

# Laurent Series and z-Transform Examples case 1.B

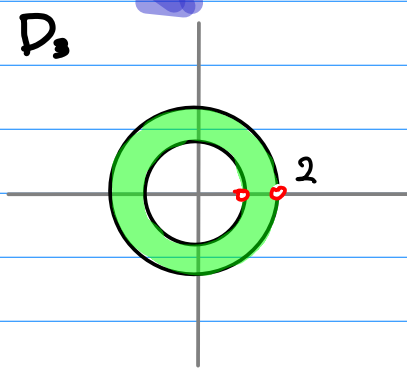
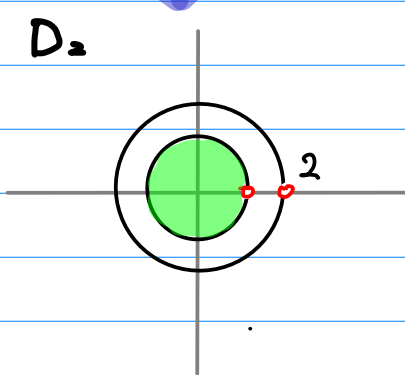
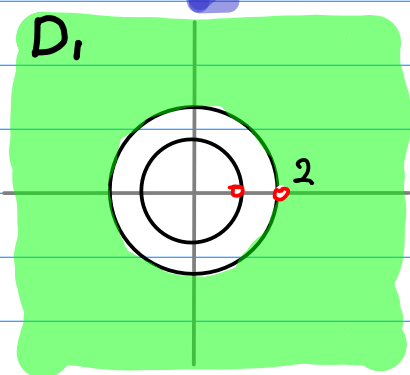
20171009

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$$X(z) = \frac{-1}{(z-1)(z-2)}$$

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$x_n?$

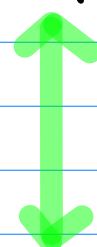
$x_n?$

$x_n?$

$\parallel$   
 $a_n$

$\parallel$   
 $a_n$

$\parallel$   
 $a_n$

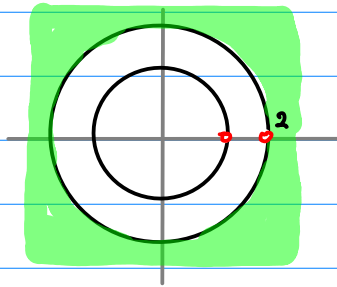


$f(z)?$

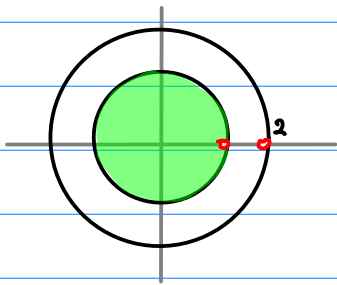
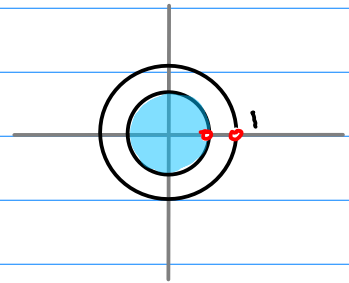
$f(z)?$

$f(z)?$

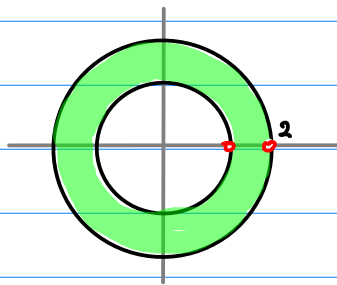
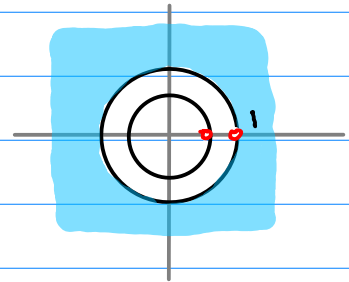
1. B  $X(z) = \frac{-1}{(z-1)(z-2)} \rightarrow f(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$



