CT Dirichlet Function (2B)

- Continuous Time Dirichlet Function

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6/10/13

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Dirichlet Function Call (1)

$$\textbf{drcl}(t, L) = \frac{\sin(L\pi t)}{L\sin(\pi t)}$$

$$t \leftarrow \hat{f}$$

$$\textbf{drcl}(\hat{f}, L) = \frac{\sin(L\pi \hat{f})}{L\sin(\pi \hat{f})}$$

$$= \frac{\sin(L 2\pi \hat{f}/2)}{L\sin(2\pi \hat{f}/2)}$$

$$\textbf{diric}(x, L) = \frac{\sin(L x/2)}{L\sin(x/2)}$$

$$x \leftarrow \hat{\omega}/2$$

$$\textbf{diric}(\hat{\omega}, L) = \frac{\sin(L \hat{\omega}/2)}{L\sin(\hat{\omega}/2)}$$

$$\textbf{drcl}(\hat{f}, L) = \textbf{diric}(2\pi \hat{f}, L)$$

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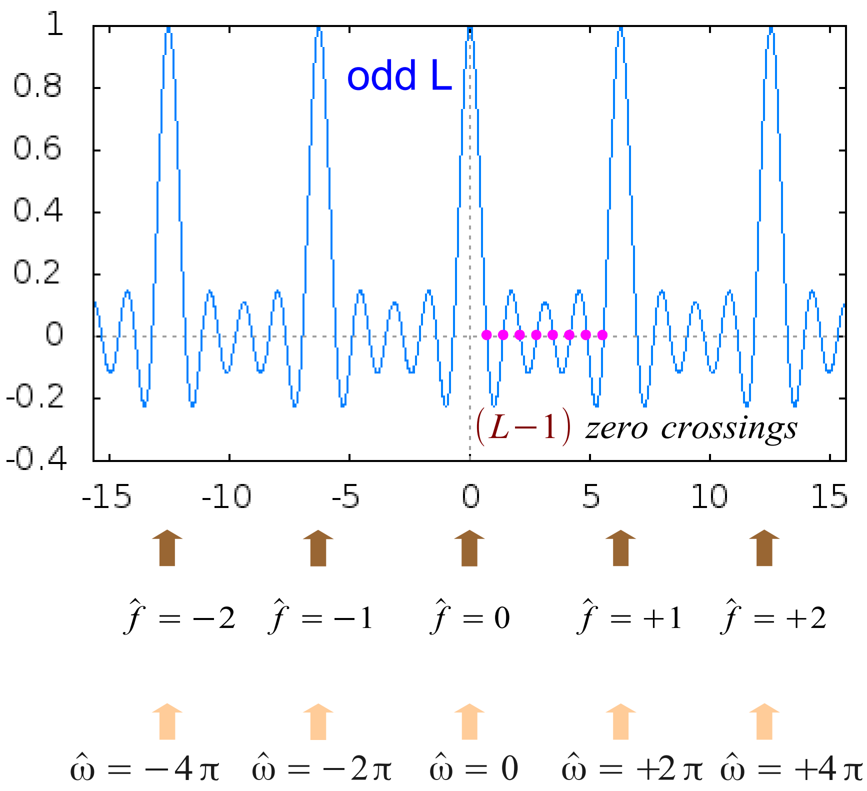
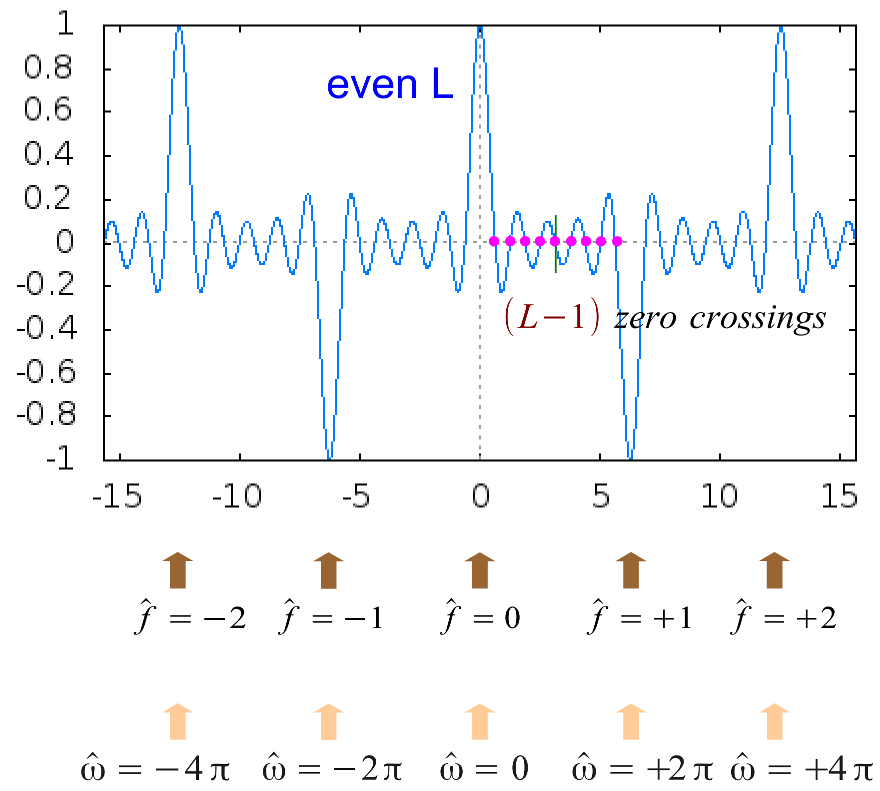
Dirichlet Function Call (2)

$$\boldsymbol{drcl}(\hat{f}, L) = \frac{\sin(L\pi\hat{f})}{L\sin(\pi\hat{f})}$$

$$\boldsymbol{diric}(\hat{\omega}, L) = \frac{\sin(L\hat{\omega}/2)}{L\sin(\hat{\omega}/2)}$$





