DTFS of Periodic Pulse Functions (3B)

- DTFS of Periodic Pulse Functions

Young Won Lim
6/10/13

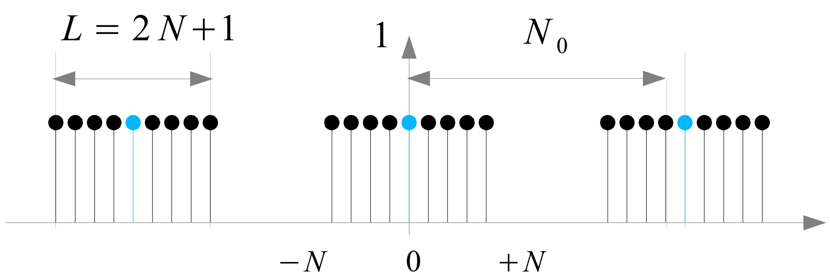
Please send corrections (or suggestions) to youngwlim@hotmail.com.

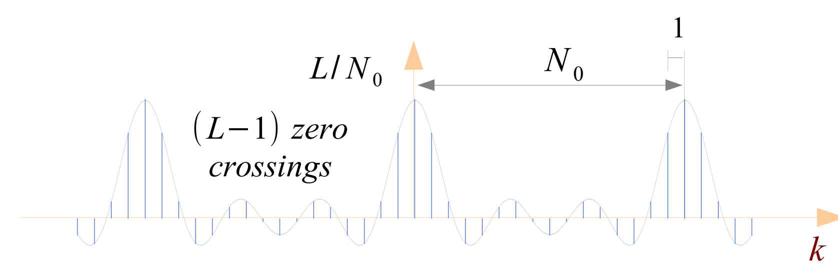
This document was produced by using OpenOffice and Octave.

Discrete Time Fourier Series

DTFS

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \iff x[n] = \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$





DTFS (Discrete Time Fourier Series)

$$X[k] = \frac{1}{N_0}\frac{\sin(\pi L k/N_0)}{\sin(\pi k/N_0)} = \frac{L}{N_0}\cdot \mathbf{drcl}(k/N_0,\ L)$$

$\mathrm{Rect}_N[n] * \delta_{N0}[n]$ DTFS (1)

Discrete Time Fourier Series

DTFS

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Longleftrightarrow \quad x[n] = \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

$$X[k] = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n] e^{-j(2\pi/N_0)kn}$$

$$= \frac{1}{N_0}\sum_{n=-N}^{+N} x[n] e^{-j(2\pi/N_0)kn}$$

$$N_0 X[k] = e^{+j(2\pi N/N_0)k} + \cdots + e^{-j(2\pi N/N_0)k}$$

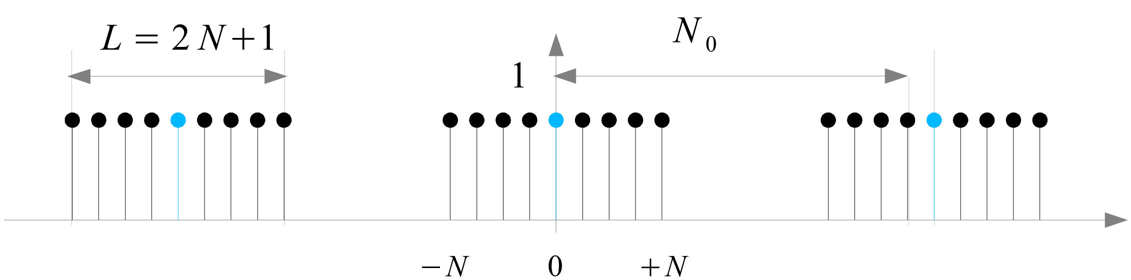
$$= e^{+j(2\pi/N_0)Nk} \cdot \frac{1 - e^{-j(2\pi/N_0)(2N+1)k}}{1 - e^{-j(2\pi/N_0)k}}$$

$$= e^{+j(m)Nk} \cdot \frac{1 - e^{-j(m)(2N+1)k}}{1 - e^{-j(m)k}} \qquad m = (2\pi/N_0)$$

$$= e^{+j(m)Nk} \cdot \frac{e^{-j(m)(2N+1)k/2}}{e^{-j(m)k/2}} \cdot \frac{e^{+j(m)(2N+1)k/2} - e^{-j(m)(2N+1)k/2}}{e^{+j(m)k/2} - e^{-j(m)k/2}}$$

$$= \frac{\sin((m)(2N+1)k/2)}{\sin((m)k/2)}$$

$$X[k] = \frac{1}{N_0}\,\frac{\sin((2\pi/N_0)(2N+1)k/2)}{\sin((2\pi/N_0)k/2)}$$



Dirichlet Function

$$drcl(t, L) = \frac{\sin(\pi L t)}{L \sin(\pi t)}$$

$\text{Rect}_N[n] * \delta_{N0}[n]$ DTFS (2)

Discrete Time Fourier Series DTFS

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Leftrightarrow \quad x[n] = \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

$$X[k] = \frac{1}{N_0}\frac{\sin((2\pi/N_0)(2N+1)k/2)}{\sin((2\pi/N_0)k/2)}$$

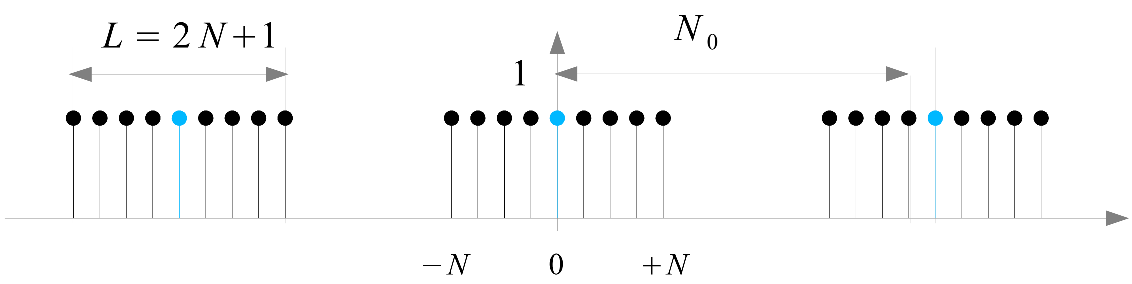
$$= \frac{1}{N_0}\frac{\sin(\pi k(2N+1)/N_0)}{\sin(\pi k/N_0)}$$