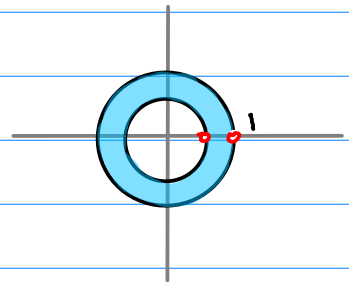
$$\sum_{n=-\infty}^{\infty} [1 - 2^{-n+1}] z^{-n}$$



$$\sum_{n=-1}^{-\infty} [-1 + 2^{n-1}] z^{-n}$$



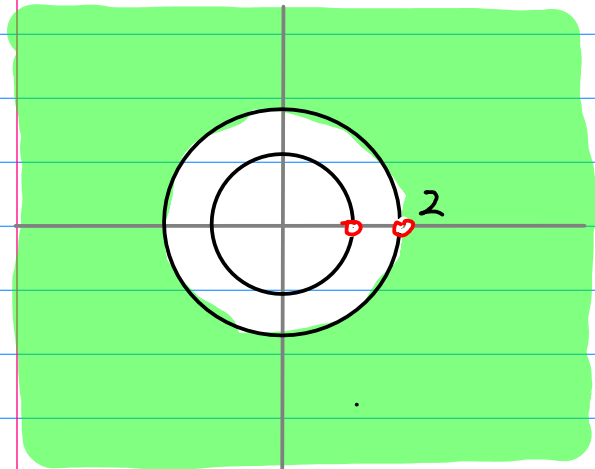
$$\sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^{-n}$$



$$X(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

Z.T. first

$$\textcircled{I} \quad 0_1 \quad |z| > 2 \quad \left[ \left| \frac{1}{z} \right| < 1, \quad \left| \frac{2}{z} \right| < 1 \right]$$



$$|z| > \frac{1}{2} \Rightarrow \left| \frac{1}{2z} \right| < 1$$

$$\left(\frac{2}{z}\right)^0 + \left(\frac{2}{z}\right)^1 + \left(\frac{2}{z}\right)^2 + \dots \longrightarrow \frac{1}{1 - \frac{2}{z}}$$

Converge

$$|z| > 1 \Rightarrow \left| \frac{1}{z} \right| < 1$$

$$\left(\frac{1}{z}\right)^0 + \left(\frac{1}{z}\right)^1 + \left(\frac{1}{z}\right)^2 + \dots \longrightarrow \frac{1}{1 - \frac{1}{z}}$$

Converge

ROC (Region of Convergence)

$$\begin{aligned} f(z) &= \frac{1}{z-1} - \frac{1}{z-2} = \frac{z}{z-1} - \frac{z}{z-2} = \frac{1}{z} \frac{z}{1-\frac{1}{z}} - \frac{1}{z} \frac{z}{1-\frac{2}{z}} \\ &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n} = \sum_{n=1}^{\infty} (1-2^{n-1}) z^{-n} \end{aligned}$$

$$\left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots + \frac{1}{2} \left\{ \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \dots \right\} \longrightarrow \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{(z-1)(z-2)}$$

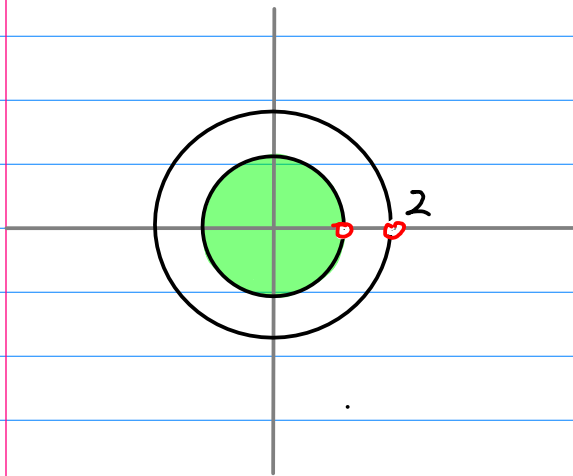
Converge

$$(1-2^0)z^{-1} + (1-2^1)z^{-2} + (1-2^2)z^{-3} + \dots \longrightarrow \frac{-1}{(z-1)(z-2)} \quad (|z| > 2)$$

Converge

$$x[n] = 1 - 2^n \quad \longleftrightarrow \quad X(z) = \frac{-1}{(z-1)(z-2)} \quad (|z| > 2)$$

III  $b_2 \quad |z| < 1 \quad \left[ \left| \frac{z}{1} \right| < 1, \quad \left| \frac{z}{2} \right| < 1 \right]$



$|z| < 2 \Rightarrow \left| \frac{z}{2} \right| < 1$

$\left(\frac{z}{2}\right)^0 + \left(\frac{z}{2}\right)^1 + \left(\frac{z}{2}\right)^2 + \dots \rightarrow \frac{1}{1 - \frac{z}{2}}$   
 Converge

$|z| < 1 \Rightarrow \left| \frac{z}{1} \right| < 1$

$\left(\frac{z}{1}\right)^0 + \left(\frac{z}{1}\right)^1 + \left(\frac{z}{1}\right)^2 + \dots \rightarrow \frac{1}{1 - z}$   
 Converge

