CMOS Delay-7 (H.7)
Elmore Delay

20170116

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References

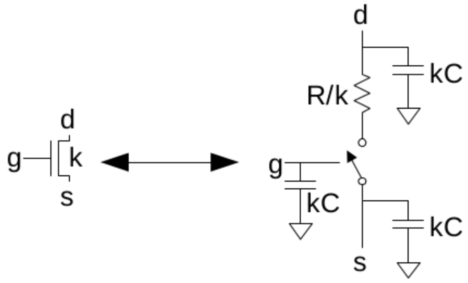
Some Figures from the following sites

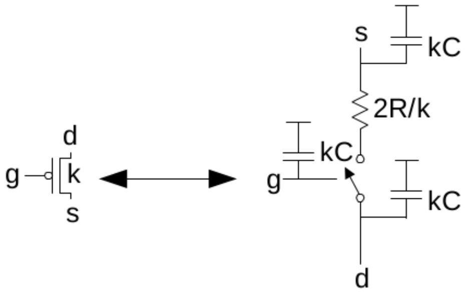
[1] http://pages.hmc.edu/harris/cmosvlsi/4e/index.html
Weste & Harris Book Site

[2] en.wikipedia.org

RC Delay Model

- Use equivalent circuits for MOS transistors
  - Ideal switch + capacitance and ON resistance
  - Unit nMOS has resistance R, capacitance C
  - Unit pMOS has resistance 2R, capacitance C
- Capacitance proportional to width
- Resistance inversely proportional to width



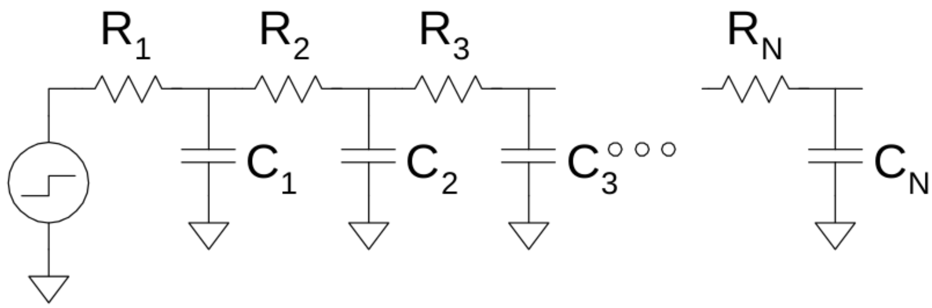


Elmore Delay

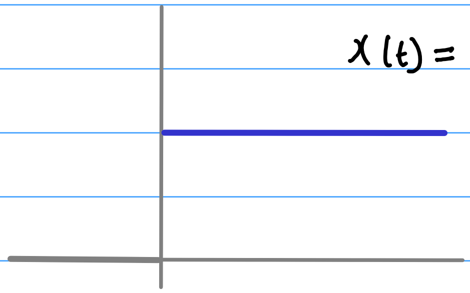
- ON transistors look like resistors
- Pullup or pulldown network modeled as *RC ladder*
- Elmore delay of RC ladder

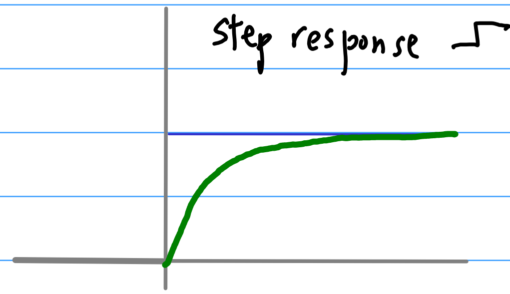
$$t_{pd} \approx \sum_{\text{nodes } i} R_{i-to-source} C_i$$

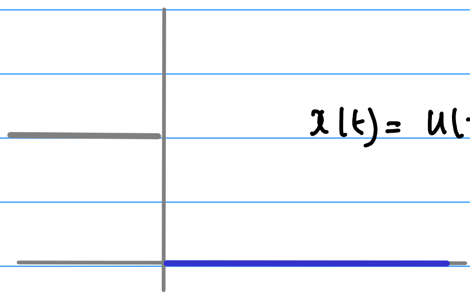
$$= R_1 C_1 + \left(R_1 + R_2\right) C_2 + \ldots + \left(R_1 + R_2 + \ldots + R_N\right) C_N$$