$$drcl(k/N_0, (2N+1)) = \frac{\sin(\pi k(2N+1)/N_0)}{(2N+1)\sin(\pi k/N_0)}$$

$$X[k] = \frac{(2N+1)}{N_0}\cdot drcl(k/N_0, (2N+1))$$

$$X[k] = \frac{1}{N_0}\frac{\sin(\pi k L/N_0)}{\sin(\pi k/N_0)}$$

$$X[k] = \frac{L}{N_0}\cdot drcl(k/N_0, L)$$



Dirichlet Function

$$drcl(t, L) = \frac{\sin(\pi L t)}{L\sin(\pi t)}$$

$$D_L(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}$$

$Rect_N[n] * \delta_{N0}[n]$ DTFS (3)

Discrete Time Fourier Series

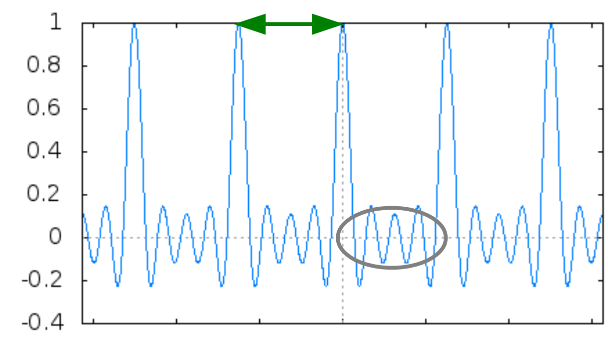
DTFS

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Longleftrightarrow \quad x[n] = \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

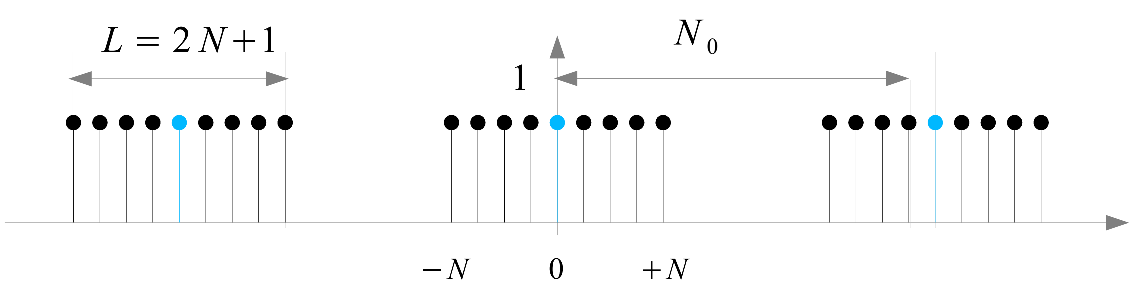
$$X[k] = \frac{1}{N_0}\frac{\sin(\pi k L/N_0)}{\sin(\pi k/N_0)}$$

$$X[k] = \frac{L}{N_0}\cdot drcl(k/N_0,\, L)$$

Period : N_0 (odd L), $2N_0$ (even L)



(L-1) zero crossings

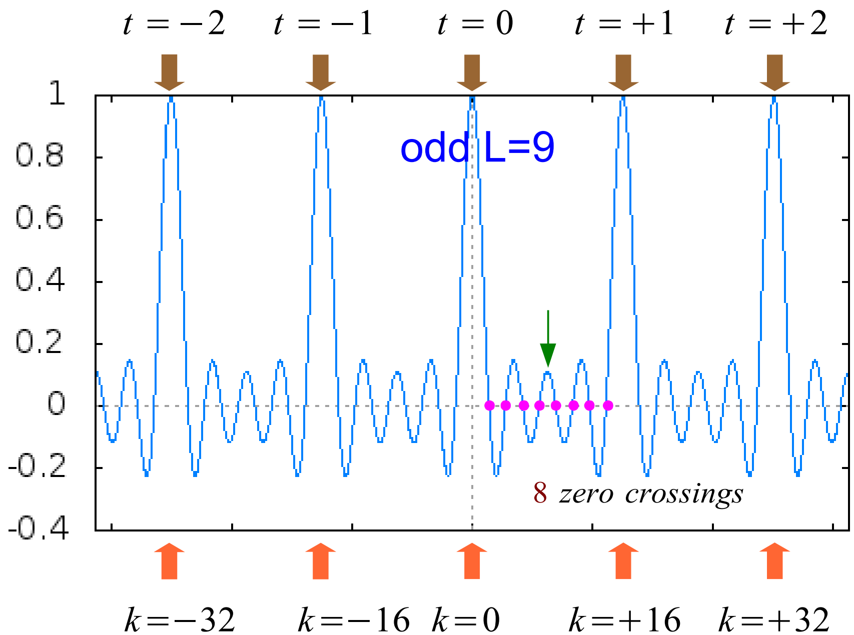


Dirichlet Function

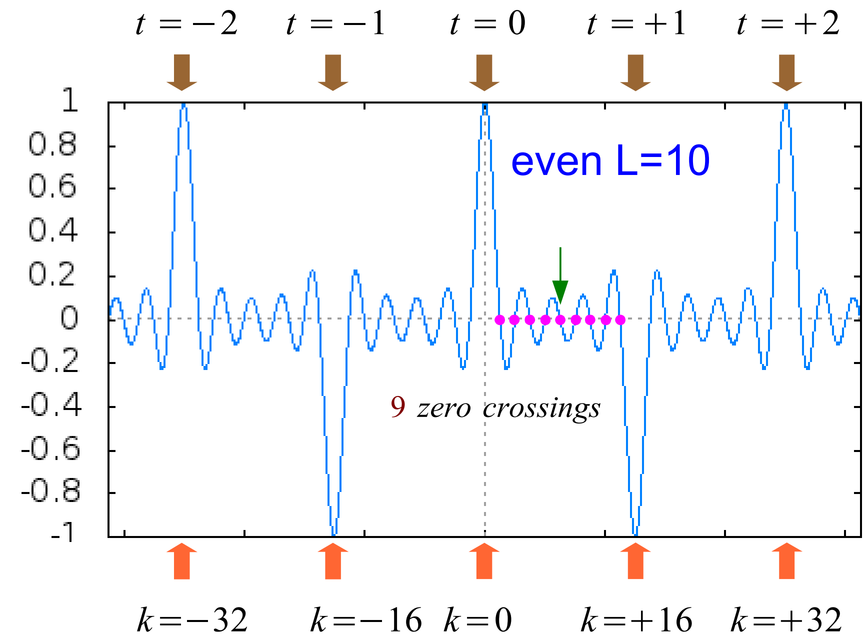
$$drcl(t, L) = \frac{\sin(\pi L t)}{L\sin(\pi t)}$$

$$D_L(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega} L/2)}{L\sin(\hat{\omega}/2)}$$