ROC (Region of convergence)

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = -\frac{1}{1-z} + \frac{1}{2} \frac{2}{2-z} = -\frac{1}{1-\left(\frac{z}{1}\right)} + \frac{1}{2} \frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} z^n + \frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^n} = \sum_{n=0}^{\infty} (-1 + 2^{-n-1}) z^n$$

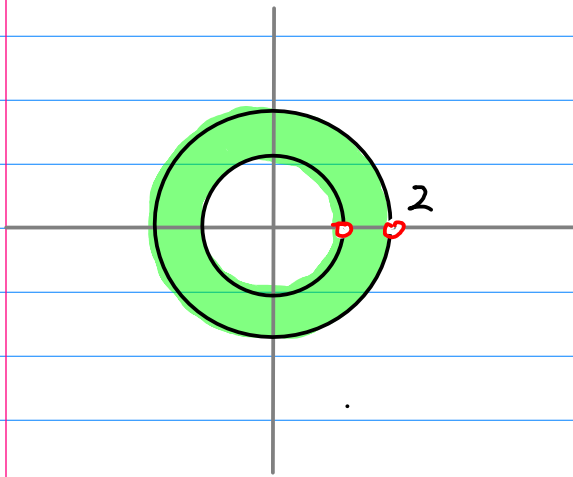
$$= \sum_{k=0}^{-\infty} (-1 + 2^{k-1}) z^{-k} = \sum_{n=0}^{-\infty} (-1 + 2^{n-1}) z^{-n}$$

$-\left\{ \left(\frac{z}{1}\right)^0 + \left(\frac{z}{1}\right)^1 + \left(\frac{z}{1}\right)^2 + \dots \right\}$   
 $+ \frac{1}{2} \left\{ \left(\frac{z}{2}\right)^0 + \left(\frac{z}{2}\right)^1 + \left(\frac{z}{2}\right)^2 + \dots \right\}$   $\rightarrow \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{(z-1)(z-2)}$   
 Converge

$(1+2^{-1})z^1 + (1+2^{-3})z^2 + (1+2^{-4})z^3 + \dots$   $\rightarrow \frac{-1}{(z-1)(z-2)} \quad (|z| < 2)$   
 Converge

$x[n] = -1 + 2^{n-1} \quad n \leq 0$   $\leftrightarrow$   $X(z) = \frac{-1}{(z-1)(z-2)} \quad (|z| > 2)$

III  $D_3 \quad 1 < |z| < 2 \quad \left[ \left| \frac{1}{z} \right| < 1, \quad \left| \frac{z}{2} \right| < 1 \right]$



$|z| < 2 \Rightarrow \left| \frac{z}{2} \right| < 1$

$\left(\frac{z}{2}\right)^0 + \left(\frac{z}{2}\right)^1 + \left(\frac{z}{2}\right)^2 + \dots \rightarrow \frac{1}{1 - \frac{z}{2}}$   
 Converge

$|z| > 1 \Rightarrow \left| \frac{1}{z} \right| < 1$

$\left(\frac{1}{z}\right)^0 + \left(\frac{1}{z}\right)^1 + \left(\frac{1}{z}\right)^2 + \dots \rightarrow \frac{1}{1 - \frac{1}{z}}$   
 Converge

ROC (Region of Convergence)

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \frac{z}{z-1} + \frac{1}{2} \frac{2}{z-2} = \frac{1}{z} \frac{1}{1 - \frac{1}{z}} + \frac{1}{2} \frac{1}{1 - \frac{z}{2}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{k=0}^{-\infty} \frac{z^{-k}}{2^{-k+1}} = \sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{-\infty} 2^{n-1} z^{-n}$$

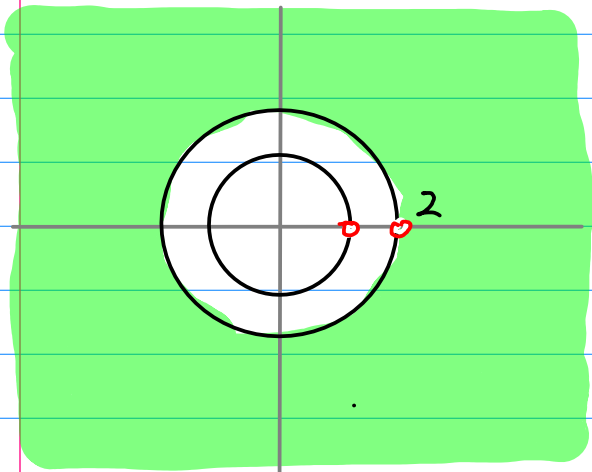
$$x_n = \begin{cases} 1 & (n > 0) \\ 2^{n-1} & (n \leq 0) \end{cases}$$

