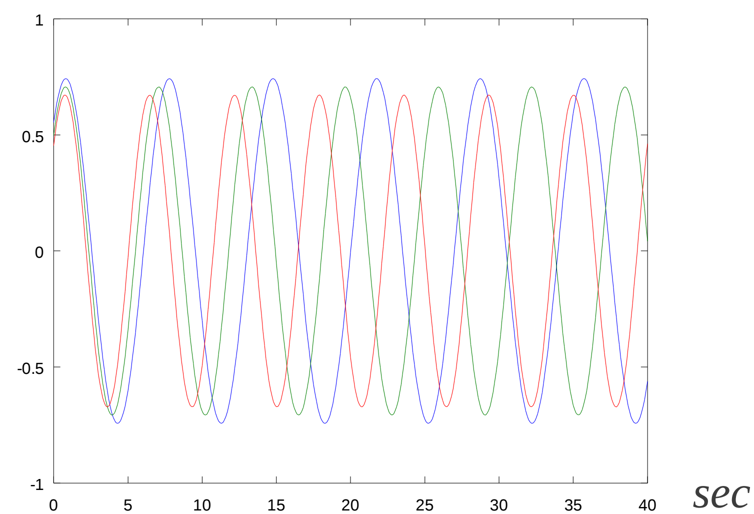
$\cos(1.1t) \Rightarrow$ LTI $\Rightarrow 0.672\cos(1.1t - 0.83)$

**Frequency Response**

$$|H(j\,0.9)| = 0.743$$
$$|H(j\,1.0)| = 0.707$$
$$|H(j\,1.1)| = 0.672$$

$$\angle H(j\,0.9) = -0.73$$
$$\angle H(j\,1.0) = -0.79$$
$$\angle H(j\,1.1) = -0.83$$