$\mathrm{Rect}_N[n] * \delta_{N0}[n]$ DTFS (4)



(L-1) zero crossings



(L-1) zero crossings

Dirichlet Function

$$drcl(t, L) = \frac{\sin(\pi L t)}{L \sin(\pi t)}$$

$$X[k] = \frac{9}{16} \cdot drcl(k/16, 9)$$

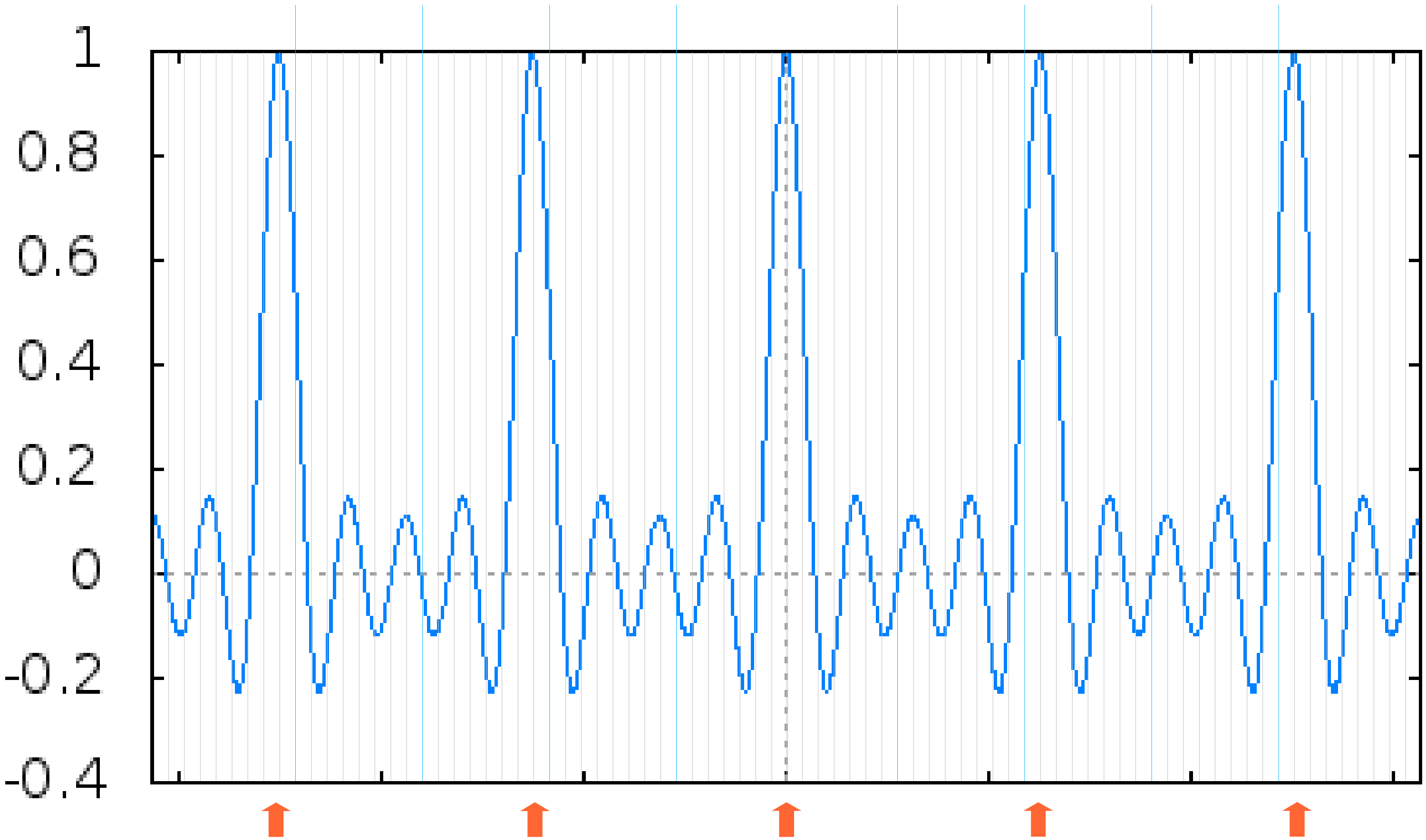
L → 9

k: $\cdots\ -3,\ -2,\ -1,\ 0,\ +1,\ +2,\ +3, \cdots$

Young Won Lim
6/10/13

$Rect_N[n] * \delta_{N0}[n]$ DTFS (5)

Period : N_0 (odd L), $2N_0$ (even L)



$k=-32$ $k=-16$ $k=0$ $k=+16$ $k=+32$

(L-1) zero crossings

 $X[k] = \frac{9}{16} \cdot drcl(k/18, 9)$

Rect$_2$[n] * δ$_8$[n] DTFS Example

Discrete Time Fourier Series

DTFS

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Longleftrightarrow \quad x[n] = \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