$\left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots$   
 $+ \frac{1}{2} \left\{ \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots \right\} \rightarrow \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{(z-1)(z-2)}$   
 Converge

$$X(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

Ⓘ  $D_1$   $|z| > 2$

causal  $x_n = 0$  ( $n \leq 0$ )

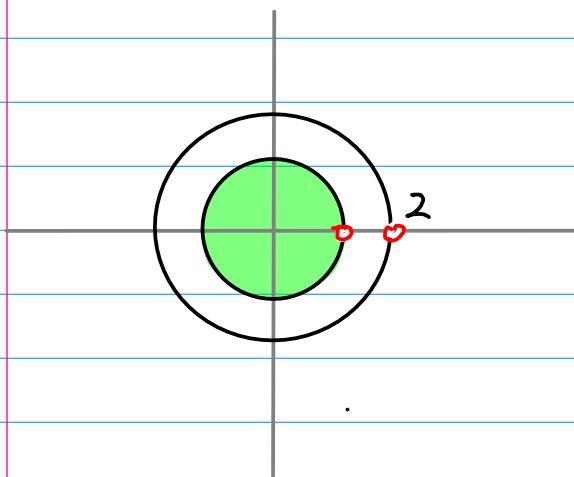


$$X(z) = \frac{-1}{(z-1)(z-2)}$$

$$x_n = \begin{cases} 1 - 2^{n-1} & (n > 0) \\ 0 & (n \leq 0) \end{cases}$$

Ⓜ  $D_3$   $|z| < 1$

anti-causal  $x_n = 0$  ( $n > 0$ )

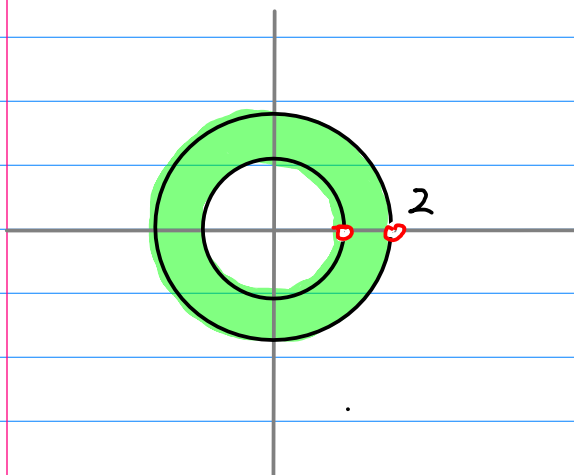


$$X(z) = \frac{-1}{(z-1)(z-2)}$$

$$x_n = \begin{cases} 0 & (n > 0) \\ 2^{n+1} - 1 & (n \leq 0) \end{cases}$$

Ⓝ  $D_2$   $1 < |z| < 2$

two-sided



$$X(z) = \frac{-1}{(z-1)(z-2)}$$

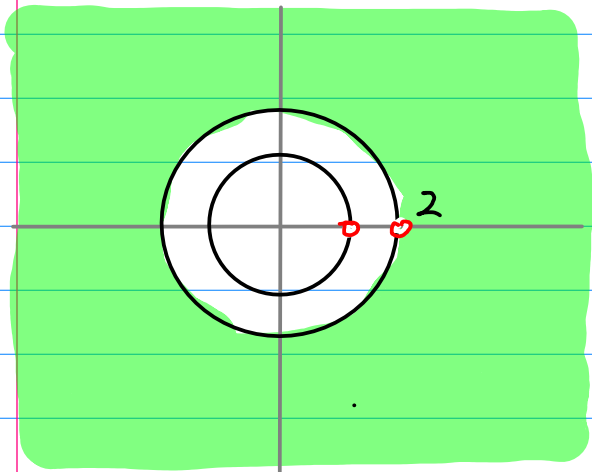
$$x_n = \begin{cases} 1 & (n > 0) \\ 2^{n+1} & (n \leq 0) \end{cases}$$