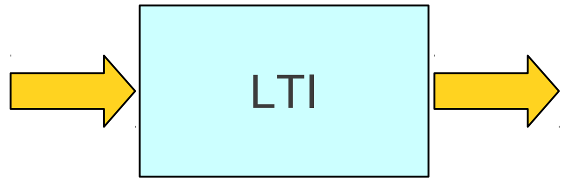


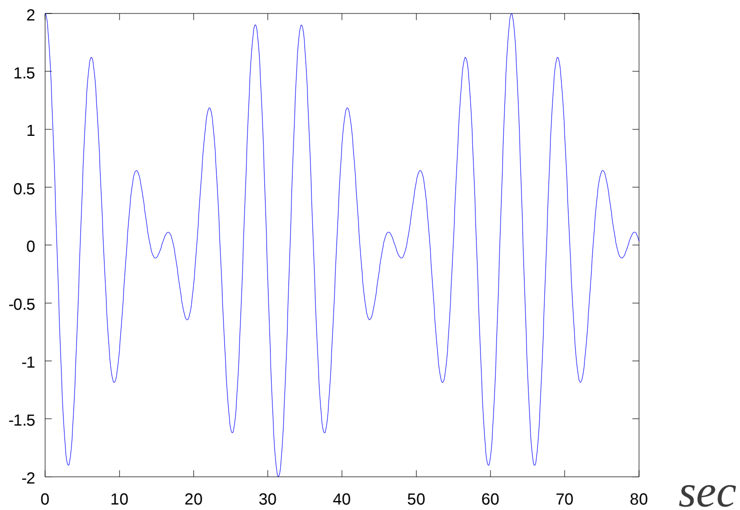
# Example – Beat Signal Response

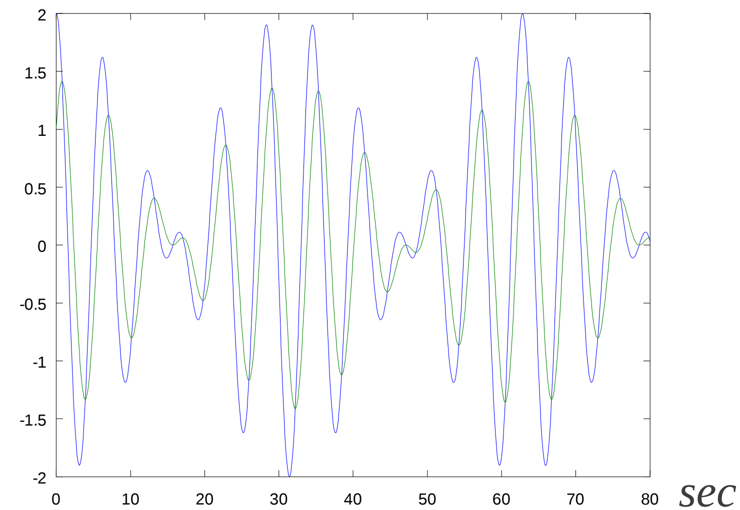
## Frequency Response

$$\cos(0.9t) + \cos(1.1t) = 2\cos(0.1t)\cos(1.0t)$$




$$0.743\cos(0.9t - 0.73) + 0.672\cos(1.1t - 0.83)$$



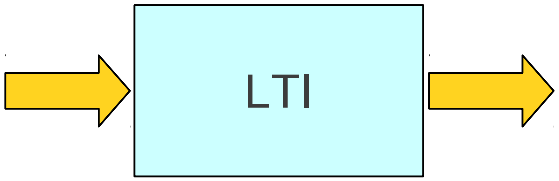



# Example – Envelopes

**Frequency Response** $\tau(\omega) = -\dfrac{d\phi}{d\omega} = \dfrac{1}{1+\omega^2}$ $\tau(1) = \dfrac{1}{2}$

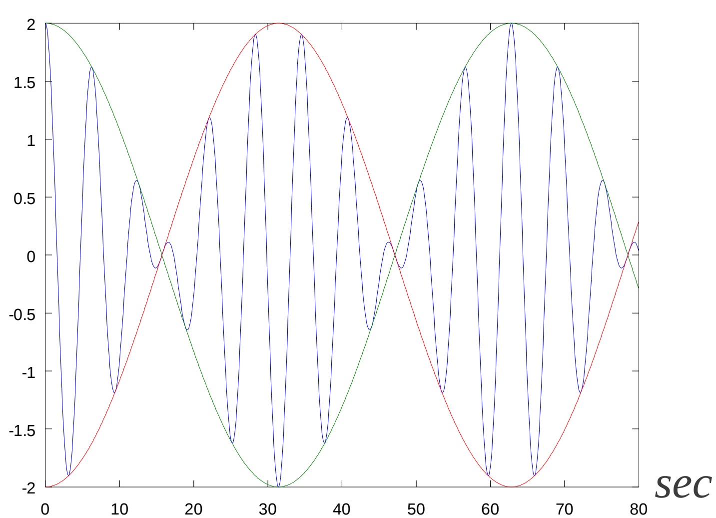