$$X[k] = \frac{1}{N_0}\frac{\sin(\pi k(2N+1)/N_0)}{\sin(\pi k/N_0)}$$

$$X[k] = \frac{L}{N_0}\cdot drcl(k/N_0,\ L)$$

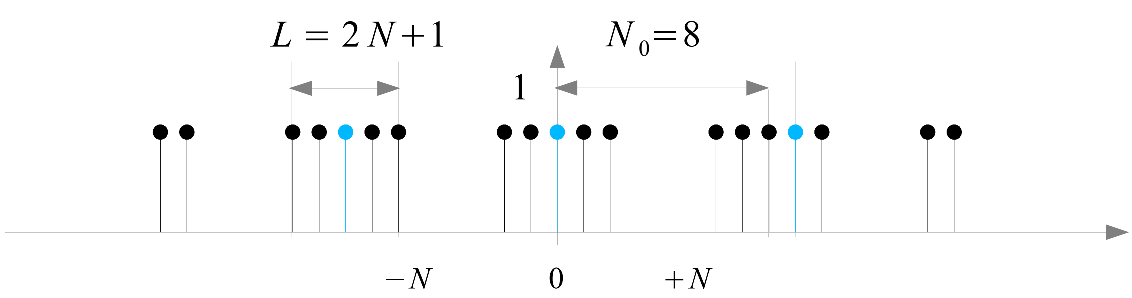
$$X[k] = \frac{1}{8}\frac{\sin(\pi k 5/8)}{\sin(\pi k/8)}$$

$$X[k] = \frac{5}{8}\cdot drcl(k/8,\ 5)$$

Period : $N_0 = 8$ (odd $L = 5$)

($L - 1$) = 4 zero crossings

$N_0 = 8 \quad L = 5 \quad (N = 2)$



Dirichlet Function

$$drcl(t, L) = \frac{\sin(\pi L t)}{L\sin(\pi t)}$$

$$D_L(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega} L/2)}{L\sin(\hat{\omega}/2)}$$

$Rect_3[n] * \delta_{16}[n]$ DTFS Example

Discrete Time Fourier Series DTFS

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Leftrightarrow \quad x[n] = \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

$$X[k] = \frac{1}{N_0}\frac{\sin(\pi k(2N+1)/N_0)}{\sin(\pi k/N_0)}$$

$$X[k] = \frac{L}{N_0}\cdot drcl(k/N_0,\, L)$$

$$X[k] = \frac{1}{16}\frac{\sin(\pi k\, 7/16)}{\sin(\pi k/16)}$$

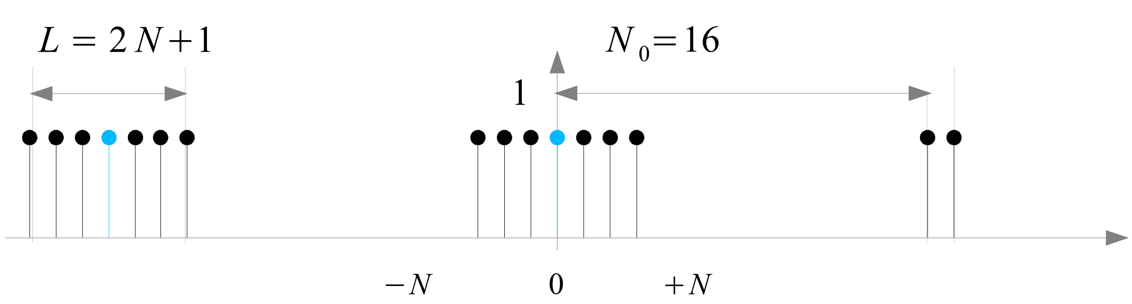
$$X[k] = \frac{7}{16}\cdot drcl(k/16,\, 7)$$

$Rect_3[n] * \delta_{16}[n]$

Period : $N_0 = 16$ (odd $L = 7$)