$$X(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

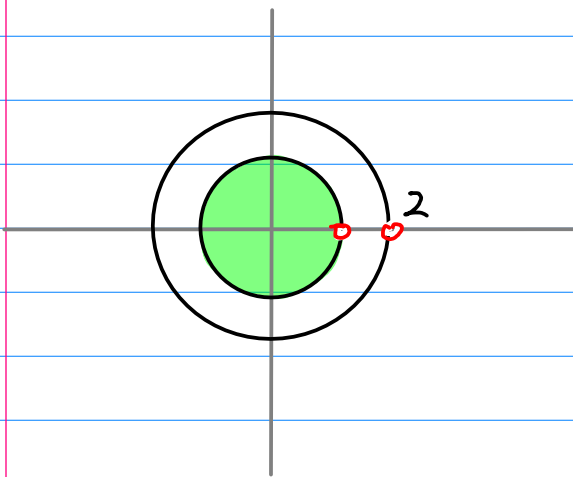
Ⓘ  $D_1$   $|z| > 2$  causal  $x_n = 0$  ( $n < 0$ )



$$\left| \frac{1}{z} \right| < 1 \quad \left| \frac{2}{z} \right| < 1$$

$$\begin{aligned} X(z) &= \frac{\left(\frac{1}{z}\right)}{1 - \left(\frac{1}{z}\right)} - \frac{\left(\frac{1}{z}\right)}{1 - \left(\frac{2}{z}\right)} \\ &= \sum_{n=0}^{\infty} 1 \cdot z^{-n-1} - \sum_{n=0}^{\infty} 2^n z^{-n-1} \\ &= \sum_{n=1}^{\infty} [1 - 2^{n-1}] z^{-n} \end{aligned}$$

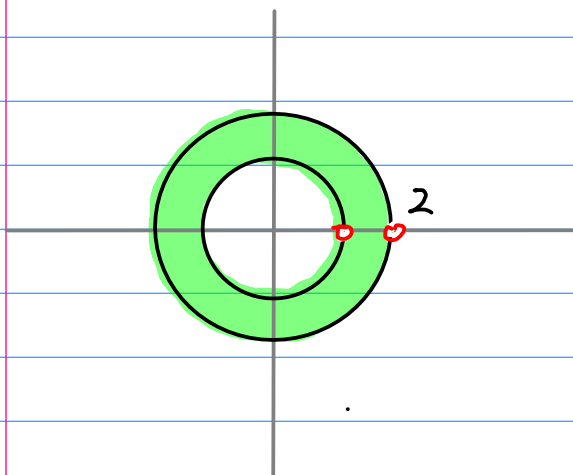
Ⓜ  $D_3$   $|z| < 1$  anti-causal  $x_n = 0$  ( $n > 0$ )



$$\left| \frac{z}{1} \right| < 1 \quad \left| \frac{z}{2} \right| < 1$$

$$\begin{aligned} X(z) &= \frac{-1}{1 - \left(\frac{z}{1}\right)} + \frac{\frac{1}{2}}{1 - \left(\frac{z}{2}\right)} \\ &= -\sum_{n=0}^{\infty} 1 \cdot z^n - \sum_{n=0}^{\infty} 2^{-n-1} z^n \\ &= \sum_{n=-1}^{-\infty} [-1 + 2^{n-1}] z^{-n} \end{aligned}$$

Ⓝ  $D_2$   $1 < |z| < 2$  two-sided



$$\left| \frac{1}{z} \right| < 1 \quad \left| \frac{z}{2} \right| < 1$$

$$\begin{aligned} X(z) &= \frac{\left(\frac{1}{z}\right)}{1 - \left(\frac{1}{z}\right)} + \frac{\left(\frac{1}{z}\right)}{1 - \left(\frac{z}{2}\right)} \\ &= \sum_{n=0}^{\infty} 1 \cdot z^{-n-1} + \sum_{n=0}^{\infty} 2^{-n-1} z^n \\ &= \sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^{-n} \end{aligned}$$

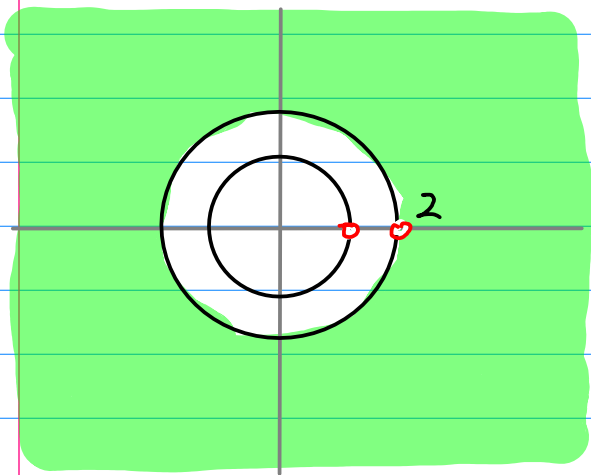
$$X(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