Dirichlet Function Call (3)

$$drcl(\hat{f}, L) = \frac{\sin(L\pi\hat{f})}{L\sin(\pi\hat{f})}$$

ODD L

Maxima at $\hat{f} = n$ n: integer

EVEN L

Maxima at $\hat{f} = n$ n: even

Minima at $\hat{f} = n$ n: odd

$$diric(\hat{\omega}, L) = \frac{\sin(L\hat{\omega}/2)}{L\sin(\hat{\omega}/2)}$$

ODD L

Maxima at $\hat{\omega}/2 = \pi n$ n: integer

$\Rightarrow$ $\hat{\omega} = 2\pi n$

EVEN L

Maxima at $\hat{\omega} = 2\pi n$ n: even

Minima at $\hat{\omega} = 2\pi n$ n: odd

Dirichlet Function Call (4)

$$drcl(t, L) = \frac{\sin(L \pi t)}{L \sin(\pi t)}$$

```
function x = drcl(t, N)
   X = diric(2*pi*t, N);
```

```
function x = drcl(t, N)
   nomi = sin(N*pi*t);
   denomi = N * sin(pi*t);

   I =
    find(abs(denomi) < 100*eps);

   nomi(I) = cos(N*pi*t(I));
   denomi(I) = cos(pi*t(I));

   x = nomi ./ denomi;
```

$$diric(x, L) = \frac{\sin(L x/2)}{L \sin(x/2)}$$

builtin function

```
      diric(x, L);
```

Zeros and Maxima/Minima (1)

Dirichlet Function

$$drcl(t, L) = \frac{\sin(\pi L t)}{L \sin(\pi t)}$$

Odd L: Period = 1
Even L: Period = 2

odd L

$$\lim_{t \to n} \frac{\cos(\pi L t)}{\cos(\pi t)} = +1$$

$$drcl(n, L) = +1$$

even L

$$\lim_{t \to n} \frac{\cos(\pi L t)}{\cos(\pi t)} = (-1)^n$$

$$drcl(n, L) = (-1)^n$$

$t = 0, \ \pm 1, \ \pm 2, \ \pm 3, \ \cdots$ t: an integer n

$\Rightarrow \sin(\pi L t) = 0$

$\Rightarrow \sin(\pi t) = 0$

$$drcl(t, L) = \frac{(\sin(\pi L t))'}{(L \sin(\pi t))'} = \lim_{t \to n} \frac{L \pi \cos(\pi L t)}{L \pi \cos(\pi t)} = \pm 1$$

odd L $\quad \dfrac{L \pi \cos(\pi L t)}{L \pi \cos(\pi t)} = \dfrac{(\mp 1)}{(\mp 1)} = +1 \Rightarrow drcl(n, L) = +1$

even L $\quad \dfrac{L \pi \cos(\pi L t)}{L \pi \cos(\pi t)} = \dfrac{(+1)}{(\mp 1)} = \mp 1 \Rightarrow drcl(n, L) = (-1)^n$

$t = -2$	$t = -1$	$t = 0$	$t = +1$	$t = +2$	$t = +3$	
↓	↓	↓	↓	↓	↓	
$\frac{\cos(-L 2\pi)}{\cos(-2\pi)}$,	$\frac{\cos(-L\pi)}{\cos(-\pi)}$,	$\frac{\cos(0)}{\cos(0)}$,	$\frac{\cos(L\pi)}{\cos(\pi)}$,	$\frac{\cos(L 2\pi)}{\cos(2\pi)}$,	$\frac{\cos(L 3\pi)}{\cos(3\pi)}$,	$\cdots$
↓	↓	↓	↓	↓	↓	
$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	odd L
$+1$	-1	$+1$	-1	$+1$	-1	even L

Zeros and Maxima/Minima (2)

Dirichlet Function

$$drcl(t, L) = \frac{\sin(L\pi t)}{L\sin(\pi t)}$$

Zeros: $t = \frac{m}{L}$

odd L: Period = 1

$$\lim_{t\to n} \frac{\cos(L\pi t)}{\cos(\pi t)} = +1$$

$$drcl(n, L) = +1$$

even L: Period = 2

$$\lim_{t\to n} \frac{\cos(L\pi t)}{\cos(\pi t)} = (-1)^n$$

$$drcl(n, L) = (-1)^n$$

$t = \pm 1/L,\ \pm 2/L,\ \pm 3/L,\ \cdots$ t: not an integer but an integer multiple of 1/L

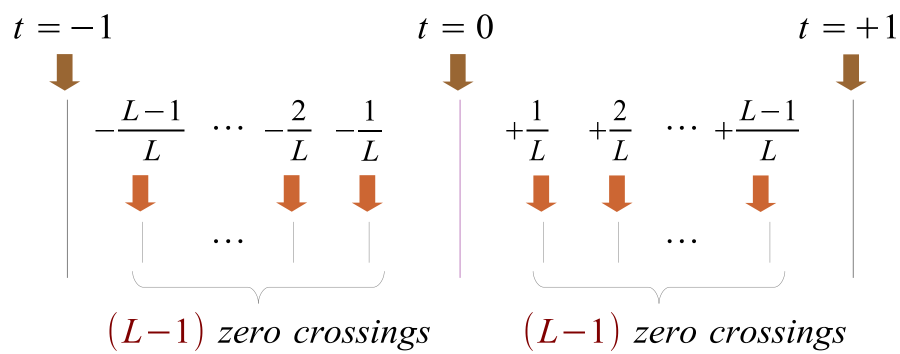
$$t = \frac{m}{L} \qquad m = \pm 1,\ \pm 2,\ \cdots,\ \pm(L-1),\ \pm(L+1),\ \cdots$$