($L - 1$) = 6 zero crossings

$N_0 = 16 \quad L = 7 \quad (N = 3)$

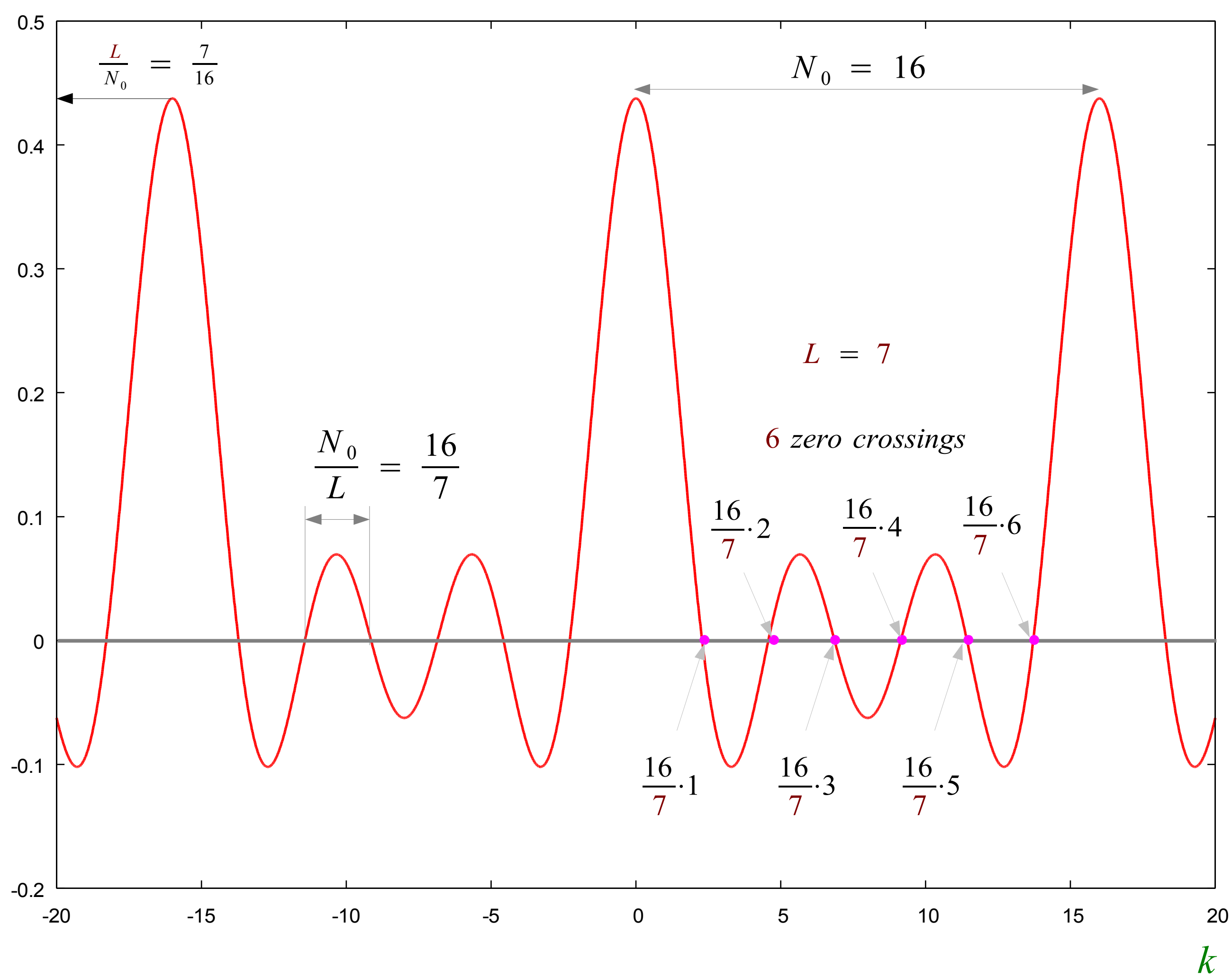


Dirichlet Function

$$drcl(t, L) = \frac{\sin(\pi L t)}{L\sin(\pi t)}$$

$$D_L(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}$$

7/16 drcl(k/16, 7)



$$\frac{1}{16}\frac{\sin(\pi 7 k/16)}{\sin(\pi k/16)}$$

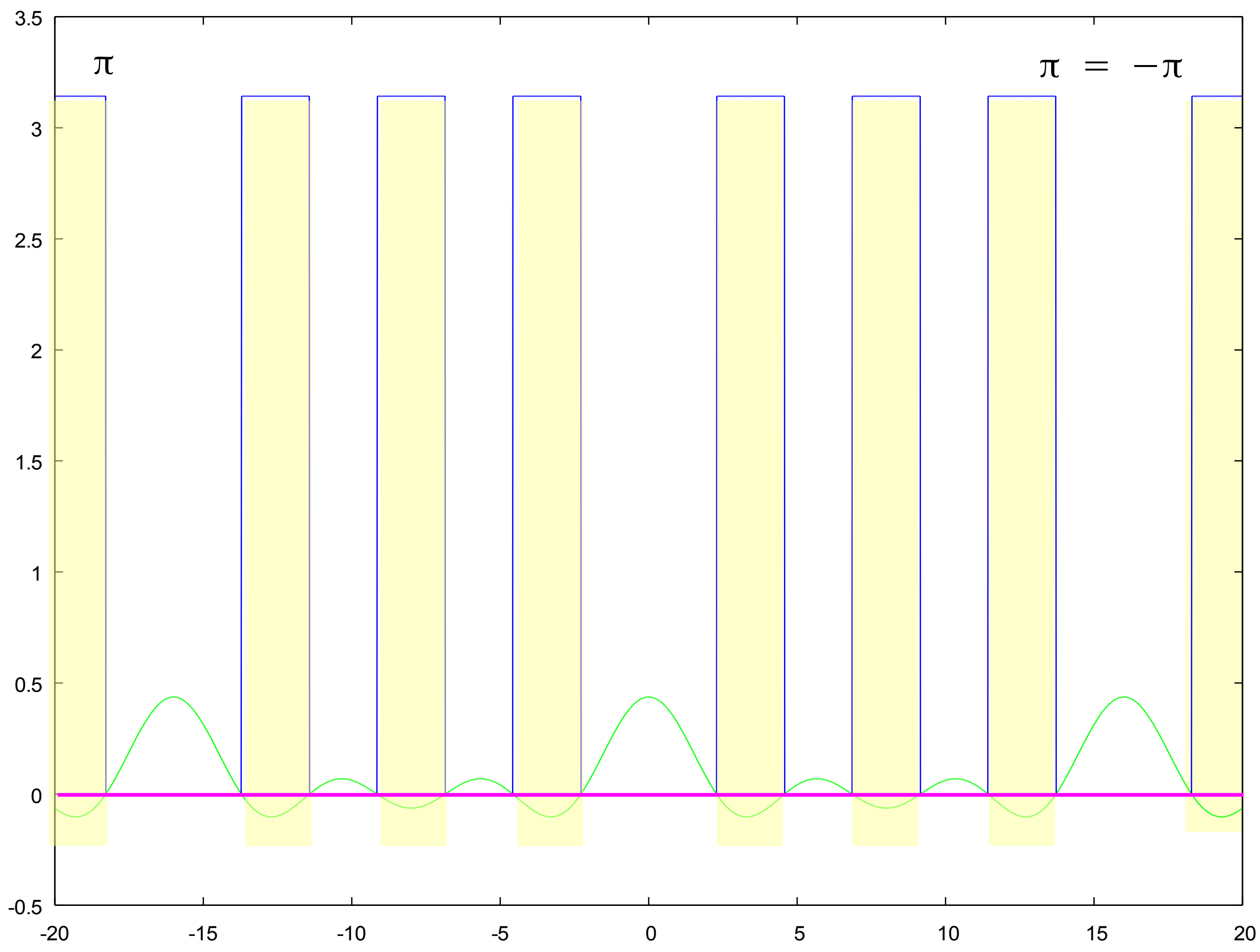
$$= \frac{7}{16}\frac{\sin(\pi 7 k/16)}{7\sin(\pi k/16)}$$

$$= \frac{7}{16}\,\mathbf{drcl}\left(\frac{k}{16}, 7\right)$$

Zeros

$$\frac{k}{16} = \frac{1}{7}m \Rightarrow k = \frac{16}{7}m$$

Phase of 7/16 drcl(k/16, 7)