Ⓘ  $D_1$

$|z| > 2$

causal

$x_n = 0 \quad (n < 0)$



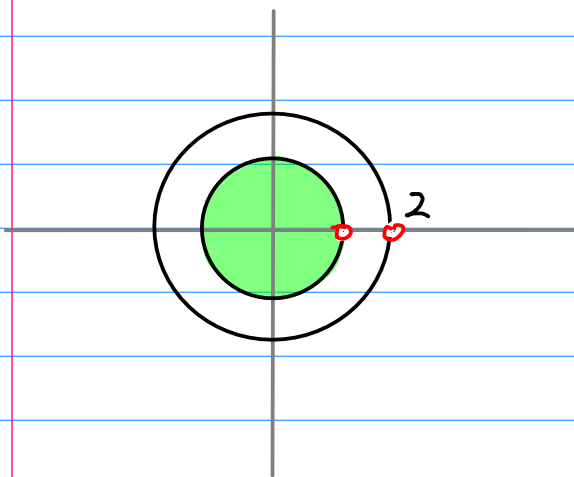
$$X(z) = \sum_{n=1}^{\infty} [1 - 2^{-n+1}] z^{-n}$$

Ⓜ  $D_3$

$|z| < 1$

anti-causal

$x_n = 0 \quad (n > 0)$

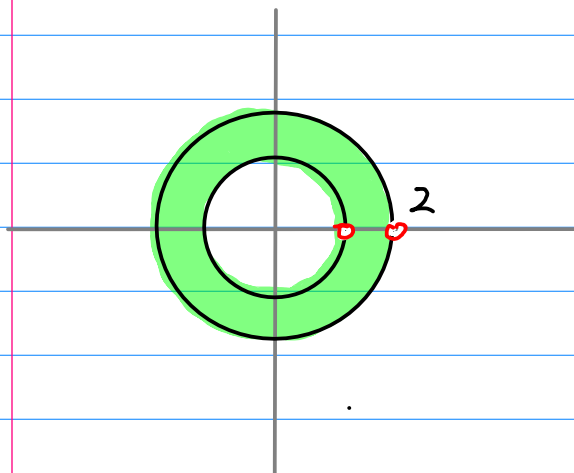


$$X(z) = \sum_{n=-1}^{\infty} [-1 + 2^{n+1}] z^{-n}$$

Ⓝ  $D_2$

$1 < |z| < 2$

two-sided



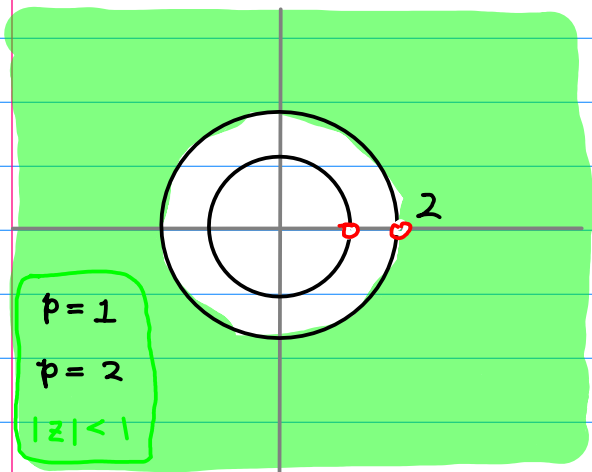
$$X(z) = \sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{-\infty} 2^{n+1} \cdot z^{-n}$$

$$X(z) = \frac{-1}{(z-1)(z-2)} \quad f(z) =$$

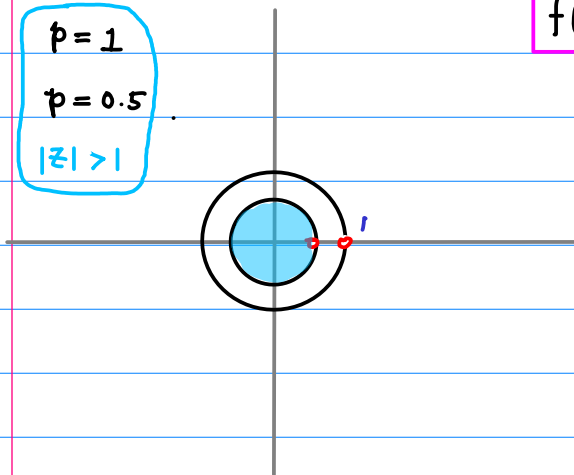
z.T. first

① - 1

$$\textcircled{1} \quad 0, \quad |z| > 2 \quad \left[ \left| \frac{1}{z} \right| < 1, \quad \left| \frac{2}{z} \right| < 1 \right]$$