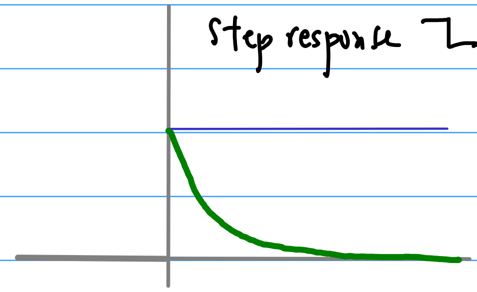


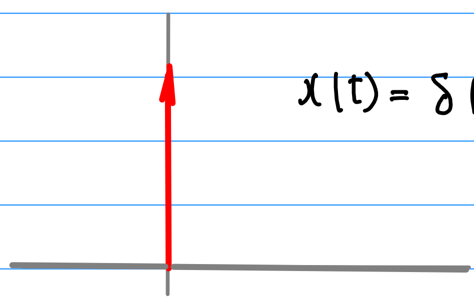
Step Response & Impulse Response

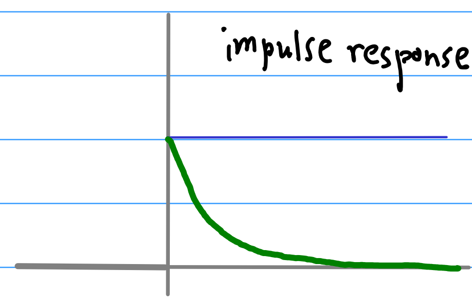




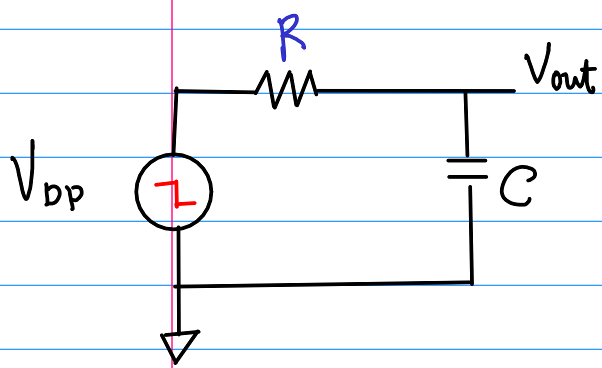








Transient Response : 1st Order RC Systems

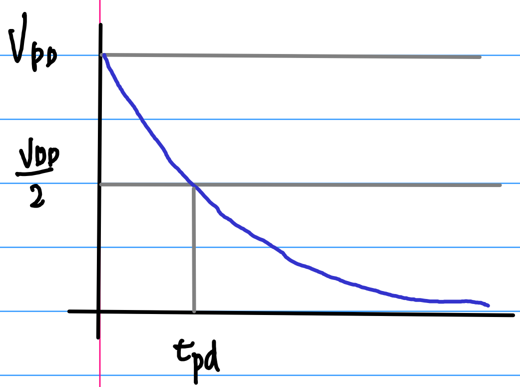


$$\frac{\frac{1}{sC}}{R+\frac{1}{sC}} = \frac{1}{1+sRC}$$

$$H(s) = \frac{1}{1+sRC}$$

$$V_{out}(t) = V_{DD}\, e^{-t/\tau}$$

$$\tau = RC$$



Step response to

$$\frac{V_{DD}}{2} = V_{DD}\, e^{-t/\tau}$$

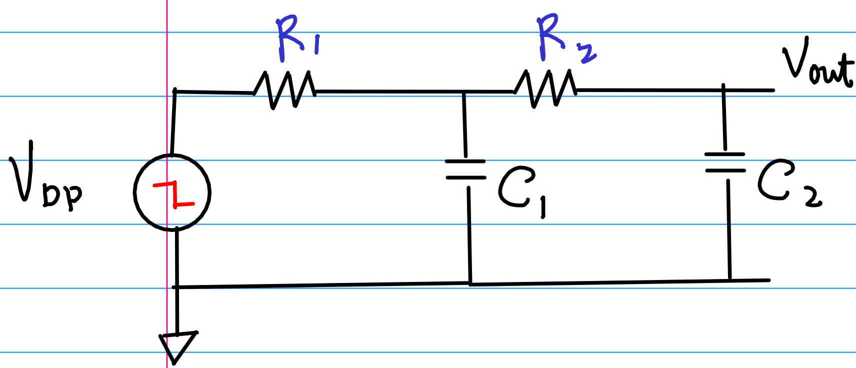
$$\frac{1}{2} = e^{-t/\tau}$$

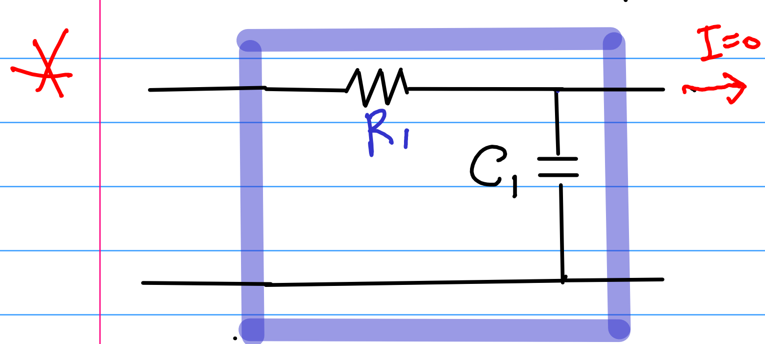
$$-t/\tau = \ln 2^{-1}$$

$$t = \ln 2\, \tau$$

$$t_{pd} = RC \ln 2$$
$$= R\ln 2 \cdot C$$
$$= R'C$$

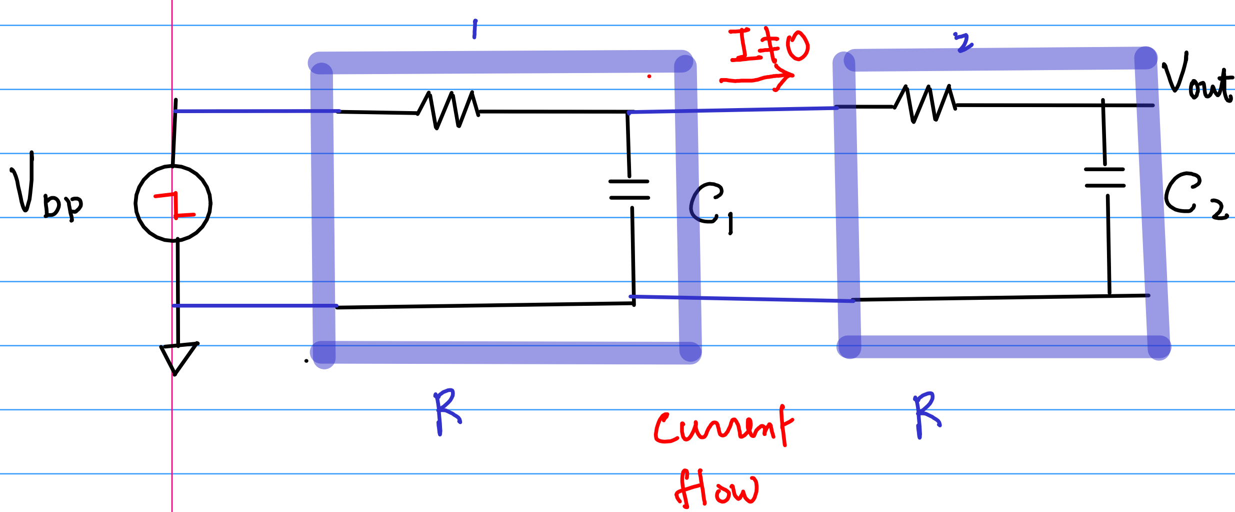
Transient Response : 2nd Order RC Systems





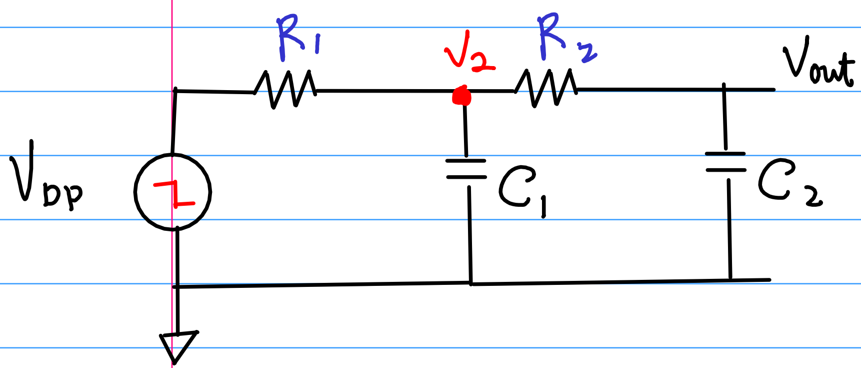
$$H(s) = \frac{1}{1+sRC}$$

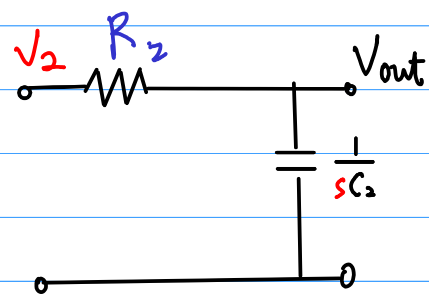
no current flow



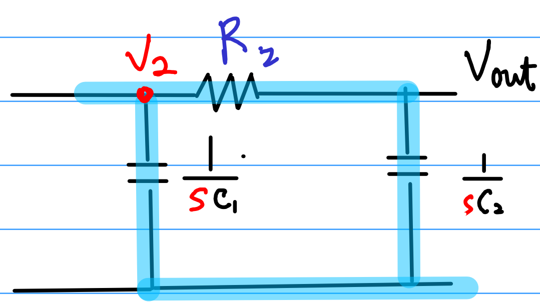
$$H(s) = \frac{1}{1+sR_1C_1} \cdot \frac{1}{1+sR_2C_2} = \frac{1}{1+s(R_1C_1+R_2C_2)+s^2R_1R_2C_1C_2}$$