⇒ $\sin(L\pi t) = 0$

⇒ $\sin(\pi t) \neq 0$

$$drcl\left(\frac{m}{L}, L\right) = \frac{\sin\left(\pi L \frac{m}{L}\right)}{L\sin\left(\pi \frac{m}{L}\right)} = \frac{\sin(\pi m)}{L\sin\left(\pi \frac{m}{L}\right)} = 0$$

zeros $t = \frac{\pm 1}{L},\ \frac{\pm 2}{L},\ \cdots,\ \frac{\pm(L-1)}{L},\ \frac{\pm(L+1)}{L},\ \cdots$



Zeros and Maxima/Minima (3)

Dirichlet Function

$$diric(x, L) = \frac{\sin(Lx/2)}{L\sin(x/2)}$$

$$D_L(e^{j\hat{\omega}}) = \frac{\sin(L\hat{\omega}/2)}{L\sin(\hat{\omega}/2)}$$

Zeros: $\hat{\omega} = 2\pi\frac{m}{L}$

Maximum, Minimum:

odd L

$$\lim_{\hat{\omega}\to 2\pi n} D_L(e^{j\hat{\omega}}) = +1$$

even L

$$\lim_{\hat{\omega}\to 2\pi n} D_L(e^{j\hat{\omega}}) = (-1)^n$$

$x = \pm 2\pi\cdot 1/L,\ \pm 2\pi\cdot 2/L,, \cdots$ x: not an integer but an integer multiple of 2π/L

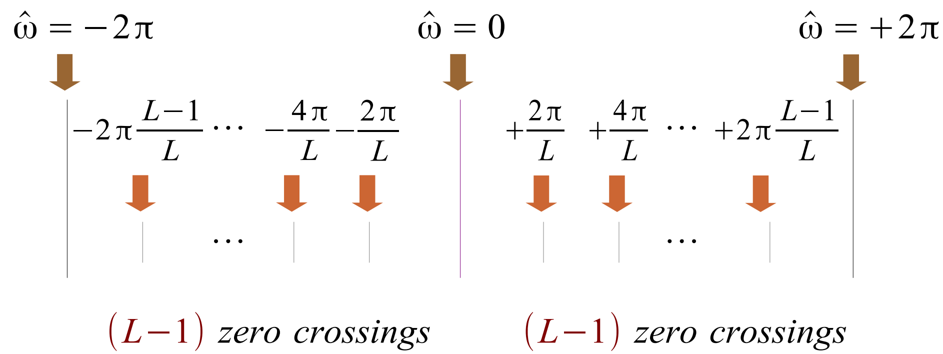
$$x = 2\pi\frac{m}{L} \qquad m = \pm 1, \pm 2, \cdots, \pm(L-1), \pm(L+1), \cdots$$

➡ $\sin(Lx/2) = 0$

➡ $\sin(x/2) \neq 0$

$$diric\left(2\pi\cdot\frac{m}{L}, L\right) = \frac{\sin\left(\pi L\frac{m}{L}\right)}{L\sin(\pi\frac{m}{L})} = \frac{\sin(\pi m)}{L\sin(\pi\frac{m}{L})} = 0$$

zeros $t = \pm 2\pi\cdot\frac{1}{L},\ \pm\pi\cdot\frac{2}{L},\ \cdots,\ \pm 2\pi\frac{(L-1)}{L},\ \pm 2\pi\cdot\frac{(L+1)}{L},\ \cdots$



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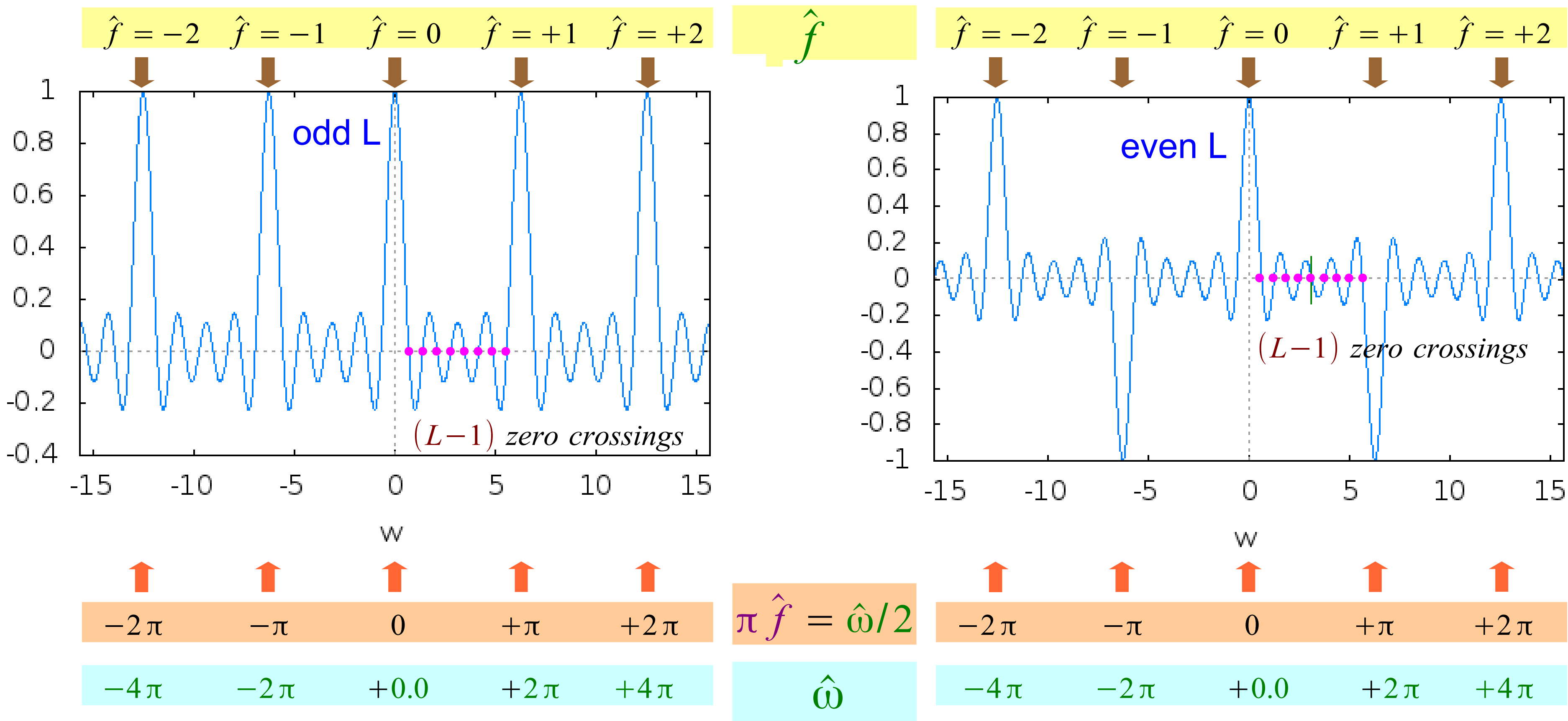
Zeros and Maxima/Minima (3)