$$\begin{aligned} X(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\ &= \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} - \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{2}{z}\right)} \\ &= \sum_{n=0}^{\infty} 1 \cdot z^{-n-1} - \sum_{n=0}^{\infty} 2^n z^{-n-1} \\ &= \sum_{n=1}^{\infty} [1 - 2^{n-1}] z^{-n} \end{aligned}$$



$$\begin{aligned} f(z) = X(z^{-1}) &= \sum_{n=1}^{\infty} [1 - 2^{n-1}] z^n \\ &= \sum_{n=1}^{\infty} 1 \cdot z^n - 2^{n-1} \cdot z^n \\ &= \frac{z}{1-\left(\frac{z}{1}\right)} - \frac{z}{1-\left(\frac{2z}{1}\right)} \\ &= -\frac{z}{z-1} + \frac{0.5z}{z-0.5} \\ &= \frac{-z+0.5+0.5z-0.5}{(z-1)(z-0.5)} z \\ &= \frac{-0.5z^2}{(z-1)(z-0.5)} \end{aligned}$$

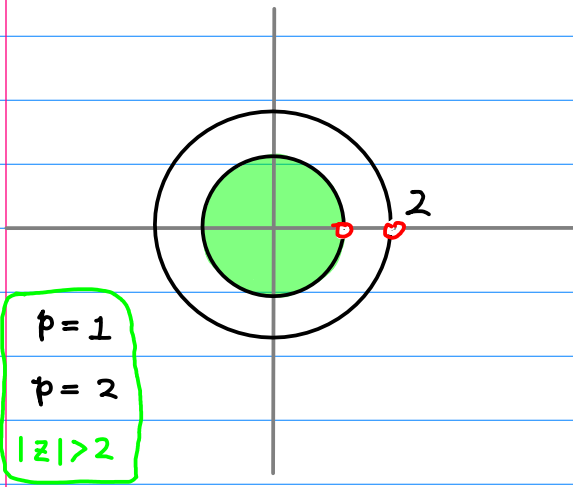
$$X(z) = \frac{-1}{(z-1)(z-2)}$$

$$X(z^{-1}) = \frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \frac{-z^2}{(1-z)(1-2z)} = \frac{-0.5z^2}{(z-1)(z-0.5)} = f(z)$$

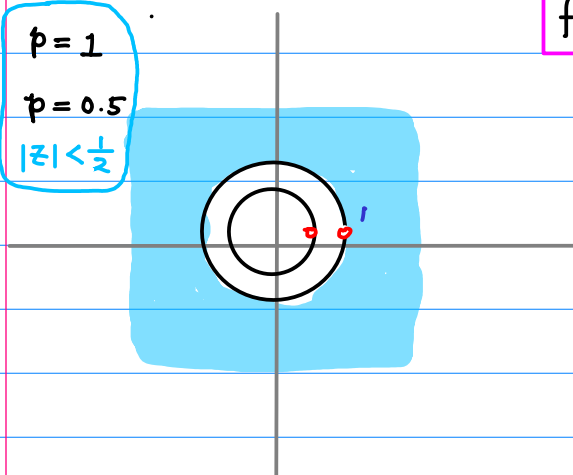
$$X(z) = \frac{-1}{(z-1)(z-2)} \quad f(z) =$$

Ⓜ -1

Ⓜ  $D_2 \quad |z| < 1 \quad [|\frac{z}{1}| < 1, |\frac{z}{2}| < 1]$



$$\begin{aligned} X(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\ &= \frac{-1}{1-\frac{z}{1}} + \frac{\frac{1}{2}}{1-\frac{z}{2}} \\ &= \sum_{n=0}^{\infty} (-1)\left(\frac{z}{1}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{z}{2}\right)^n \\ &= -\sum_{n=0}^{\infty} 1 \cdot z^n + \sum_{n=0}^{\infty} 2^{-n-1} z^n \\ &= \sum_{n=0}^{\infty} [-1 + 2^{n-1}] z^{-n} \end{aligned}$$



$$\begin{aligned} f(z) = X(z^{-1}) &= \sum_{n=0}^{\infty} [-1 + 2^{n-1}] z^n \\ &= \sum_{n=0}^{\infty} -1 \cdot z^{-n} + \sum_{n=0}^{\infty} 2^{-n-1} \cdot z^{-n} \\ &= -\frac{1}{1-\left(\frac{1}{z}\right)} + \frac{\frac{1}{2}}{1-\left(\frac{1}{2z}\right)} \\ &= -\frac{z}{z-1} + \frac{0.5z}{z-0.5} \\ &= \frac{-z+0.5+0.5z-0.5}{(z-1)(z-0.5)} z \\ &= \frac{-0.5z^2}{(z-1)(z-0.5)} \end{aligned}$$