Voltage Divider

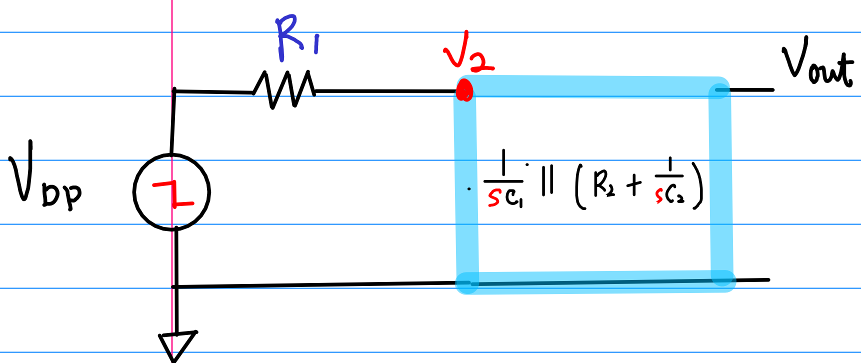




$$\boxed{V_{out}} = \frac{\frac{1}{sC_2}}{R_2 + \frac{1}{sC_2}} V_2$$



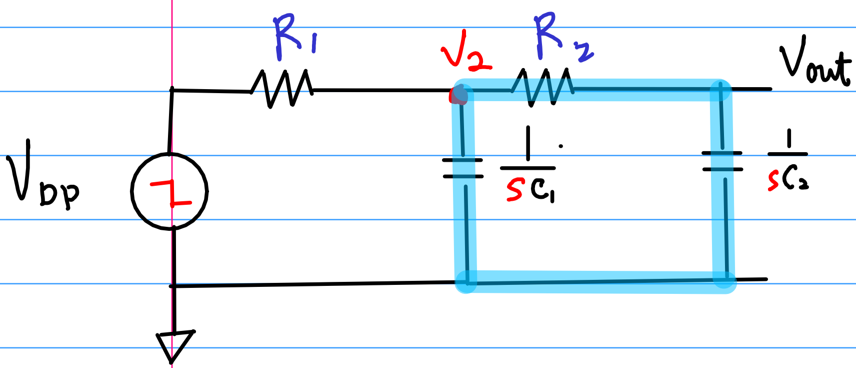
$$\boxed{\frac{1}{sC_1} \parallel \left(R_2 + \frac{1}{sC_2}\right)}$$



$$V_2 = \frac{\boxed{\frac{1}{sC_1} \parallel \left(R_2 + \frac{1}{sC_2}\right)}}{R_1 + \boxed{\frac{1}{sC_1} \parallel \left(R_2 + \frac{1}{sC_2}\right)}} V_{DD}$$

$$\Rightarrow \boxed{V_{out}} = \frac{1}{1 + sR_2C_2} V_2$$

Transfer Function H(s)



$$\boxed{V_{out}} = \frac{1}{1+sR_2C_2} V_2$$

$$\boxed{\frac{1}{sC_1} \,\|\, \left(R_2 + \frac{1}{sC_2}\right)} = \frac{1}{sC_1 + \frac{1}{R_2 + \frac{1}{sC_2}}} = \frac{1}{sC_1 + \frac{sC_2}{sR_2C_2+1}}$$

$$= \frac{sR_2C_2 + 1}{s^2R_2C_1C_2 + sC_1 + sC_2} = \frac{sR_2C_2 + 1}{s^2R_2C_1C_2 + s(C_1+C_2)}$$

$$V_2 = \frac{\frac{sR_2C_2 + 1}{s^2R_2C_1C_2 + s(C_1+C_2)}}{R_1 + \frac{sR_2C_2 + 1}{s^2R_2C_1C_2 + s(C_1+C_2)}} V_{pp} = \frac{sR_2C_2 + 1}{s^2R_1R_2C_1C_2 + s(R_1C_1 + R_1C_2 + R_2C_2) + 1} V_{pp}$$

$$\boxed{V_{out}} = \frac{1}{1+sR_2C_2} V_2 = \frac{1}{\cancel{(1+sR_2C_2)}} \cdot \frac{\cancel{(sR_2C_2 + 1)}}{s^2R_1R_2C_1C_2 + s(R_1C_1 + R_1C_2 + R_2C_2) + 1} V_{pp}$$

$$H(s) = \frac{1}{1 + s\left[R_1C_1 + (R_1+R_2)C_2\right] + s^2 R_1C_1R_2C_2}$$

Quadratic Equations   the reciprocals of the roots

$$a s^2 + b s + c = 0$$

$$\boxed{\begin{aligned} s &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ \frac{1}{s} &= \frac{2a}{-b \pm \sqrt{b^2-4ac}} \Rightarrow \frac{1}{2c}\left[-b \mp \sqrt{b^2-4ac}\right] \end{aligned}}$$

$$\frac{1}{-b \pm \sqrt{b^2-4ac}} = \frac{-b \mp \sqrt{b^2-4ac}}{(-b)^2 - (b^2-4ac)} = \frac{-b \mp \sqrt{b^2-4ac}}{4ac}$$

$$\frac{2a}{-b \pm \sqrt{b^2-4ac}} = 2a \cdot \frac{-b \mp \sqrt{b^2-4ac}}{4ac} = \frac{1}{2c}\left[-b \mp \sqrt{b^2-4ac}\right]$$

the reciprocals of the poles of H(s)

$$a s^2 + b s + c = 0$$

$$\frac{1}{s} = \frac{2a}{-b \pm \sqrt{b^2 - 4ac}} \Rightarrow \frac{1}{2c}\left[-b \mp \sqrt{b^2 - 4ac}\right]$$

$$1 + s\left[R_1 C_1 + (R_1 + R_2) C_2\right] + s^2 R_1 C_1 R_2 C_2 = (1 + s\tau_1)(1 + s\tau_2) = 0$$

$$s = -\frac{1}{\tau_1}, -\frac{1}{\tau_2} \qquad \frac{1}{s} = \tau_1, \tau_2$$

$$a = R_1 C_1 R_2 C_2$$
$$b = \left[R_1 C_1 + (R_1 + R_2) C_2\right]$$
$$c = 1$$

$$\frac{1}{s} = \frac{1}{2c}\left[-b \mp \sqrt{b^2 - 4ac}\right] \Rightarrow$$

$$\frac{1}{s} = \frac{1}{2}\left[-\left[R_1 C_1 + (R_1 + R_2) C_2\right] \mp \sqrt{\left[R_1 C_1 + (R_1 + R_2) C_2\right]^2 - 4 R_1 C_1 R_2 C_2}\right]$$

$$\frac{1}{s} = -\frac{1}{2}\left[R_1 C_1 + (R_1 + R_2) C_2\right]\left[1 \pm \sqrt{1 - \frac{4 R_1 C_1 R_2 C_2}{\left[R_1 C_1 + (R_1 + R_2) C_2\right]^2}}\right]$$

Time constants τ_1 & τ_2

$$\boxed{1 + s\left[R_1C_1 + (R_1+R_2)C_2\right] + s^2 R_1C_1R_2C_2 = (1+s\tau_1)(1+s\tau_2) = 0 \qquad s = -\frac{1}{\tau_1},\ -\frac{1}{\tau_2}}$$

$$\frac{1}{s} = -\frac{1}{2}\left[R_1C_1 + (R_1+R_2)C_2\right]\left[1 \pm \sqrt{1 - \frac{4R_1C_1R_2C_2}{\left[R_1C_1 + (R_1+R_2)C_2\right]^2}}\right]$$

$$\frac{R_2}{R_1} \equiv R' \qquad \frac{C_2}{C_1} \equiv C'$$

$$\sqrt{1 - \frac{4R_1C_1R_2C_2}{\left[R_1C_1 + (R_1+R_2)C_2\right]^2}} = \sqrt{1 - \frac{4\frac{R_2}{R_1}\frac{C_2}{C_1}}{\left[1 + \left(1+\frac{R_2}{R_1}\right)\frac{C_2}{C_1}\right]^2}} = \sqrt{1 - \frac{4R'C'}{\left[1 + (1+R')C'\right]^2}}$$

$$\boxed{\tau_1, \tau_2 = \frac{1}{2}\left[R_1C_1 + (R_1+R_2)C_2\right]\left[1 \pm \sqrt{1 - \frac{4R'C'}{\left[1 + (1+R')C'\right]^2}}\right]}$$