$$\cos(0.9t) + \cos(1.1t)$$

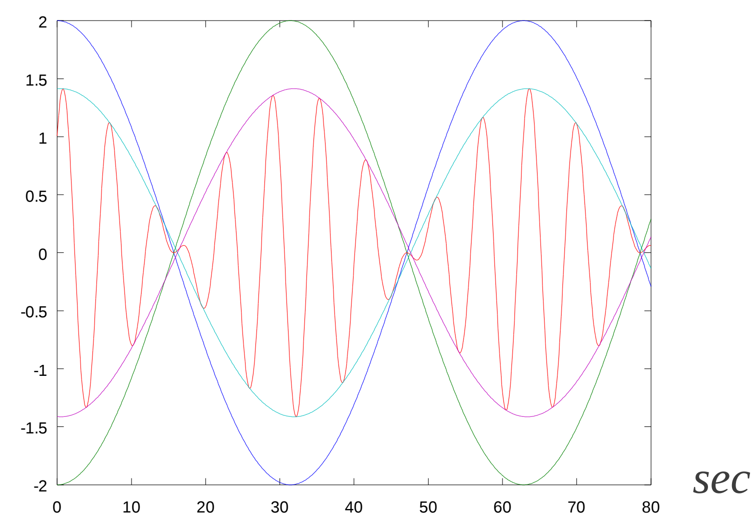
$$= 2\cos(0.1t)\cos(1.0t)$$



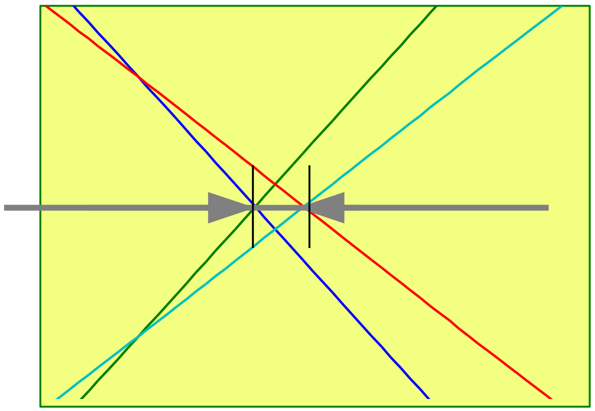

$$0.743\cos(0.9t - 0.73) + 0.672\cos(1.1t - 0.83)$$

$$\approx 1.41\cos(0.1(t-0.5))\cos(1.0t - \pi/4)$$

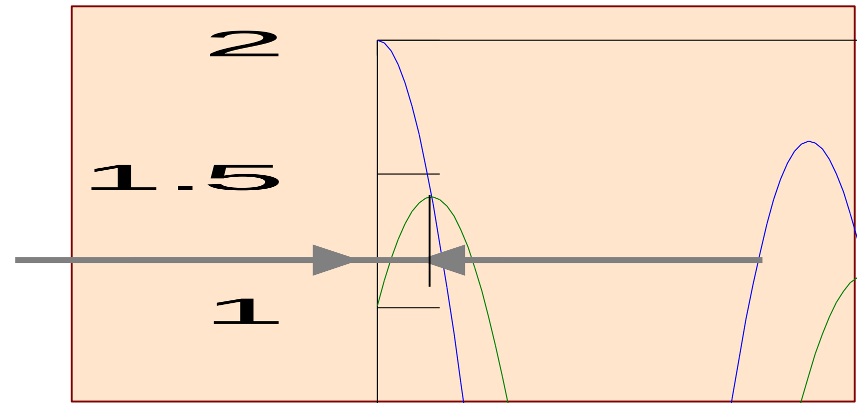


# Example – Group & Phase Delay



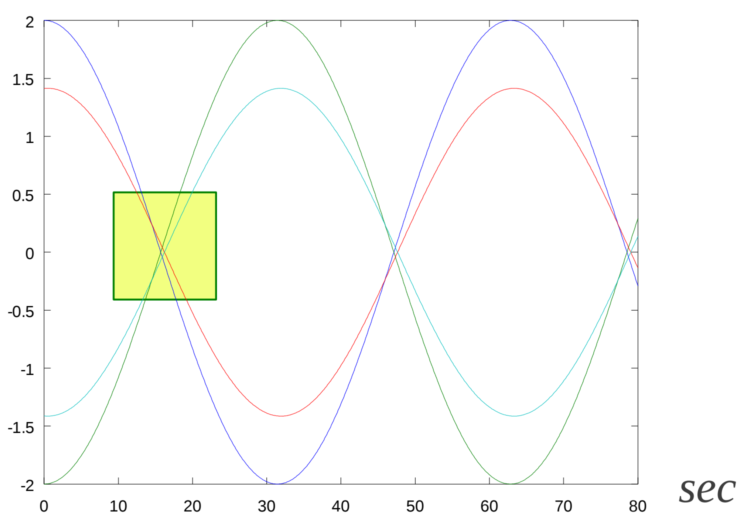
$$\tau(\omega) = -\frac{d\phi}{d\omega}$$
$$= \frac{1}{1+\omega^2}$$
$$\tau(1) = \frac{1}{2}$$