Dirichlet Function

$$\boldsymbol{drcl}(\hat{f}, L) = \frac{\sin(L\pi\hat{f})}{L\sin(\pi\hat{f})}$$

$$\boldsymbol{diric}(\hat{\omega}, L) = \frac{\sin(L\hat{\omega}/2)}{L\sin(\hat{\omega}/2)}$$

Dirichlet Function Properties

Dirichlet Function

$$\textbf{drcl}(t, L) = \frac{\sin(\pi L t)}{L \sin(\pi t)}$$

$$D_L(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega} L/2)}{L \sin(\hat{\omega}/2)}$$

$$\textbf{diric}(x, N) = \frac{\sin(N x/2)}{N \sin(x/2)}$$

$$D_L(e^{j(\hat{\omega}+2\pi)}) = \frac{\sin((\hat{\omega}+2\pi)L/2)}{L\sin((\hat{\omega}+2\pi)/2)} = \frac{\sin(\hat{\omega}L/2 + L\pi)}{L\sin(\hat{\omega}/2+\pi)}$$

$$\begin{cases} +D_L(e^{j\hat{\omega}}) & \textit{for an odd } L \\ -D_L(e^{j\hat{\omega}}) & \textit{for an even } L \end{cases}$$

(period: 2π)

$$0 \le \hat{\omega} \le +\pi$$

$$0 \le \hat{\omega}/2 \le +\frac{\pi}{2} \qquad 0 \le \hat{\omega}L/2 \le +L\frac{\pi}{2}$$

$$0 \le \sin(\hat{\omega}/2) \le +1 \qquad -1 \le \sin(\hat{\omega}L/2) \le +1$$

a quarter period — L quarter periods

Envelope: $\frac{1}{\sin(\hat{\omega}/2)}$

Zeros: $\hat{\omega} = \frac{2\pi}{L}k$

$$\sin(\hat{\omega}L/2) = 0$$

$$D_L(e^{-j\hat{\omega}}) = \frac{\sin(-\hat{\omega}L/2)}{L\sin(-\hat{\omega}/2)} = D_L(e^{j\hat{\omega}})$$

