First Order Logic – Semantics (3A)

Young W. Lim
6/7/17

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using LibreOffice

Based on

Contemporary Artificial Intelligence,
R.E. Neapolitan & X. Jiang

Logic and Its Applications,
Burkey & Foxley

A Signature and a Language

First specify a **signature**

Constant Symbols $\{c_1, c_2, \dots c_n\} = D$

Predicate Symbols $\{\mathbf{P}_1, \mathbf{P}_2, \dots \mathbf{P}_m\}$

Function Symbols $\{\mathbf{f}_1, \mathbf{f}_2, \dots \mathbf{f}_l\}$

Determines the **language**

Given a language

A **model** is specified

A **domain of discourse**

a set of entities $\{\text{entity}_1, \text{entity}_2, \dots \text{entity}_n\}$

An **interpretation**

constant assignments $\{c_1, c_2, \dots c_n\} = D$

function assignments $\mathbf{f}_1(\ ), \mathbf{f}_2(\ ), \dots \mathbf{f}_l(\ )$

truth value assignments $\mathbf{P}_1(\ ), \mathbf{P}_2(\ ), \dots \mathbf{P}_m(\ )$

Model – domain of discourse

1. a nonempty set D of **entities** called a **domain of discourse**
   - this domain is a set
   - each element in the set : entity
   - each constant symbol : one entity in the domain

If we considering all individuals in a class,
The constant symbols might be
‘Mary’, - an entity
‘Fred’, - an entity
‘John’, - an entity
‘Tom’ - an entity

Model – interpretation

2. an **interpretation**

(a) an entity in D is assigned to each of the constant symbols.

Normally, every entity is assigned to a constant symbol.

(b) for each **function**,

an entity is assigned to each possible input of entities to the **function**

(c) the predicate '**True**' is always assigned the value T

The predicate '**False**' is always assigned the value F

(d) for every other **predicate**,

the value T or F is assigned

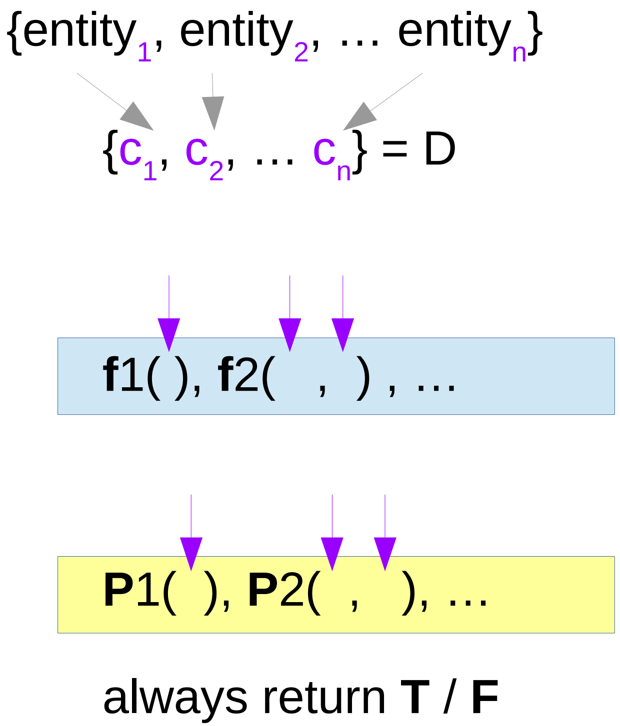
to each possible input of entities to the **predicate**

Interpretation

Constant assignments

Function assignments

Truth value assignments



Interpretation

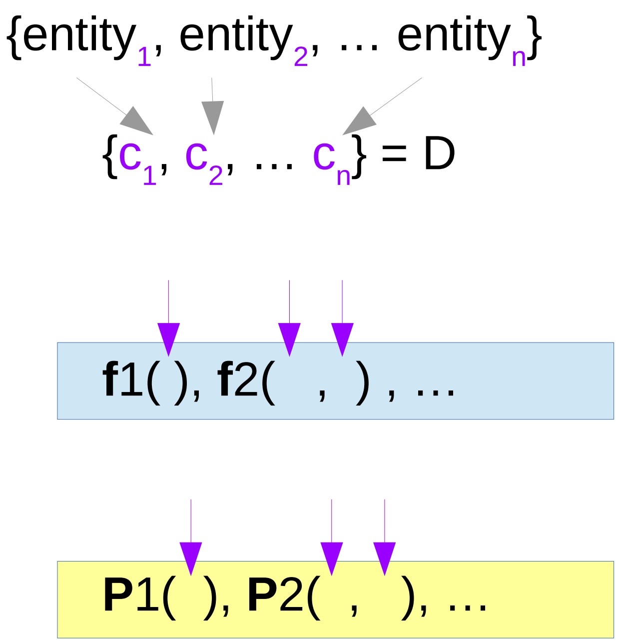
Propositional Logic

	A	B	
Interpretation I_1 ➡	T	T	
Interpretation I_2 ➡	T	F	
Interpretation I_3 ➡	F	T	
Interpretation I_4 ➡	F	F	