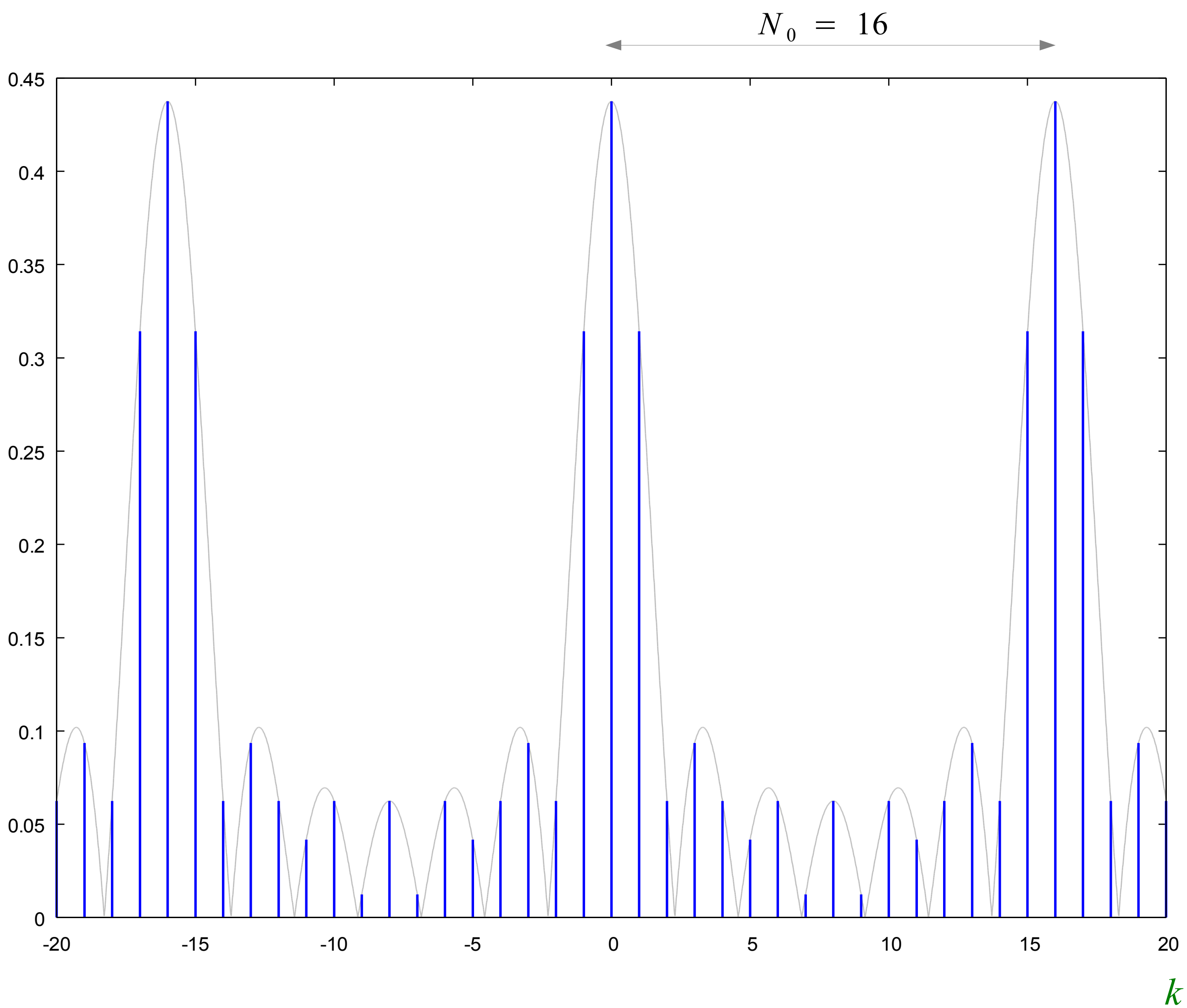


$$\frac{1}{16}\frac{\sin(\pi 7 k/16)}{\sin(\pi k/16)}$$

$$= \frac{7}{16}\frac{\sin(\pi 7 k/16)}{7\sin(\pi k/16)}$$

$$= \frac{7}{16}\,\boldsymbol{drcl}\left(\frac{k}{16}, 7\right)$$

Magnitude Response

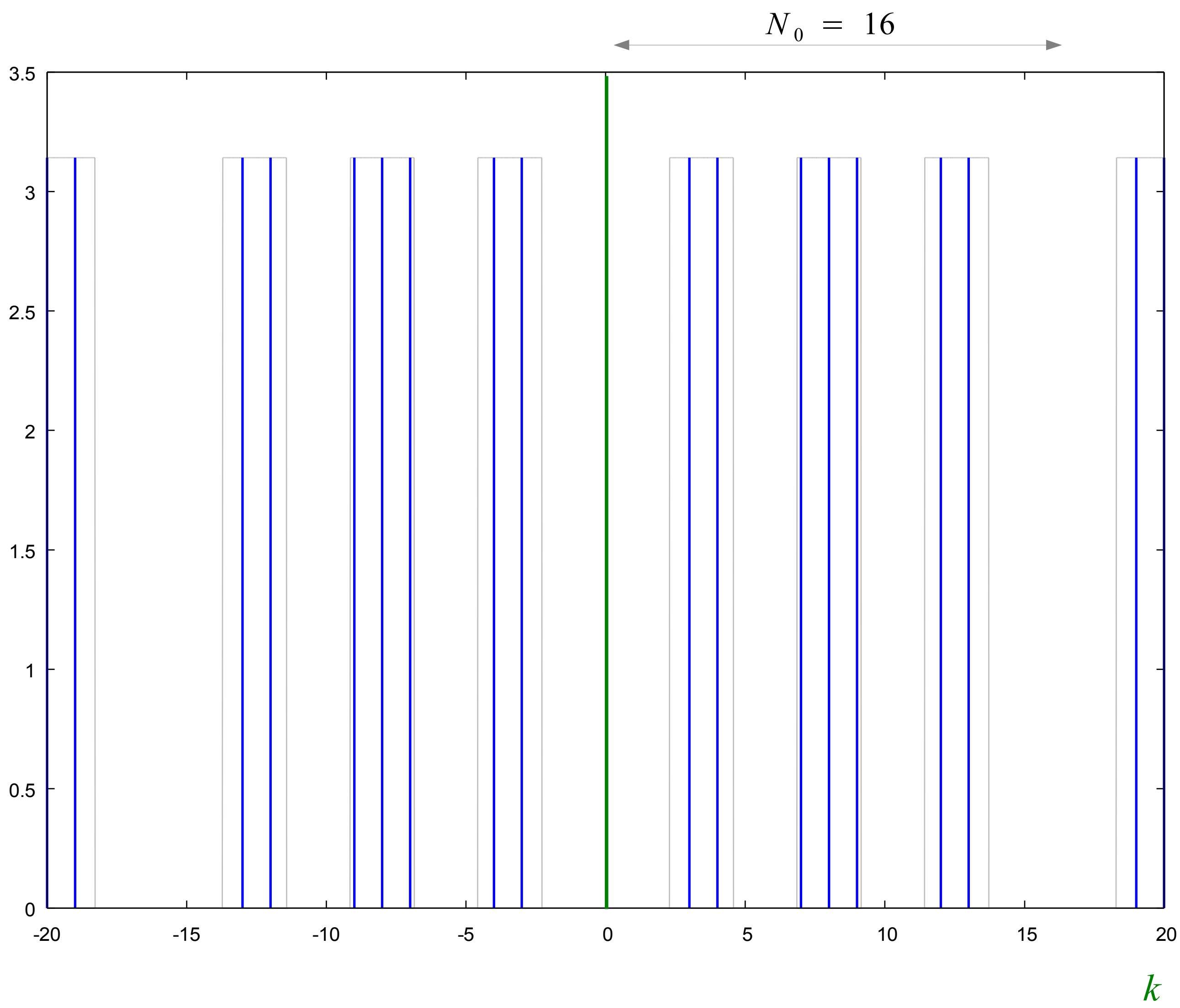


$$\frac{1}{16}\,\frac{\sin(\pi 7 k/16)}{\sin(\pi k/16)}$$