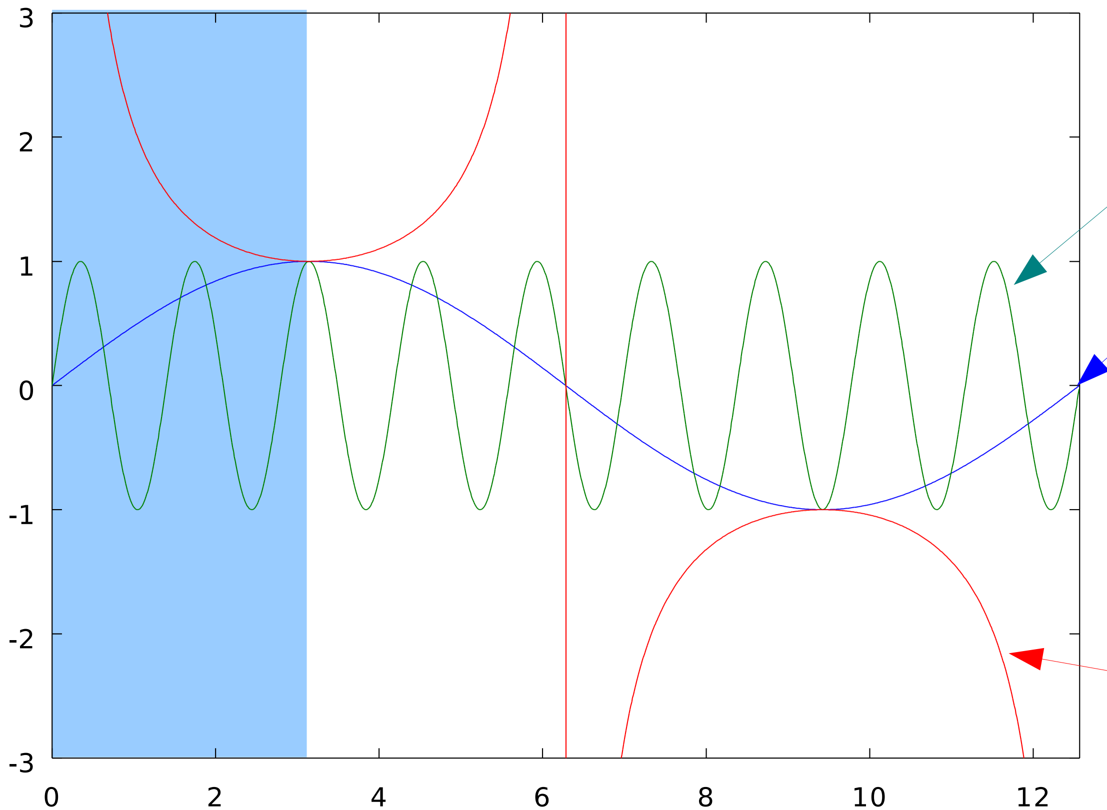
an even function

$$\lim_{\hat{\omega}\to 0} D_L(e^{j\hat{\omega}}) = \lim_{\hat{\omega}\to 0} \frac{L/2\cos(\hat{\omega}L/2)}{L/2\cos(\hat{\omega}/2)} = 1$$

Maximum: $D_L(e^{j\hat{\omega}}) = 1 \quad \textit{when } \hat{\omega} = 0$

A Dirichlet Function L=9 (1)

$$D_9(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}9/2)}{9\sin(\hat{\omega}/2)}$$



$\sin(\hat{\omega}9/2)$

$\sin(\hat{\omega}/2)$

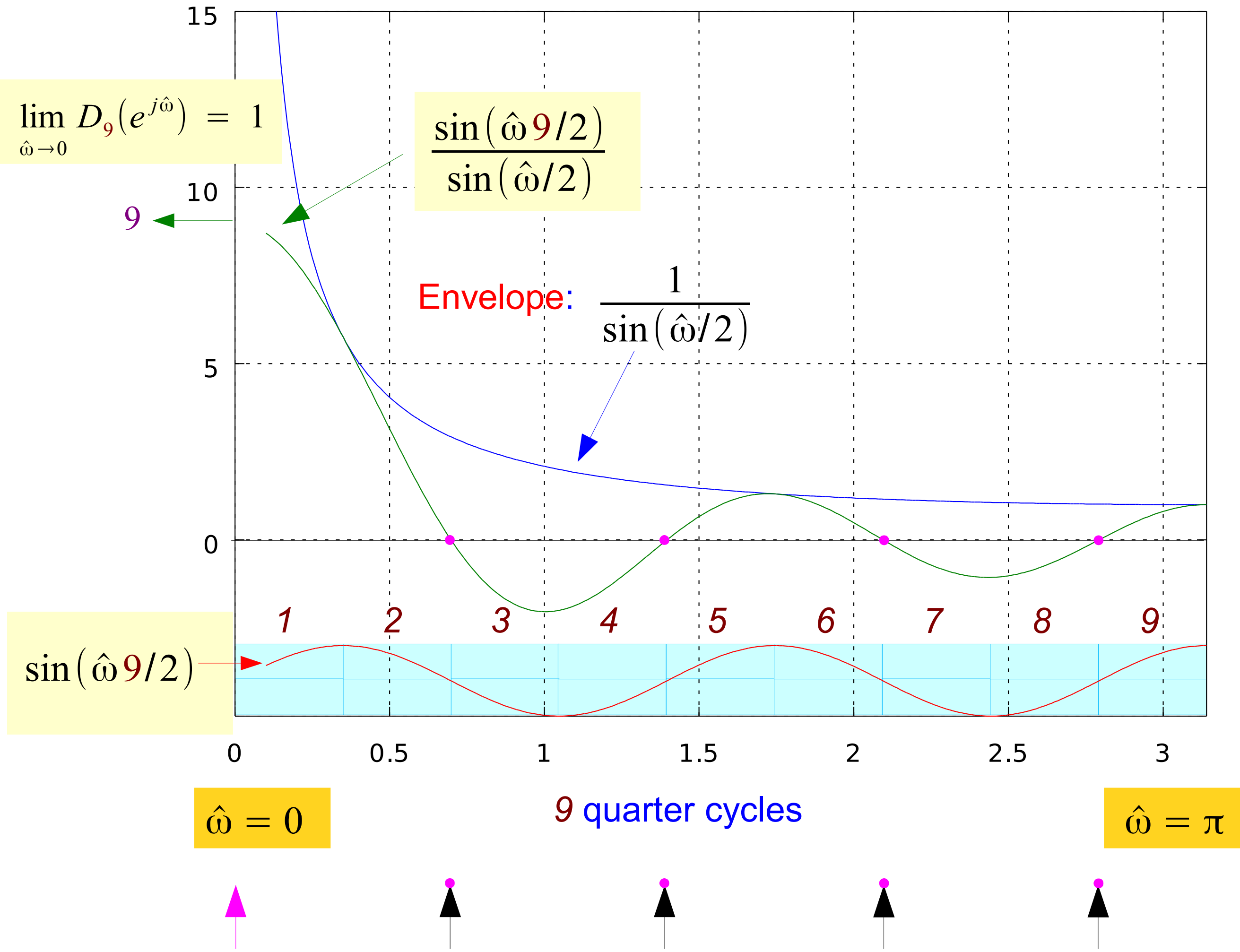
mutliplication

$\frac{1}{\sin(\hat{\omega}/2)}$

becomes an envelope

$\hat{\omega} = 0$ $\hat{\omega} = \pi$ $\hat{\omega} = 2\pi$ $\hat{\omega} = 3\pi$ $\hat{\omega} = 4\pi$