First Order Logic

Sentences

	P1( )	P2( )	…	S1	S2
Interpretation I_1 ➡	T	T			
Interpretation I_2 ➡	T	F			
Interpretation I_3 ➡	F	T			
Interpretation I_4 ➡	F	F			

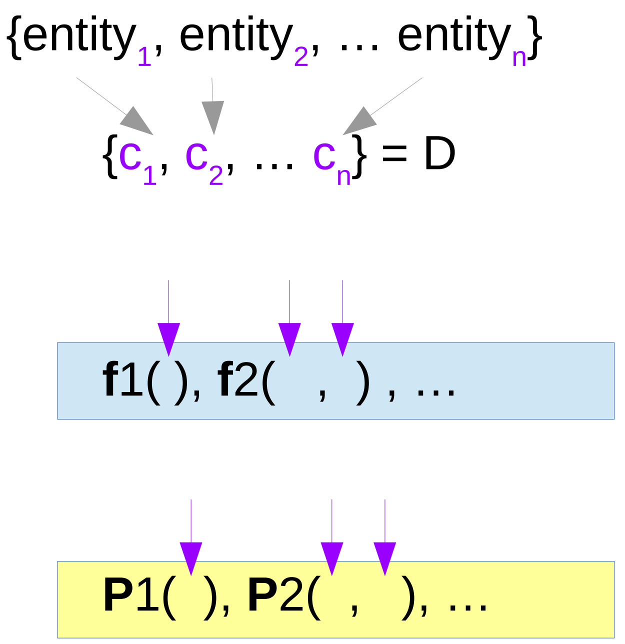


always return **T** / **F**

Each possible input of entities

Arity one: C(n, 1)
Arity two: C(n, 2)
Arity three: C(n, 3)
…

Arity one functions & predicates: C(n, 1)
Arity two: C(n, 2)
Arity three: C(n, 3)
…



always return **T** / **F**

Interpretation

Constant assignments

(a) an entity → the constant symbols.

Function assignments

(b) an entity → each possible input of entities to the **function**

Truth value assignments

(c) the value T →the predicate '**True**'
the value F → the predicate '**False**'

(d) for every other **predicate**,
the value T or F is assigned → every other predicate
to each possible input of entities to the **predicate**

Signature Model Examples A – (1)

Signature

1. constant symbols = { Mary, Fred, Sam }
2. predicate symbols = { married, young }
   married(x, y) : arity two
   young(x) : arity one

Model

1. domain of discourse D : the set of three particular *individuals*

   - this domain is a set
   - each element in the set : entity (= individuals)
   - each constant symbol : one entity in the domain (= one individual)

2. interpretation
   (a) a different individual is assigned to each of the constant symbols

   (a) an entity in D is assigned to each of the constant symbols.
   Normally, every entity is assigned to a constant symbol.

Signature Model Examples A – (2)

(b) for each **function**,
an entity is assigned to each possible input of entities to the **function**

(c) the predicate '**True**' is always assigned the value T
The predicate '**False**' is always assigned the value F

(d) the truth value assignments for every predicate

young(Mary) = F, young(Fred) = F, young(Sam) = T

married(Mary, Mary) = F, married(Mary, Fred) = T, married(Mary, Sam) = F
married(Fred, Mary) = T, married(Fred, Fred) = F, married(Fred, Sam) = F
married(Sam, Mary) = F, married(Sam, Fred) = F, married(Sam, Sam) = F

(d) for every other **predicate**,
the value T or F is assigned
to each possible input of entities to the **predicate**

(Mary, Mary), (Mary, Fred), (Mary, Sam)
(Fred, Mary), (Fred, Fred), (Fred, Sam)
(Sam, Mary), (Sam, Fred), (Sam, Sam)

Signature Model Examples B – (1)

Signature

1. constant symbols = { Fred, Mary, Sam }
2. predicate symbols = { love }   love(x, y) : arity two
3. function symbols = { mother }   mother(x) : arity one