Unit Step Response

$$H(s) = \frac{1}{1 + s[R_1C_1 + (R_1+R_2)C_2] + s^2 R_1C_1R_2C_2}$$

$$= \frac{1}{(1+s\tau_1)(1+s\tau_2)} = \left[\frac{A}{(1+s\tau_1)} + \frac{B}{(1+s\tau_2)}\right]$$

$$\boxed{\begin{aligned} A &= \left.\frac{1}{(1+s\tau_2)}\right|_{s=-\frac{1}{\tau_1}} = \frac{1}{\left(1-\frac{\tau_2}{\tau_1}\right)} = \frac{\tau_1}{\tau_1-\tau_2} \\ B &= \left.\frac{1}{(1+s\tau_1)}\right|_{s=-\frac{1}{\tau_2}} = \frac{1}{\left(1-\frac{\tau_1}{\tau_2}\right)} = \frac{\tau_2}{\tau_2-\tau_1} \end{aligned}}$$

$$H(s) = \frac{1}{\tau_1-\tau_2}\left[\frac{\tau_1}{(1+s\tau_1)} - \frac{\tau_2}{(1+s\tau_2)}\right]$$

$$h(t) = \frac{1}{\tau_1-\tau_2}\left[\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}\right]$$

Step response to $\rightharpoondown$

$$V_{out}(t) = \frac{1}{\tau_1-\tau_2}\left[\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}\right] V_{DD}$$

$$\boxed{\tau_1, \tau_2 = \frac{1}{2}\left[R_1C_1 + (R_1 + R_2)C_2\right]\left[1 \pm \sqrt{1 - \frac{4R'C'}{\left[1 + (1 + R')C'\right]^2}}\right]}$$

$$\tau = \tau_1 + \tau_2 = \left[R_1C_1 + (R_1 + R_2)C_2\right]$$

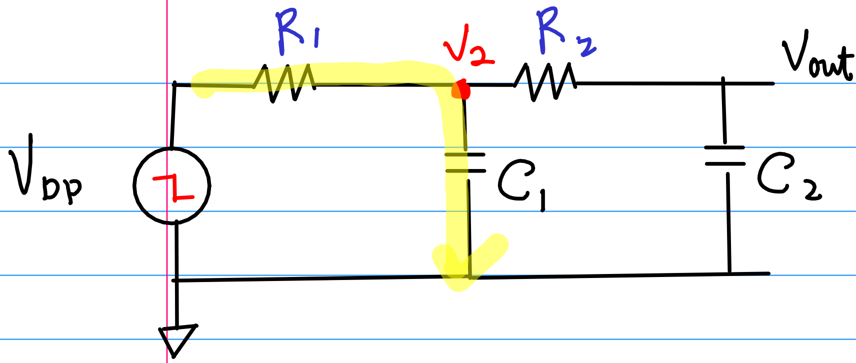
$R = R_1 = R_2$

$C = C_1 = C_2$

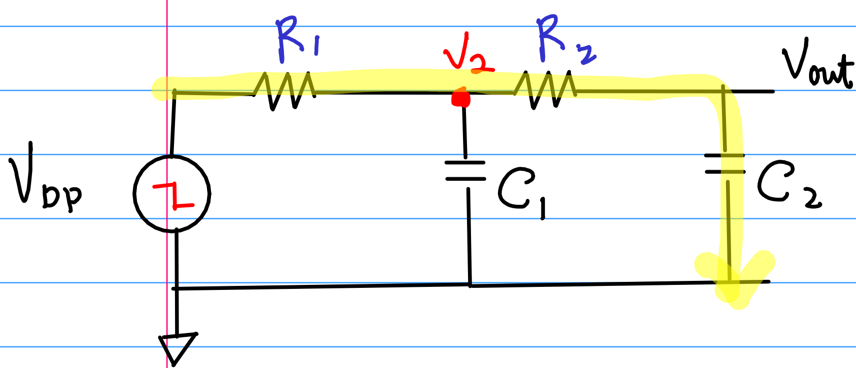
$\tau_1 = 2.6\,RC$

$\tau_2 = 0.4\,RC$

$\tau = 3.0\,RC$

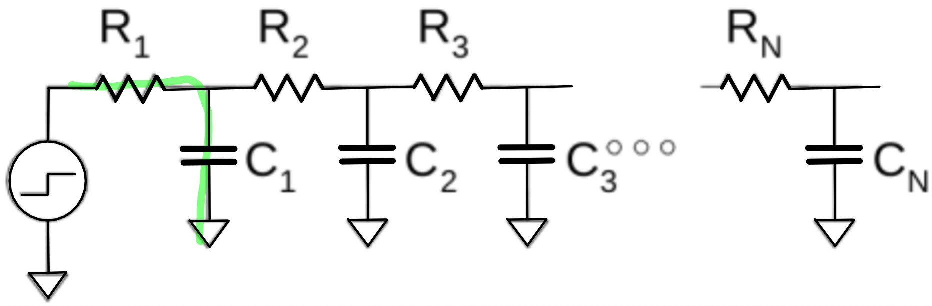


$\tau_1 = R_1 C_1$

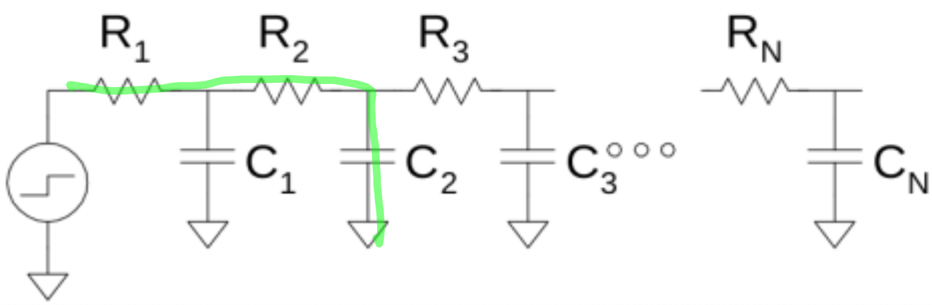


$\tau_2 = (R_1 + R_2) C_2$

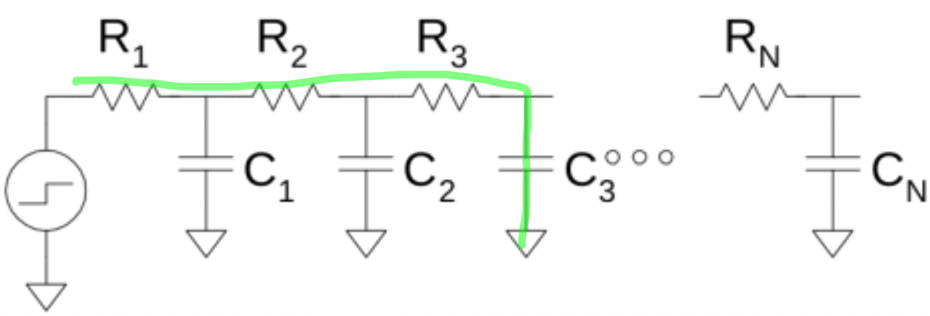
$$\tau = \tau_1 + \tau_2 = [R_1 C_1 + (R_1 + R_2) C_2]$$