A Dirichlet Function L=9 (2)



approaches
to the unit

Zeros: $\hat\omega = \dfrac{2\pi}{9} k$

$$D_9(e^{j\hat\omega}) = \frac{\sin(\hat\omega 9/2)}{9\sin(\hat\omega/2)}$$

$\hat\omega = \pi$

$\dfrac{\hat\omega}{2} = \dfrac{\pi}{2}$ a quarter cycle

$\dfrac{\hat\omega 9}{2} = 9\dfrac{\pi}{2}$ 9 quarter cycles

$\sin(\hat\omega 9/2) = 0$ Zeros

$\hat\omega 9/2 = k\pi$ → $\hat\omega = k\dfrac{2\pi}{9}$

$\lim_{\hat\omega \to 0} D_L(e^{j\hat\omega})$ Maximum

$$\lim_{\hat\omega \to 0} \frac{(\sin(\hat\omega 9/2))'}{(9\sin(\hat\omega/2))'} = 1$$

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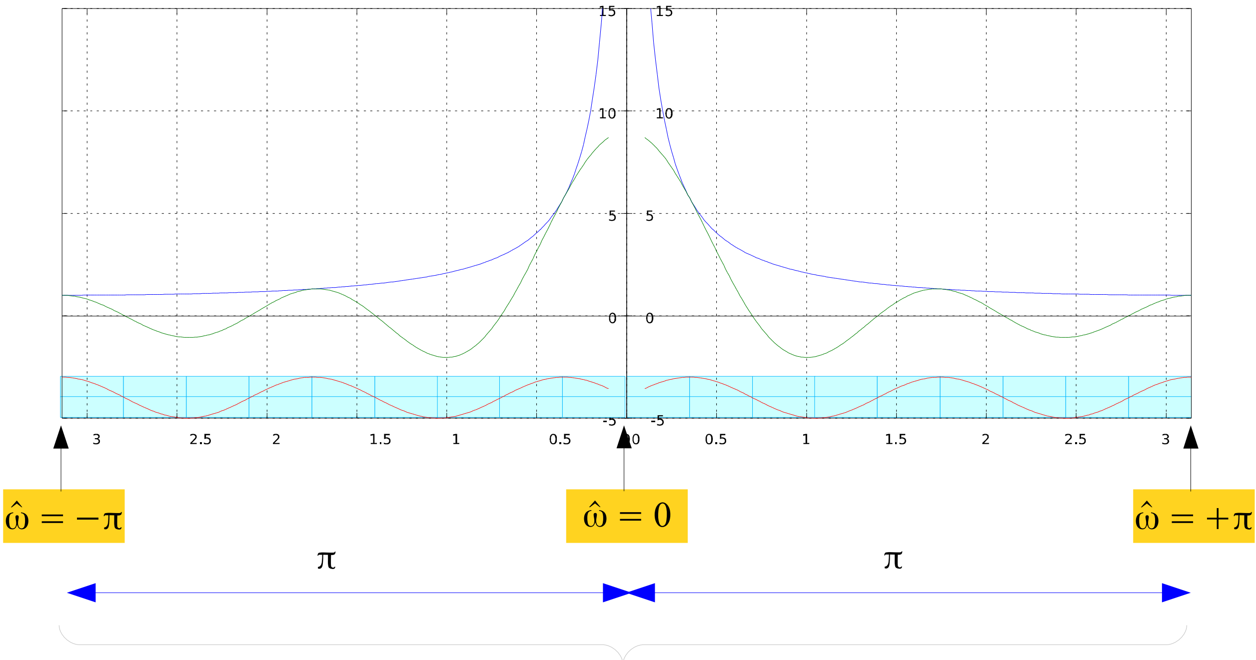
A Dirichlet Function L=9 (3)

an even function

symmetric along the y axis

$$D_L(e^{-j\hat{\omega}}) = \frac{\sin(-\hat{\omega}L/2)}{L\sin(-\hat{\omega}/2)} = D_L(e^{j\hat{\omega}})$$

$$D_9(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}9/2)}{9\sin(\hat{\omega}/2)}$$



$$D_L(e^{j(\hat{\omega}+2\pi)}) = \frac{\sin(\hat{\omega}L/2 + L\pi)}{L\sin(\hat{\omega}/2+\pi)} = \begin{cases} +D_L(e^{j\hat{\omega}}) & \textit{for an odd } L \\ -D_L(e^{j\hat{\omega}}) & \textit{for an even } L \end{cases}$$

(period: 2π)