$$= \frac{7}{16}\,\frac{\sin(\pi 7 k/16)}{7\sin(\pi k/16)}$$

$$= \frac{7}{16}\,\boldsymbol{drcl}\left(\frac{k}{16}, 7\right)$$

Phase Response (1)

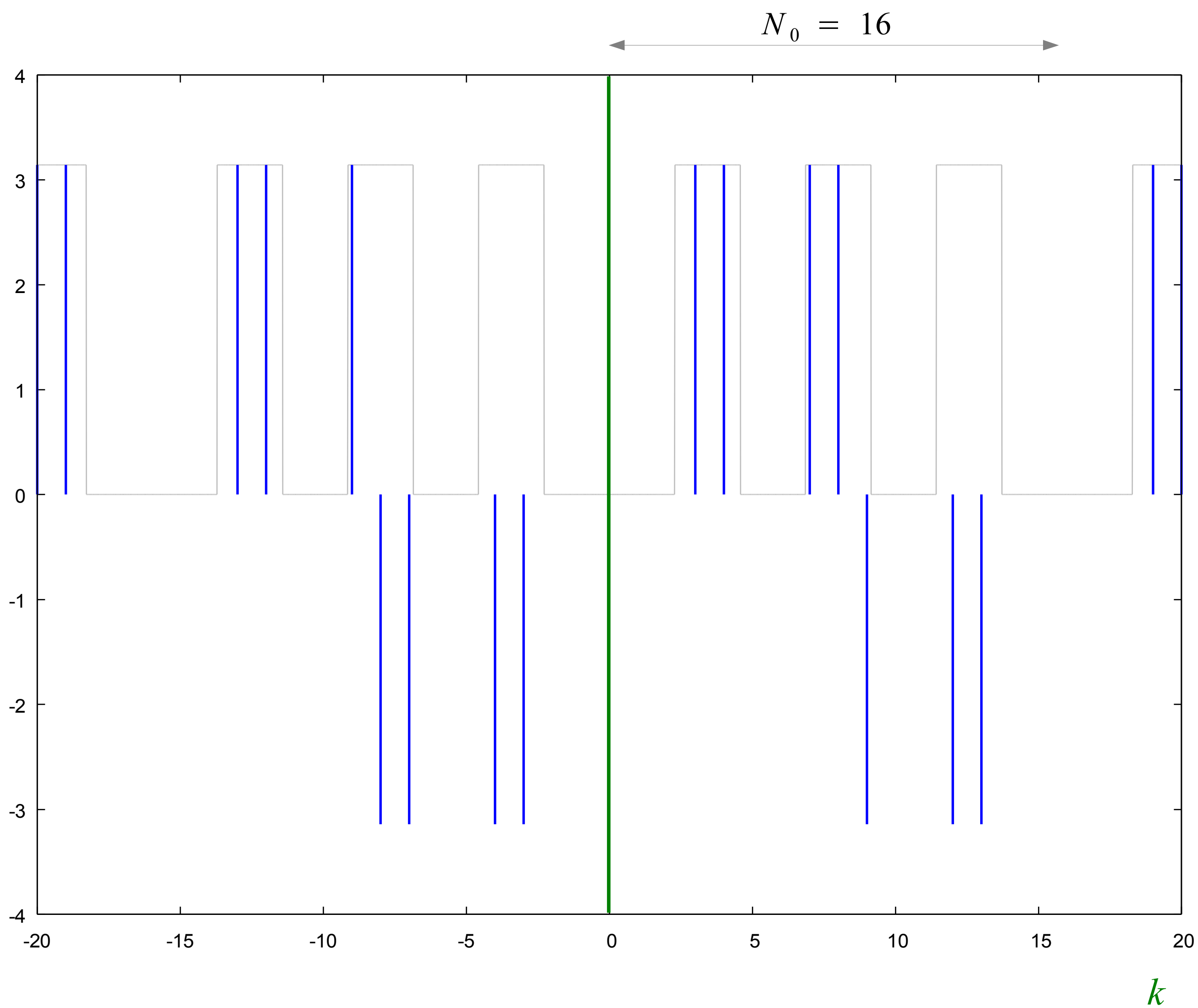


$$\frac{1}{16}\frac{\sin(\pi 7 k/16)}{\sin(\pi k/16)}$$

$$= \frac{7}{16}\frac{\sin(\pi 7 k/16)}{7\sin(\pi k/16)}$$

$$= \frac{7}{16}\,\boldsymbol{drcl}\left(\frac{k}{16}, 7\right)$$

Phase Response (2)