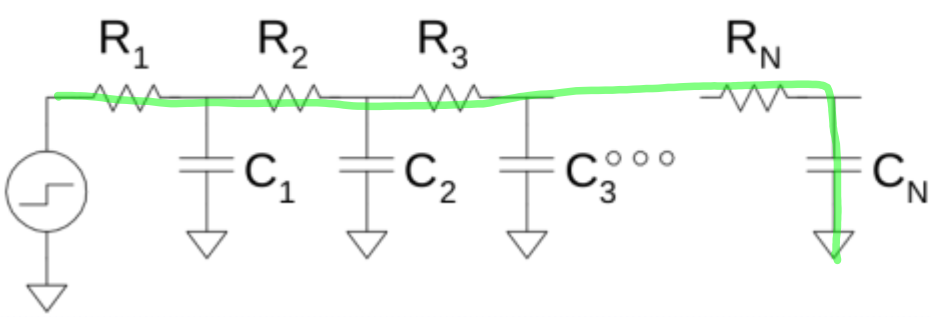
$R_1 C_1$



$(R_1 + R_2)\, C_2$

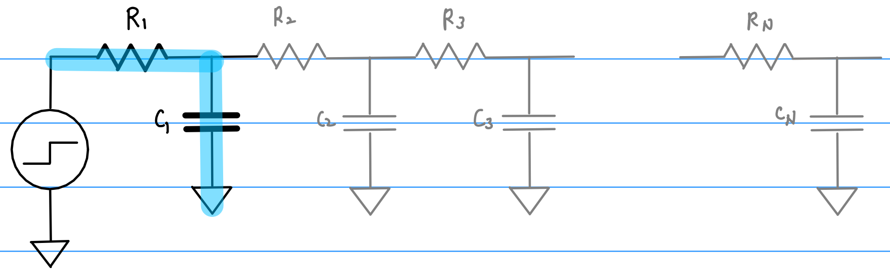


$(R_1 + R_2 + R_3)\, C_3$

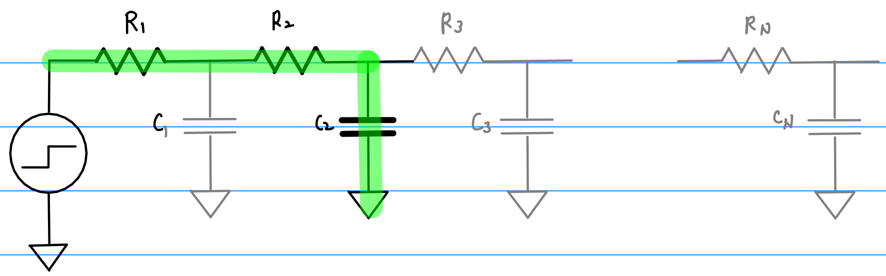


$(R_1 + R_1 + \cdots + R_N)\, C_N$

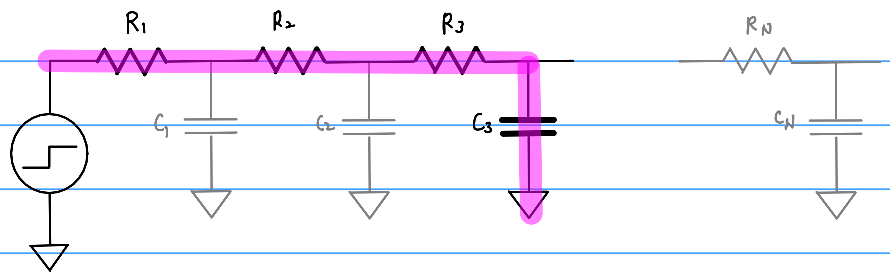
$$t_{pd} = R_1 C_1 + (R_1 + R_2) C_2 + \ldots + (R_1 + R_2 + \ldots + R_N) C_N$$



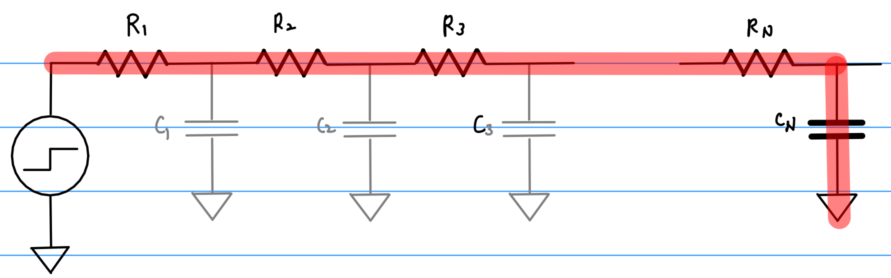
$(R_1)C_1$



$(R_1+R_2)C_2$



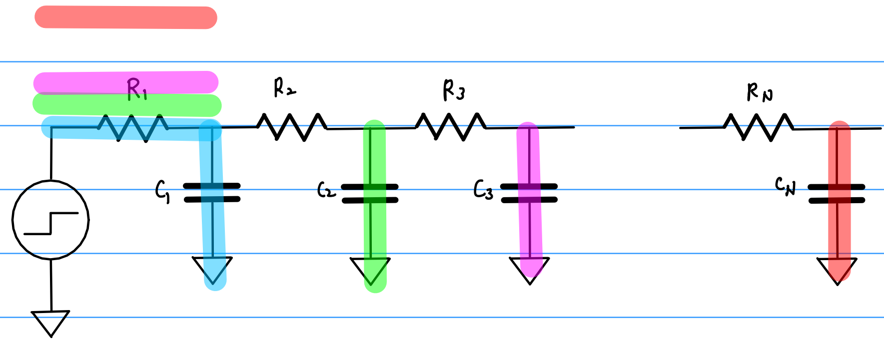
$(R_1+R_2+R_3)C_3$



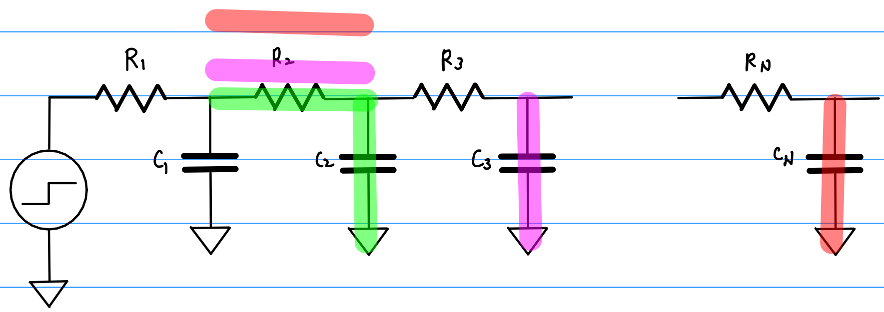
$(R_1+R_2+\cdots+R_N)C_N$

$$t_{pd} = (R_1)C_1 + (R_1+R_2)C_2 + (R_1+R_2+R_3)C_3 + \cdots + (R_1+R_2+\cdots+R_N)C_N$$

