

Characteristics of Multiple Random Variables

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles, Jr. and B. Shi

Outline

- 1 Expected Value of a Function with Multiple Random Variables

Expected Value

two random variables

Definition

the expected value of $g(x,y)$ is given by

$$\bar{g} = E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X, Y}(x, y) dx dy$$

where $g(x,y)$ is some function of two random variables X and Y

Expected Value

N random variables

Definition

for N random variables X_1, X_2, \dots, X_N ,
the expected value of $g(X_1, X_2, \dots, X_N)$ is given by

$$\bar{g} = E[g(X_1, X_2, \dots, X_N)]$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \dots, x_N) f_{X_1, \dots, X_N}(x_1, \dots, x_N) dx_1 \cdots dx_N$$

where $g(X_1, X_2, \dots, X_N)$ is some function of
 N random variables X_1, X_2, \dots, X_N

Expected Value

N random variables to a single random variable

If $g(X_1, X_2, \dots, X_N) = g(X_1)$, then

$$\begin{aligned}\bar{g} &= E[g(X_1, X_2, \dots, X_N)] \\ &= \int_{-\infty}^{\infty} g(x_1) f_{X_1}(x_1) dx_1 = E[g(X_1)] \\ \bar{g} &= E[g(X_1, X_2, \dots, X_N)] = E[g(X_1)]\end{aligned}$$

Joint Moments about the Origin

2 random variables

Definition

joint moment about the origin $m_{\{nk\}}$ is defined by

$$m_{\{nk\}} = E[X^n Y^k] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^n y^k f_{X,Y}(x,y) dx dy$$

the second moment $m_{\{11\}} = E[XY]$ is called the correlation $R_{\{XY\}}$ of X and Y

$$R_{\{XY\}} = m_{\{11\}} = E[X^1 Y^1] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^1 y^1 f_{X,Y}(x,y) dx dy$$

Joint Moments about the Origin

N random variables

Definition

For N random variables X_1, X_2, \dots, X_N , the $(n_1 + n_2 + \dots + n_N)$ -order joint moment about the origin $m_{\{n_1, n_2, \dots, n_N\}}$ is defined by

$$m_{\{n_1, n_2, \dots, n_N\}} = E[X_1^{n_1} X_2^{n_2} \dots X_N^{n_N}]$$

$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} x_1^{n_1} \dots x_N^{n_N} f_{X_1 \dots X_N}(x_1, \dots, x_N) dx_1 \dots dx_N$$

Correlation

2 random variables

Definition

the correlation $R_{\{XY\}}$ of X and Y

$$R_{\{XY\}} = m_{\{11\}} = E[X^1 Y^1] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^1 y^1 f_{X,Y}(x, y) dx dy$$

- if the correlation can be written as $R_{\{XY\}} = E[X]E[Y]$, then X and Y are **uncorrelated**
- **statistical independence** of X and Y is sufficient to guarantee they are **uncorrelated**
- the **converse** of this statement is not generally **true**
- If $R_{\{XY\}} = 0$, then X and Y are **orthogonal**

Joint Central Moments

2 random variables

Definition

The joint central moments for two random variables X and Y

$$\begin{aligned}\mu_{nk} &= E[(X - \bar{X})^n (Y - \bar{Y})^k] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{X})^n (y - \bar{Y})^k f_{X,Y}(x, y) dx dy\end{aligned}$$

2nd Order Joint Central Moments

2 random variables

Example

the second order moments μ_{20} and μ_{02} are just variances of X and Y

$$\mu_{20} = E[(X - \bar{X})^2]$$

$$\mu_{02} = E[(Y - \bar{Y})^2]$$

the second order moments μ_{11} is called the covariance of X and Y

$$\mu_{11} = E[(X - \bar{X})(Y - \bar{Y})]$$

Covariance

2 random variables

Definition

The covariance of two random variables X and Y

$$\begin{aligned}C_{XY} &= \mu_{11} = E[(X - \bar{X})(Y - \bar{Y})] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{X})(y - \bar{Y}) f_{X,Y}(x, y) dx dy\end{aligned}$$

Covariance

2 random variables

Theorem

note that $(x - \bar{X})(y - \bar{Y}) = xy - x\bar{Y} - \bar{X}y + \bar{X}\bar{Y}$

$$C_{XY} = R_{XY} - \bar{X}\bar{Y} = R_{XY} - E[X]E[Y]$$

if X and Y are independent or **uncorrelated**

$$C_{XY} = 0$$

if X and Y are **orthogonal**

$$C_{XY} = -E[X]E[Y]$$

if X or Y has a **zero mean value**

$$C_{XY} = 0$$

Normalized Second Order Moment

2 random variables

Definition

The normalized second order moment

$$\rho = \mu_{11} / \sqrt{\mu_{20}\mu_{02}} = C_{XY} / \sigma_X \sigma_Y$$

$$\rho = E \left[\frac{(X - \bar{X})}{\sigma_X} \frac{(Y - \bar{Y})}{\sigma_Y} \right]$$

ρ is also known as the correlation coefficient of X and Y

$$-1 \leq \rho \leq 1$$

Joint Central Moments

N random variables

Definition

The $(n_1 + n_2 + \dots + n_N)$ -order joint central moments for N random variables X_1, X_2, \dots, X_N

$$\begin{aligned} \mu_{n_1 n_2 \dots n_N} &= E[(X_1 - \bar{X}_1)^{n_1} (X_2 - \bar{X}_2)^{n_2} \dots (X_N - \bar{X}_N)^{n_N}] \\ &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} (x_1 - \bar{X}_1)^{n_1} \dots (x_N - \bar{X}_N)^{n_N} \\ &\quad \cdot f_{X_1, \dots, X_N}(x_1, \dots, x_N) dx_1 \dots dx_N \end{aligned}$$