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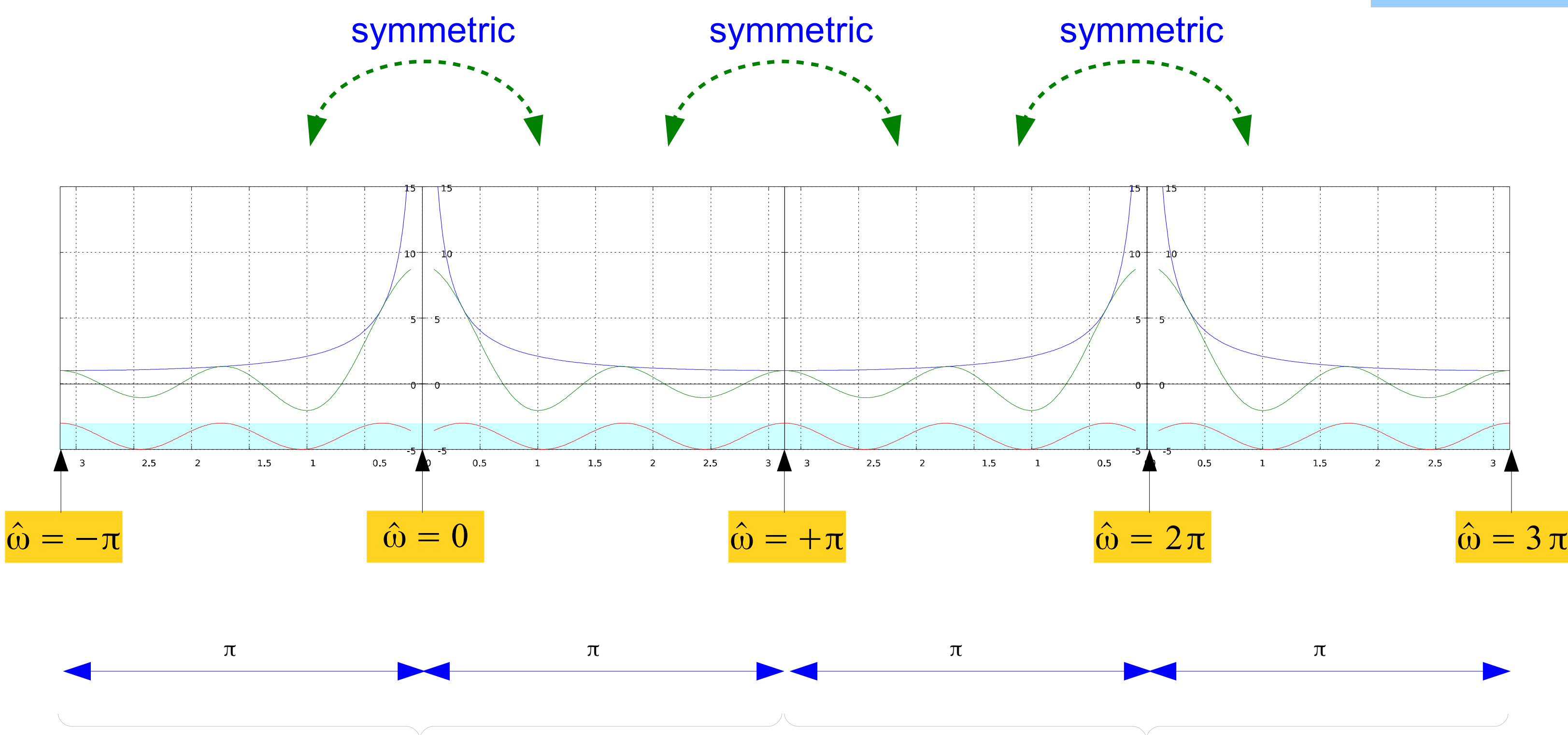
A Dirichlet Function L=9 (4)

$$D_L(e^{-j\hat{\omega}}) = \frac{\sin(-\hat{\omega}L/2)}{L\sin(-\hat{\omega}/2)} = D_L(e^{j\hat{\omega}})$$

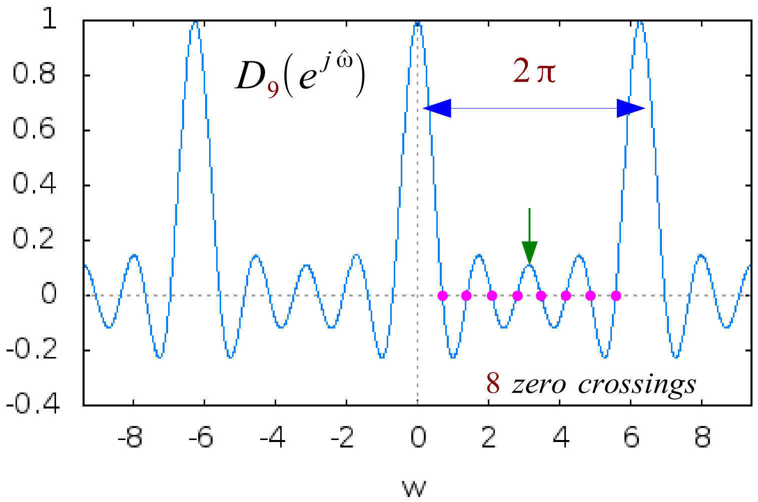
$$D_9(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}9/2)}{9\sin(\hat{\omega}/2)}$$



$$D_L(e^{j(\hat{\omega}+2\pi)}) = \frac{\sin(\hat{\omega}L/2 + L\pi)}{L\sin(\hat{\omega}/2 + \pi)} = \begin{cases} +D_L(e^{j\hat{\omega}}) & \text{for an odd } L \\ -D_L(e^{j\hat{\omega}}) & \text{for an even } L \end{cases}$$

(period: 2π)

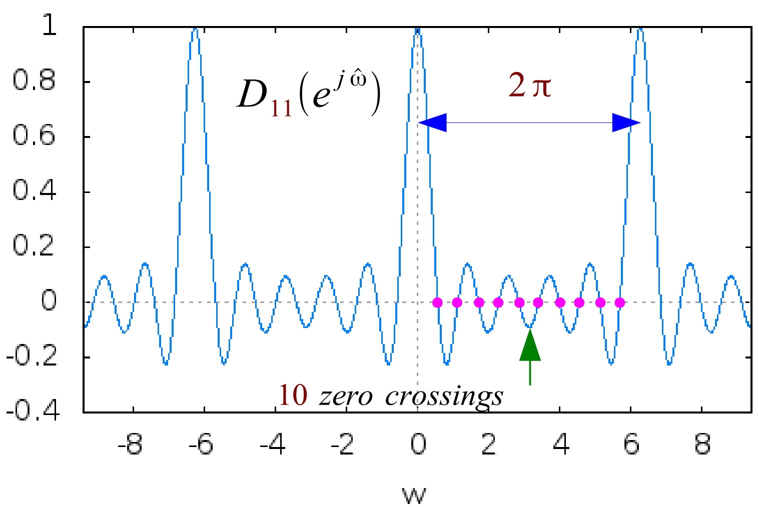
Dirichlet Functions (L: Odd)