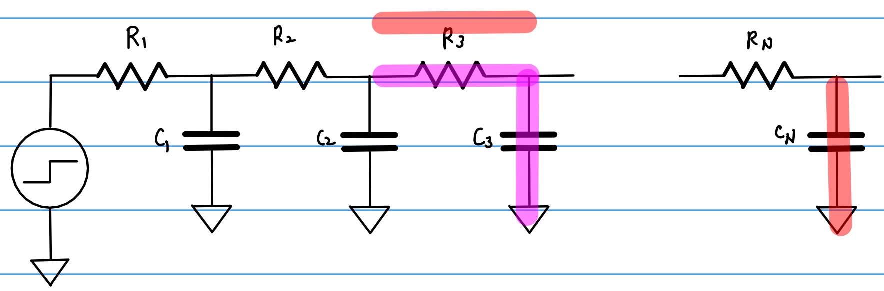
$$\boxed{t_{pd} = \sum_{i=1}^{n}\left(\sum_{j=1}^{i} R_j\right) C_i}$$



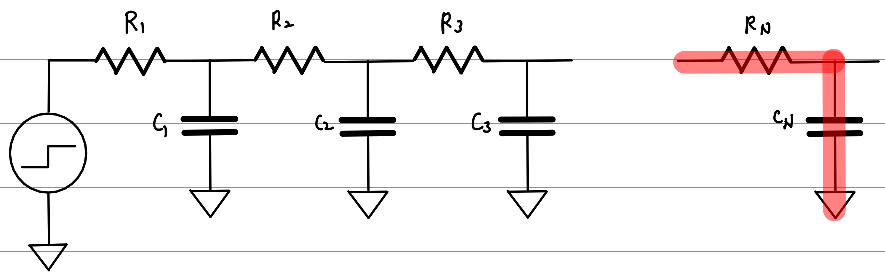
$$(C_1 + C_2 + \cdots + C_N)\,R_1$$



$$(C_2 + C_3 + \cdots + C_N)\,R_2$$



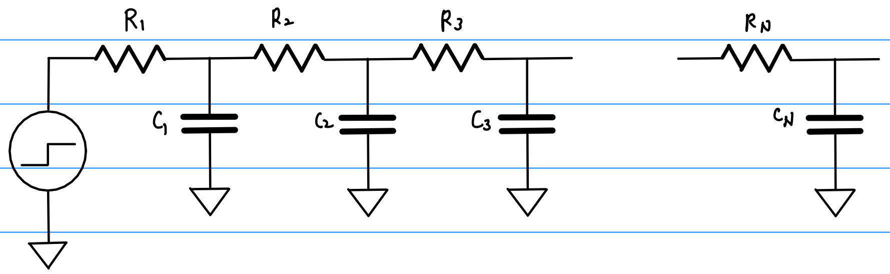
$$(C_3 + \cdots + C_N)\,R_3$$

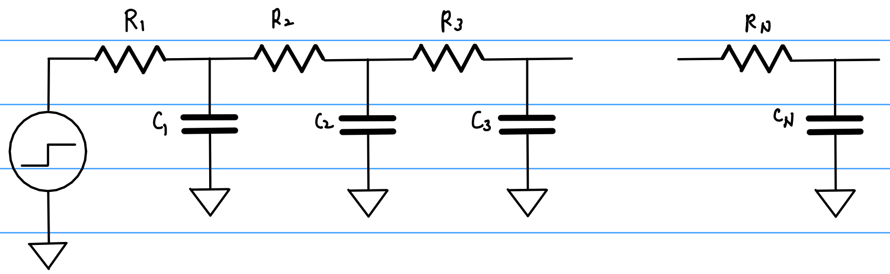


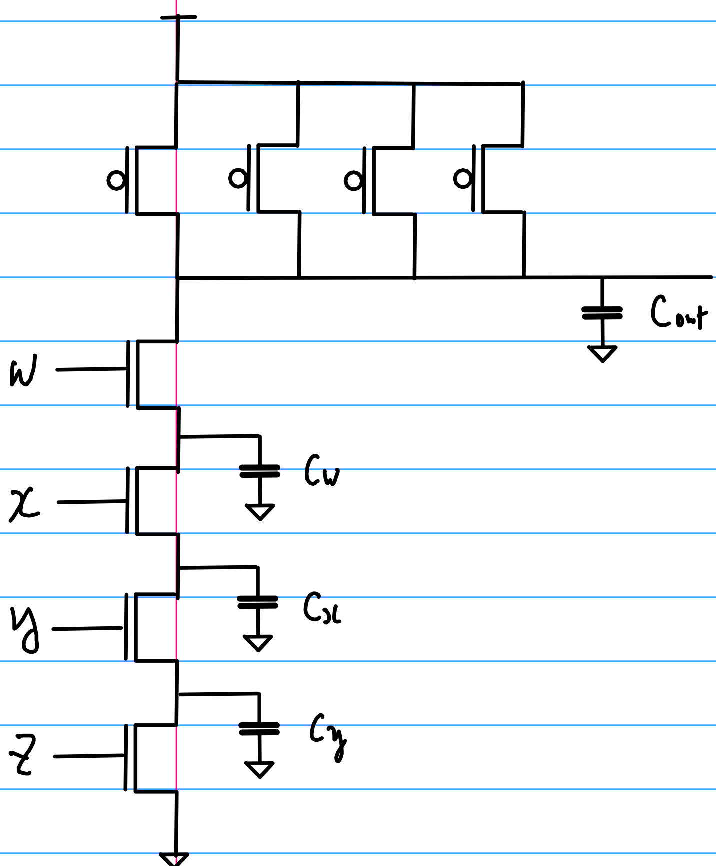
$$t_{pd} = R_1(C_1 + C_2 + \cdots + C_N) + R_2(C_2 + C_3 + \cdots + C_N) + R_3(C_3 + C_4 + \cdots + C_N) + \cdots + R_N C_N$$

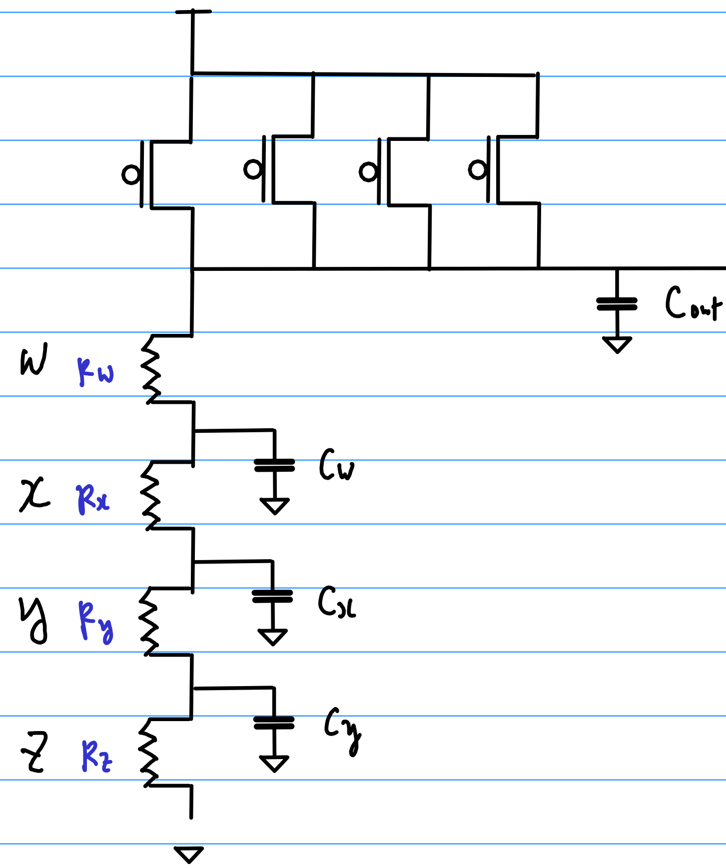
$$= (C_1 + C_2 + \cdots + C_N)R_1 + (C_2 + C_3 + \cdots + C_N)R_2 + (C_3 + C_4 + \cdots + C_N)R_3 + \cdots + C_N R_N$$