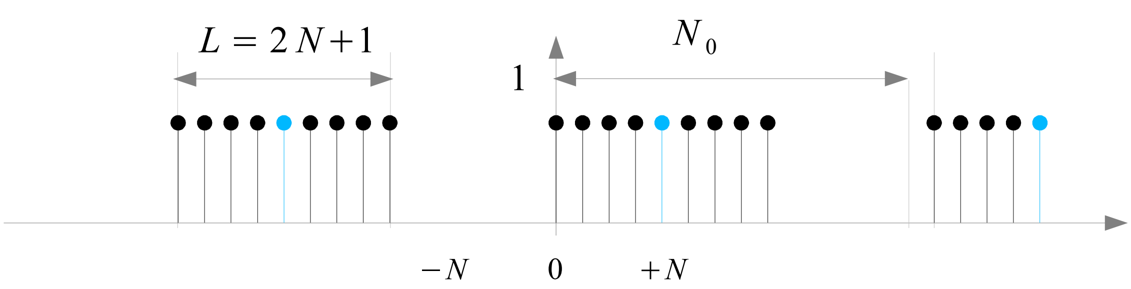
$$\frac{1}{16}\frac{\sin(\pi 7 k/16)}{\sin(\pi k/16)} = \frac{7}{16}\frac{\sin(\pi 7 k/16)}{7\sin(\pi k/16)} = \frac{7}{16}\,\textbf{\textit{drcl}}\left(\frac{k}{16}, 7\right)$$

$\text{Rect}_N[n-N] * \delta_{N0}[n]$ DTFS (1)

Discrete Time Fourier Series

DTFS

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Leftrightarrow \quad x[n] = \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$



Dirichlet Function

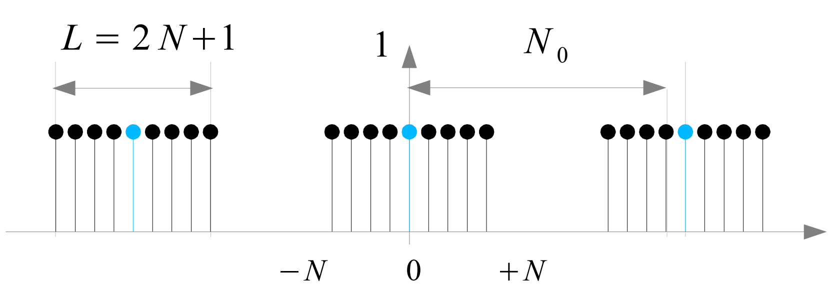
$$drcl(t, L) = \frac{\sin(\pi L t)}{L \sin(\pi t)}$$

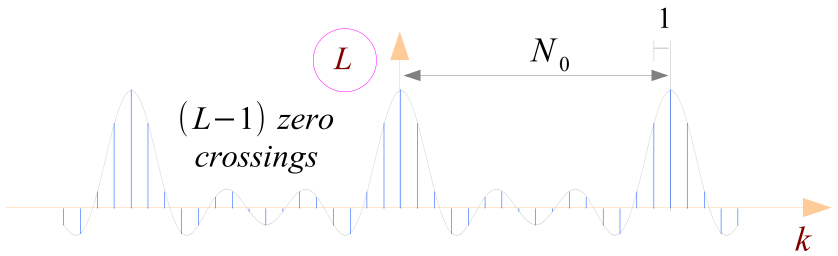
DFT

Discrete Fourier Transform

DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Longleftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$





$$X[k] = \frac{\sin(\pi L k / N_0)}{\sin(\pi k / N_0)}$$

$$= L \cdot \mathbf{drcl}(k/N_0,\ L)$$

DFT (Discrete Fourier Transform)

$\text{Rect}_N[n] * \delta_{N0}[n]$ DFT

Discrete Fourier Transform

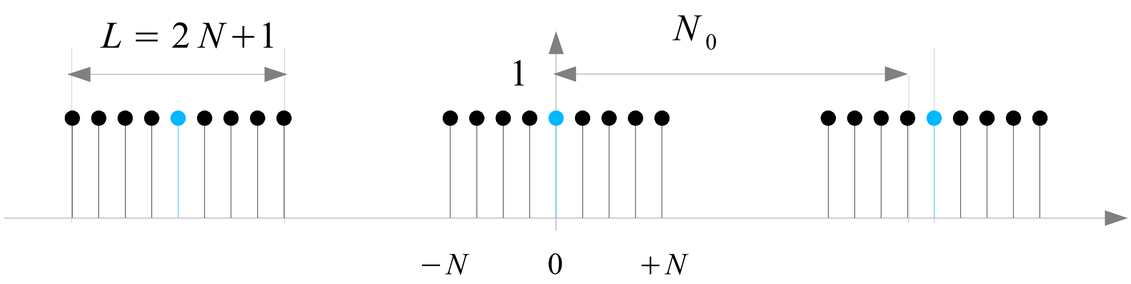
$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Longleftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

$$X[k] = \frac{\sin((2\pi/N_0)(2N+1)k/2)}{\sin((2\pi/N_0)k/2)} = \frac{\sin(\pi k/N_0(2N+1))}{\sin(\pi k/N_0)}$$

$$drcl(k/N_0, (2N+1)) = \frac{\sin(\pi k/N_0(2N+1))}{(2N+1)\sin(\pi k/N_0)}$$

$$X[k] = (2N+1)\cdot drcl(k/N_0, (2N+1))$$

$$= \frac{\sin(\pi k/N_0 L)}{\sin(\pi k/N_0)} = L\cdot drcl(k/N_0, L)$$



Dirichlet Function

$$drcl(t, L) = \frac{\sin(\pi L t)}{L\sin(\pi t)}$$

$$D_L(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}$$

$\text{Rect}_N[n] * \delta_{N0}[n]$ DTFS & DFT

Discrete Time Fourier Series

DTFS

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Longleftrightarrow \quad x[n] = \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

$$X[k] = \frac{1}{N_0}\frac{\sin(\pi k L/N_0)}{\sin(\pi k/N_0)}$$

$$X[k] = \frac{L}{N_0}\cdot drcl(k/N_0,\, L)$$

Discrete Fourier Transform

DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Longleftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

$$X[k] = \frac{\sin(\pi k L/N_0)}{\sin(\pi k/N_0)}$$

$$X[k] = L\cdot drcl(k/N_0,\, L)$$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] G. Beale, http://teal.gmu.edu/~gbeale/ece_220/fourier_series_02.html
[4] C. Langton, http://www.complextoreal.com/chapters/fft1.pdf
[5] M. J. Roberts, Fundamentals of Signals and Systems