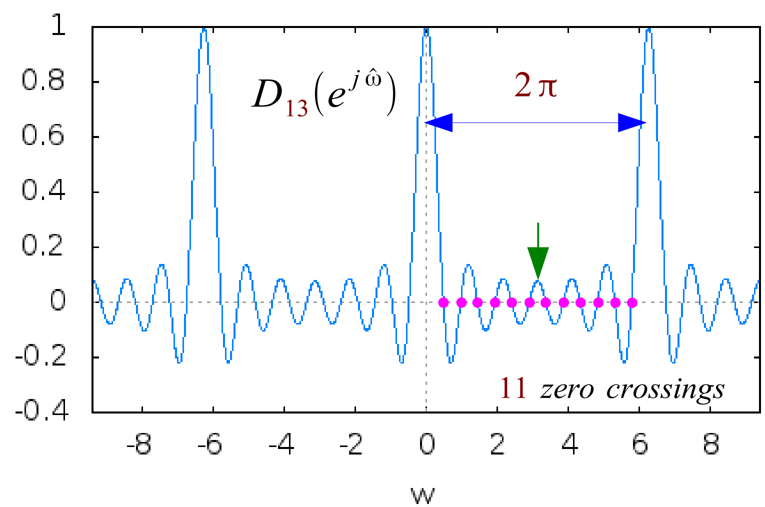


$$D_9(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}\,9/2)}{9\sin(\hat{\omega}/2)}$$ 8 zero crossings

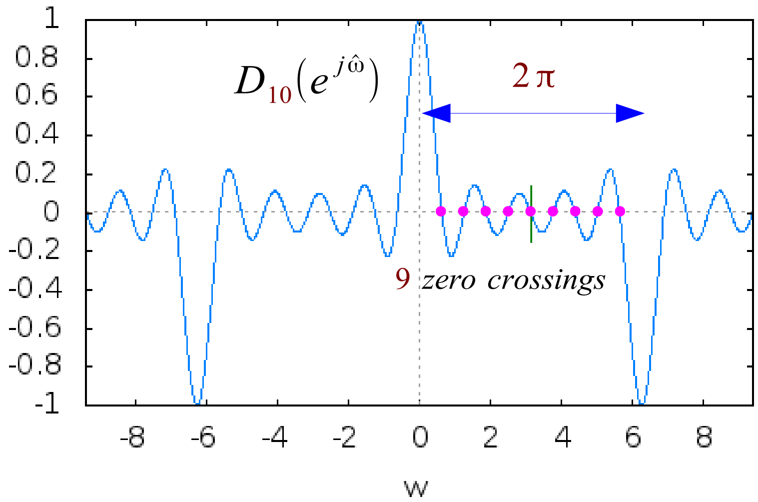
$$D_{11}(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}\,11/2)}{11\sin(\hat{\omega}/2)}$$ 10 zero crossings

$$D_{13}(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}\,13/2)}{13\sin(\hat{\omega}/2)}$$ 12 zero crossings





Dirichlet Functions (L: Even)



$$D_{10}(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}\, 10/2)}{10 \sin(\hat{\omega}/2)}$$

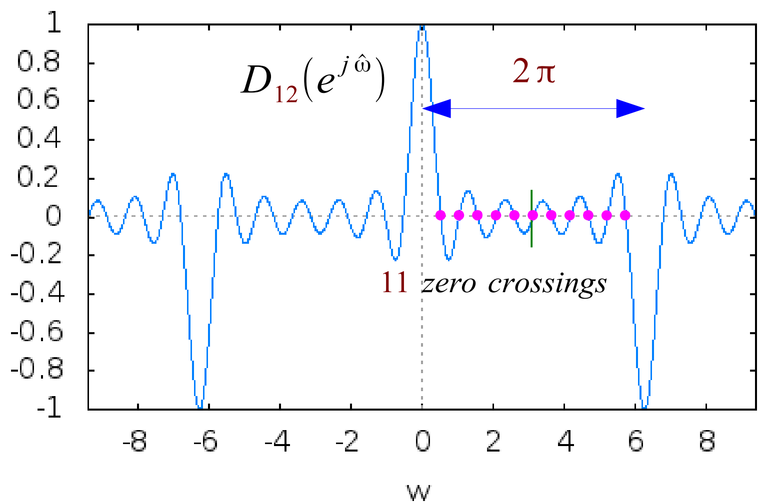
9 *zero crossings*

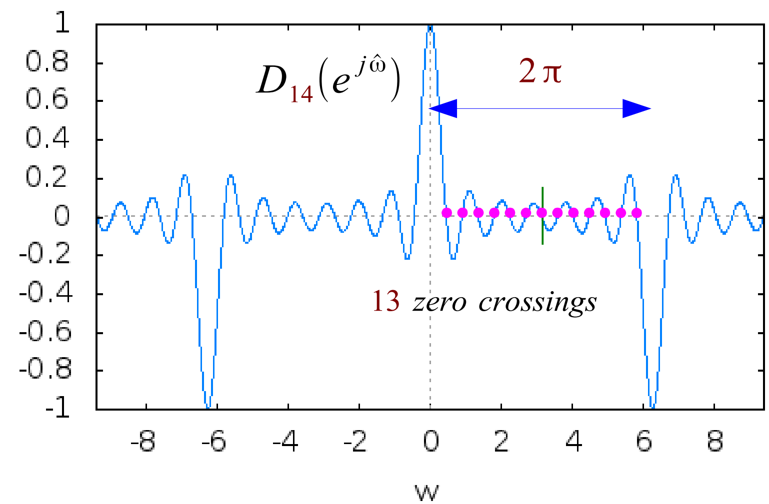
$$D_{12}(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}\, 12/2)}{12 \sin(\hat{\omega}/2)}$$

11 *zero crossings*

$$D_{14}(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}\, 14/2)}{14 \sin(\hat{\omega}/2)}$$

13 *zero crossings*





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