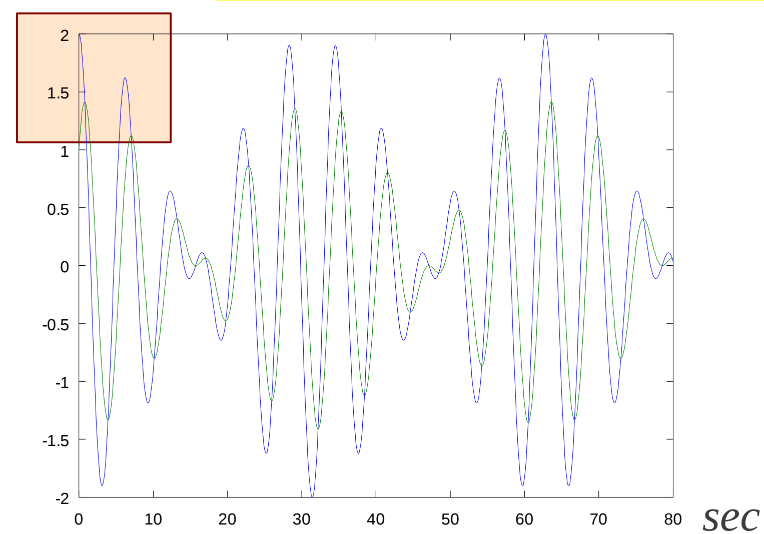



$t_g = 0.5$

$arg\{H(1)\} = -\tan^{-1}(1)$

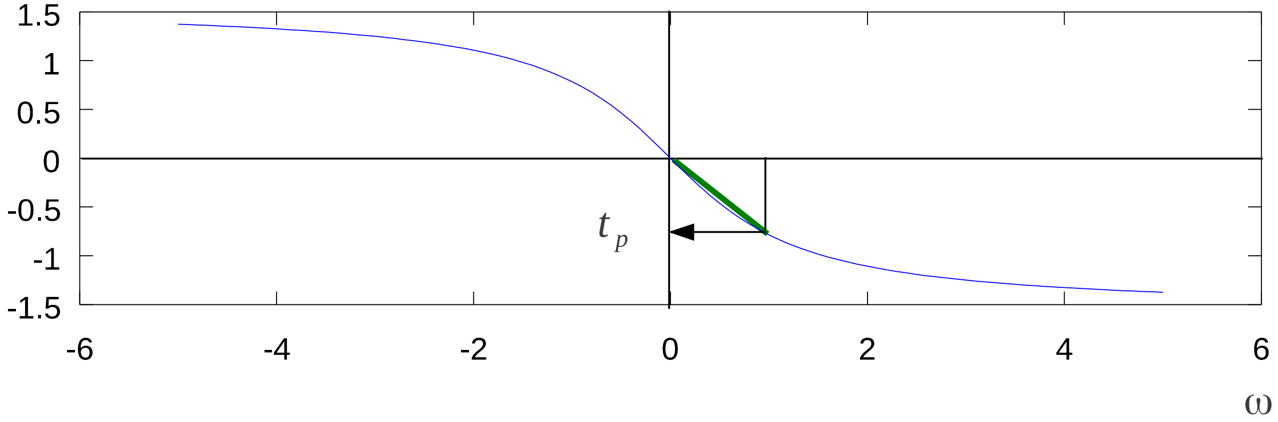
$t_p = -\pi/4 = -0.78$



# Phase & Group Delay from Phase Response

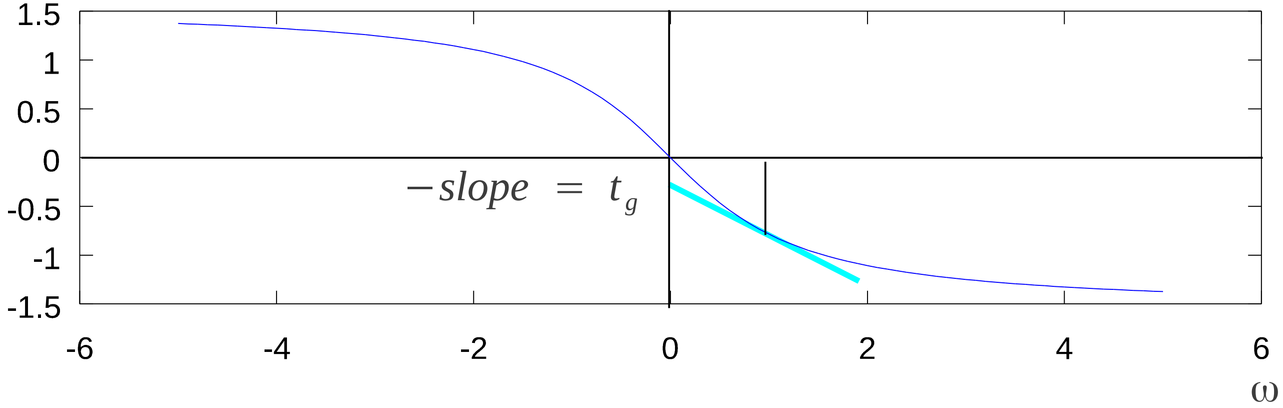
Phase Response




Phase Delay

$$t_p = -\tan^{-1}(1)$$
$$= -\pi/4 = -0.785$$