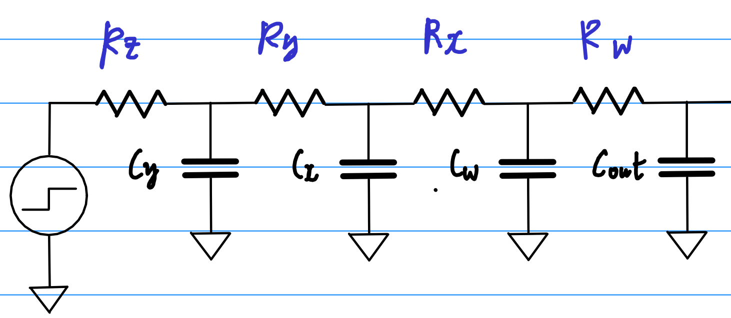
$$\boxed{t_{pd} = \sum_{i=1}^{n} \left(\sum_{j=i}^{n} C_j \right) R_i}$$


R1
R2
R3
RN
C1
C2
C3
CN

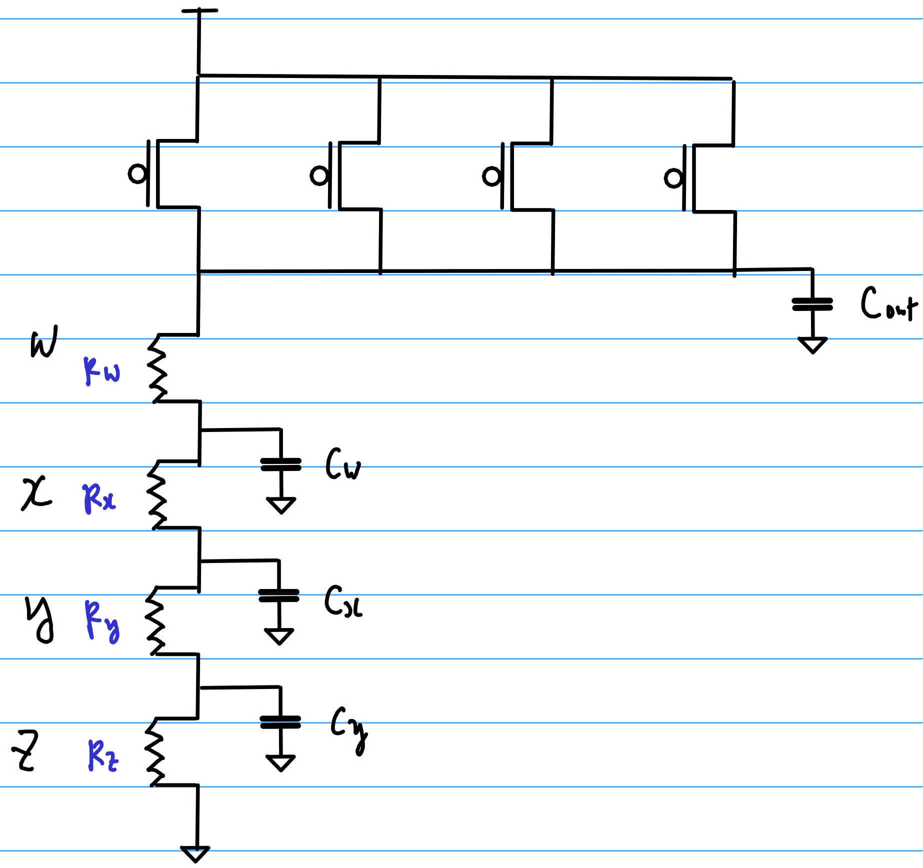

R1
R2
R3
RN
C1
C2
C3
CN



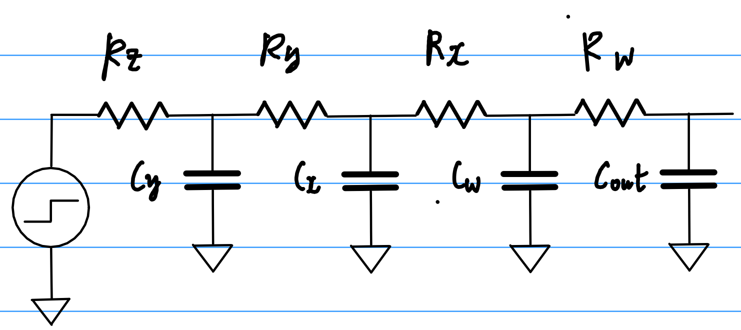




$$t_{pd} = C_y R_z + C_x (R_y + R_z) + C_w (R_x + R_y + R_z) + C_{out} (R_w + R_x + R_y + R_z)$$



W
Rw
Cw
x
Rx
Cx
y
Ry
Cy
Z
Rz
Cout



Rz
Ry
Rx
Rw
Cy
Cx
Cw
Cout

RC-Tree Delay Model

the propagation delay at node i

Sum of all the time constants formed
by each capacitor C_k and its associated resistance $R_{i,k}$

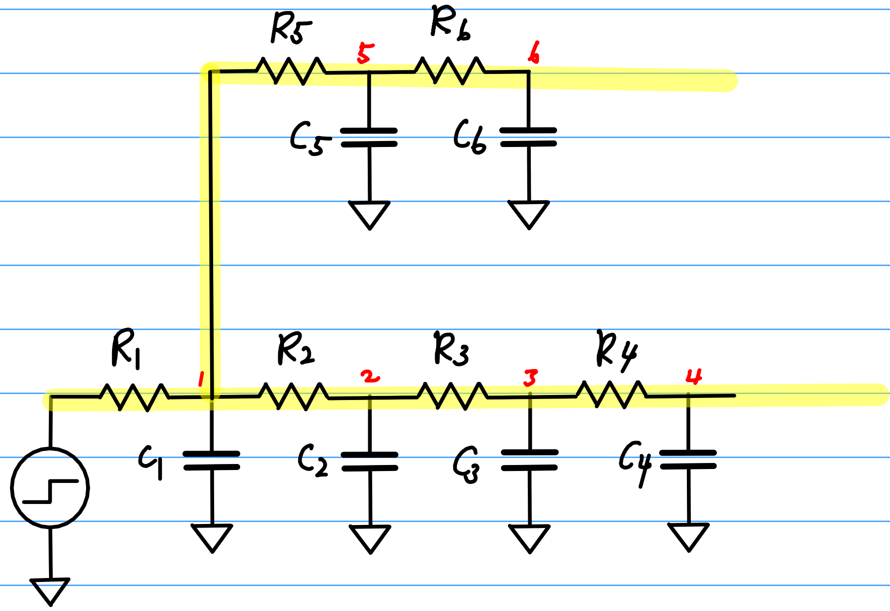
$$t_{pd\,i} = \sum_{k=1}^{n} C_k R_{i,k}$$

$$R_{i,k} = \sum R_j$$

$$\Rightarrow \quad R_j \in [\, path(i \to s) \cap path(k \to s) \,]$$

common path segment
from node i to source node s
from node k to source node s

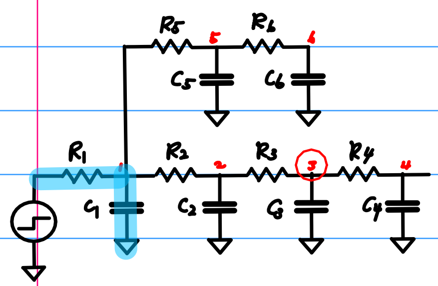
node i : node of interest
node k : $1 \leq k \leq n$ nodes to which capacitors are connected