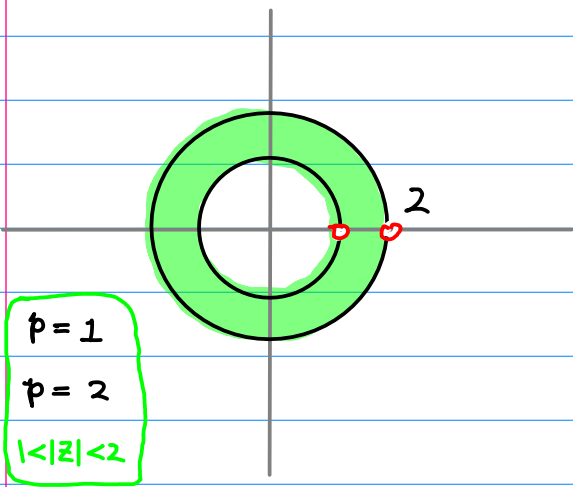
$$\begin{aligned} X(z) &= \frac{-1}{(z-1)(z-2)} \\ X(z^{-1}) &= \frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \frac{-z^2}{(1-z)(1-2z)} = \frac{-0.5z^2}{(z-1)(z-0.5)} = f(z) \end{aligned}$$

$$X(z) = \frac{-1}{(z-1)(z-2)} \quad f(z) =$$

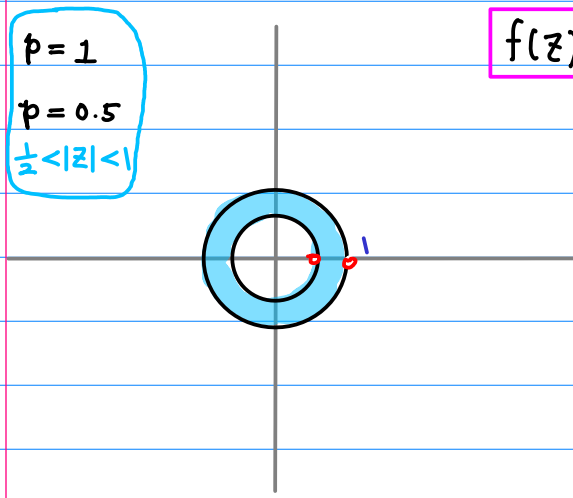
z.T. first

III -1

$$\text{III } D_3 \quad 1 < |z| < 2 \quad \left[ \left| \frac{1}{z} \right| < 1, \quad \left| \frac{z}{2} \right| < 1 \right]$$



$$\begin{aligned} X(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\ &= \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} + \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{z}{2}\right)} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{z}\right) \left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{z}\right) \left(\frac{z}{2}\right)^n \\ &= \sum_{n=0}^{\infty} z^{-n-1} + \sum_{n=0}^{\infty} 2^{-n-1} z^{-n} \\ &= \sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{\infty} 2^{-n-1} \cdot z^{-n} \end{aligned}$$



$$\begin{aligned} f(z) = X(z^{-1}) &= \sum_{n=1}^{\infty} z^n + \sum_{n=0}^{\infty} 2^{-n-1} \cdot z^n \\ &= \sum_{n=1}^{\infty} z^n + \sum_{n=0}^{\infty} 2^{-n-1} \cdot z^{-n} \\ &= \frac{\left(\frac{z}{1}\right)}{1-\left(\frac{z}{1}\right)} + \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{2z}\right)} \\ &= \frac{-z}{z-1} + \frac{0.5z}{z-0.5} \\ &= \frac{-z+0.5+0.5z-0.5}{(z-1)(z-0.5)} z \\ &= \frac{-0.5z^2}{(z-1)(z-0.5)} \end{aligned}$$

$$\begin{aligned} X(z) &= \frac{-1}{(z-1)(z-2)} \\ X(z^{-1}) &= \frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \frac{-z^2}{(1-z)(1-2z)} = \frac{-0.5z^2}{(z-1)(z-0.5)} = f(z) \end{aligned}$$