Phase Response

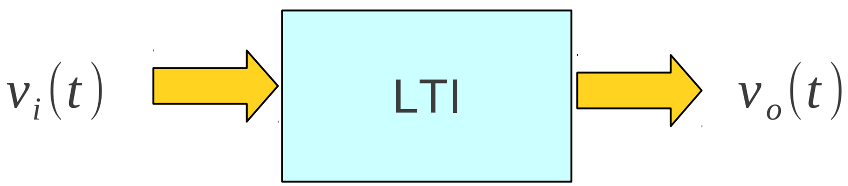


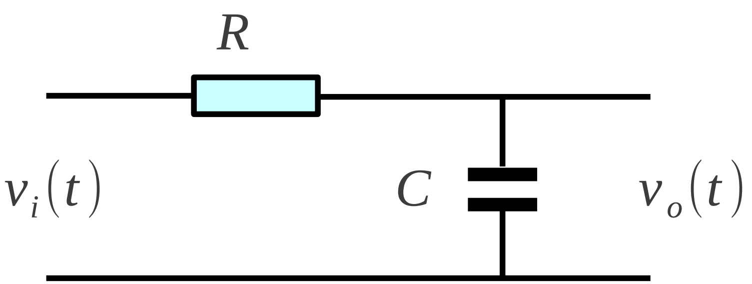

Group Delay

$$t_g = 0.5$$

# Simple LPF – Step Response

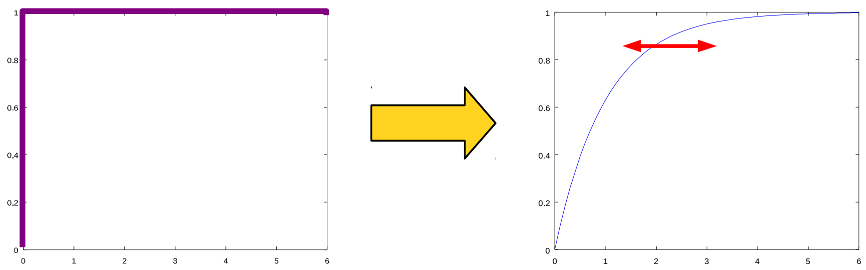
Frequency Response




$$v_o(t) = 1 - e^{-\frac{t}{\tau}} \qquad \omega_0 = \frac{1}{RC} = \frac{1}{\tau}$$