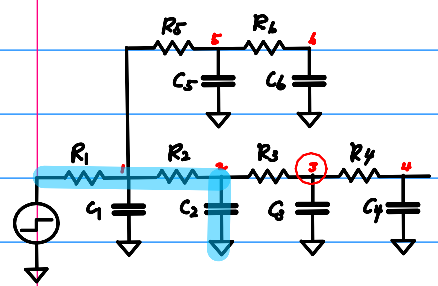
at node i

$$t_{pd\,i} = \sum_{k=1}^{n} C_k R_{i,k}$$

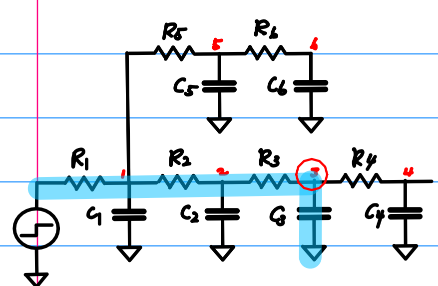
$$R_{i,k} = \sum R_j \Rightarrow R_j \in [\,\text{path}(i \to s) \cap \text{path}(k \to s)\,]$$



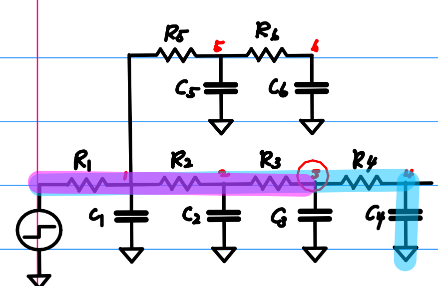
$R_1 C_1$



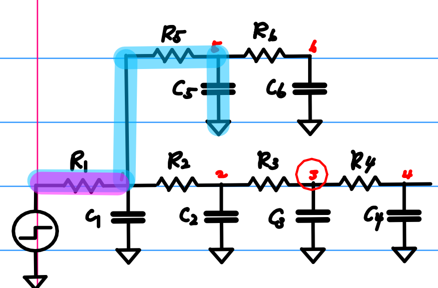
$(R_1 + R_2)\, C_2$



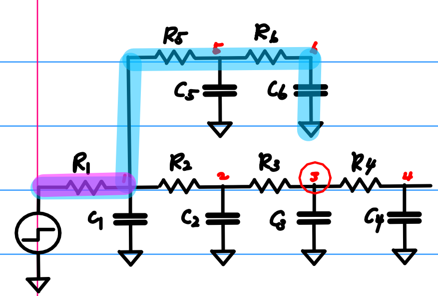
$(R_1 + R_2 + R_3)\, C_3$



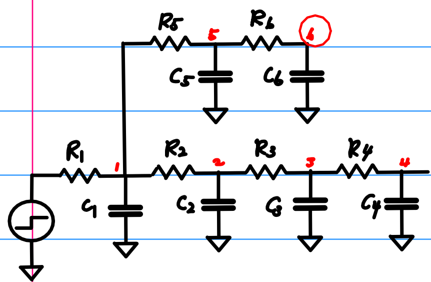
$(R_1 + R_2 + R_3)\, C_4$

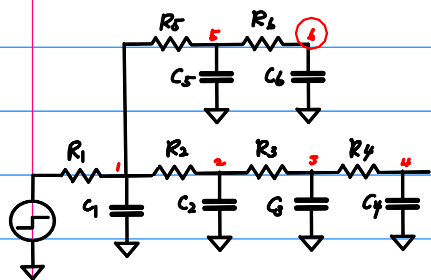


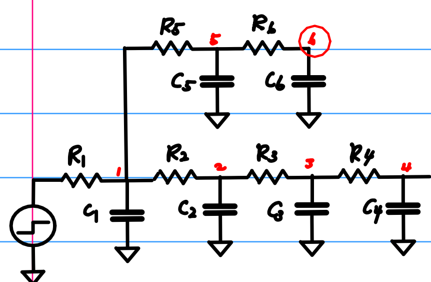
$R_1 C_5$

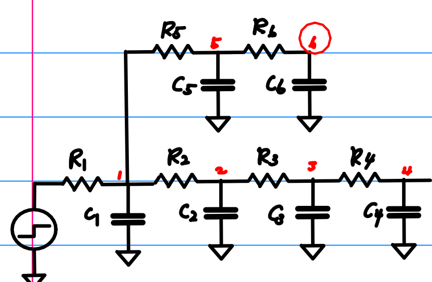


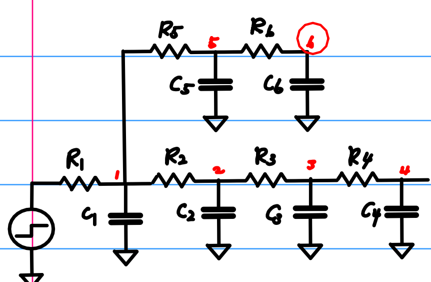
$R_1 C_6$

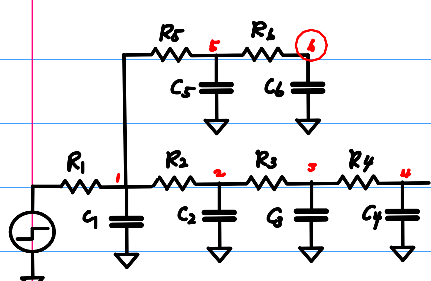

R5
5
R6
6
C5
C6
R1
1
R2
2
R3
3
R4
4
C1
C2
C3
C4


R5
5
R6
6
C5
C6
R1
1
R2
2
R3
3
R4
4
C1
C2
C3
C4


R5
5
R6
6
C5
C6
R1
1
R2
2
R3
3
R4
4
C1
C2
C3
C4


R5
5
R6
6
C5
C6
R1
1
R2
2
R3
3
R4
4
C1
C2
C3
C4


R5
5
R6
6
C5
C6
R1
1
R2
2
R3
3
R4
4
C1
C2
C3
C4


R5
5
R6
6
C5
C6
R1
1
R2
2
R3
3
R4
4
C1
C2
C3
C4

$$t_{pd3} = R_1C_1 + (R_1+R_2)C_2 + (R_1+R_2+R_3)C_3 + (R_1+R_2+R_3)C_4 + R_1C_5 + R_1C_6$$

$$t_{pd5} = R_1C_1 + R_1C_2 + R_1C_3 + R_1C_4 + (R_1+R_5)C_5 + (R_1+R_5)C_6$$