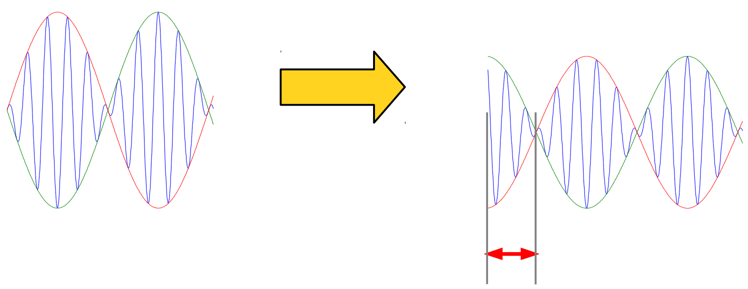
*Which is the group delay ?*



Group delay is not constant

Dispersion

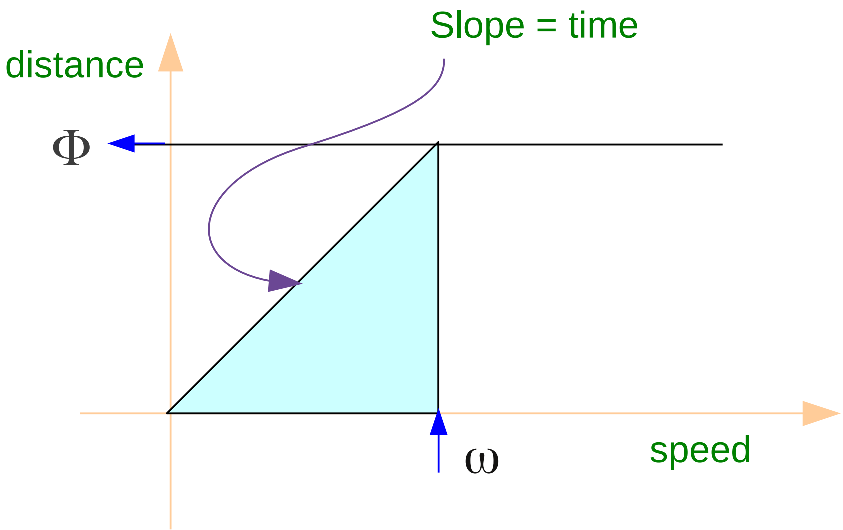
*Group delay with a narrow band signal*



*Bandlimited* *Group delay*

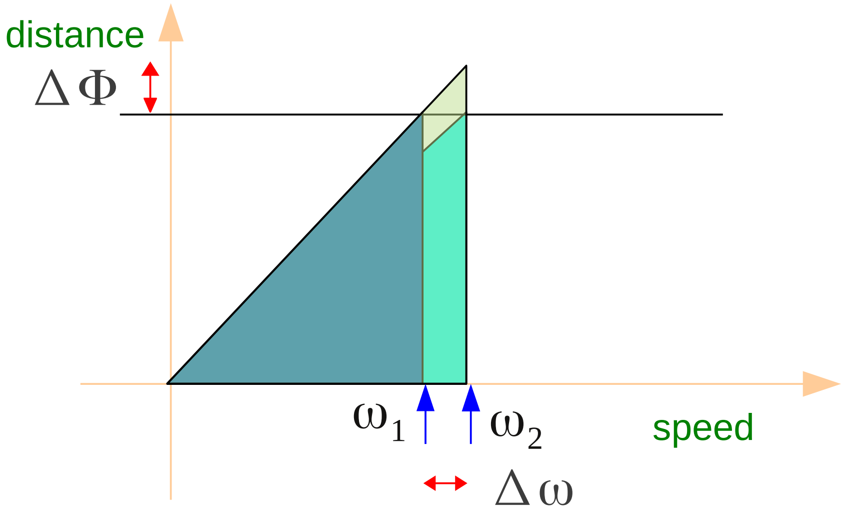
*Narrow band*

# Angle and Angular Speed




$$\Phi = \omega \cdot t$$

$$t = \frac{\Phi}{\omega}$$




$$\Delta\Phi = \Delta\omega \cdot \Delta t$$

$$\Delta t = \frac{\Delta\Phi}{\Delta\omega}$$

## References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] http://www.libinst.com/tpfd.htm
[4] K.Shin, J.K. Hammond, “Fundamentals of Signal Processing for Sound and Vibration Engineers”
[5] www.radiolab.com.au/DesignFile/DN004.pdf