Model

1. domain of discourse D : the set of three particular individuals
2. interpretation
   (a) a different individual is assigned to each of the constant symbols

   (b) the truth value assignments for every predicate
      love(Fred, Fred) = F, love(Fred, Mary) = F, love(Fred, Ann) = F
      love(Mary, Fred) = T, love(Mary, Mary) = F, love(Mary, Ann) = T
      love(Ann, Fred) = T, love(Ann, Mary) = T, love(Ann, Ann) = F

   (c) the function assignments
      mother(Fred) = Mary, mother(Mary) = Ann, mother(Ann) = - (no assignment)

Signature Model Examples B – (2)

2. interpretation

(a) a different individual is assigned to each of the constant symbols

(a) an entity in D is assigned to each of the constant symbols.
Normally, every entity is assigned to a constant symbol.

(b) the truth value assignments

(b) for each **function**,
an entity is assigned to each possible input of entities to the **function**

love(Fred, Fred) = F, love(Fred, Mary) = F, love(Fred, Ann) = F
love(Mary, Fred) = T, love(Mary, Mary) = F, love(Mary, Ann) = T
love(Ann, Fred) = T, love(Ann, Mary) = T, love(Ann, Ann) = F

(c) the function assignments

(d) for every other **predicate**,
the value T or F is assigned
to each possible input of entities to the **predicate**

mother(Fred) = Mary, mother(Mary) = Ann, mother(Ann) = - (no assignment)

The truth value of sentences

The truth values of **all sentences** are assigned :

1. the truth values for **sentences** developed with the symbols $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$ are assigned as in propositional logic.

2. the truth values for two terms connected by the = symbol is **T** if both terms refer to the same entity; otherwise it is **F**

3. the truth values for $\forall x\ p(x)$ has value **T** if $p(x)$ has value **T** for **every assignment** to x of an entity in the domain D; otherwise it has value **F**

4. the truth values for $\exists x\ p(x)$ has value **T** if $p(x)$ has value **T** for **at least one assignment** to x of an entity in the domain D; otherwise it has value **F**

5. the operator **precedence** is as follows $\neg$, $=$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$

6. the **quantifiers** have precedence over the operators

7. **parentheses** change the order of the precedence

Terms

Terms

1. **Variables**. Any variable is a term.
2. **Functions**. Any expression $\mathbf{f}(t_1,...,t_n)$ of n arguments is a term
   where each argument t_i is a term and **f** is a function symbol of valence n
   In particular, symbols denoting individual constants are
   0-ary function symbols, and are thus terms.

Only expressions which can be obtained
by finitely many applications of rules 1 and 2 are terms.

no expression involving a predicate symbol is a term.

https://en.wikipedia.org/wiki/First-order_logic#Formation_rules

Formulas

Formulas (wffs)

Predicate symbols. If **P** is an n-ary predicate symbol and t_1, ..., t_n are terms then **P**$(t_1,...,t_n)$ is a formula.

Equality. If the equality symbol is considered part of logic, and t_1 and t_2 are terms, then $t_1 = t_2$ is a formula.

Negation. If φ is a formula, then ¬φ is a formula.

Binary connectives. If φ and ψ are formulas, then (φ → ψ) is a formula. Similar rules apply to other binary logical connectives.

Quantifiers. If φ is a formula and x is a variable, then ∀x φ (for all x, holds) and ∃x φ (there exists x such that φ) are formulas.

Only expressions which can be obtained by finitely many applications of rules 1–5 are formulas.
The formulas obtained from the first two rules are said to be **atomic formulas**.

https://en.wikipedia.org/wiki/First-order_logic#Formation_rules

Atoms and Compound Formulas

a formula that contains no logical connectives
a formula that has no strict subformulas

Atoms :
the simplest well-formed formulas of the logic.

Compound formulas :
formed by combining the atomic formulas using the logical connectives.

https://en.wikipedia.org/wiki/Atomic_formula

Atomic Formula

for **propositional logic**
the atomic formulas are the propositional variables

for **predicate logic**
the atoms are predicate symbols together with their arguments,
each argument being a term.

In **model theory**
atomic formula are merely strings of symbols with a given signature
which may or may not be satisfiable with respect to a given model

https://en.wikipedia.org/wiki/Atomic_formula

Formulas and Sentences

An **formula**

- A **atomic formula**
- The operator $\neg$ followed by a **formula**
- Two formulas separated by $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$
- A quantifier following by a variable followed by a formula

A **sentence**

- A **formula** with no free variables

$\forall x$ love(x,y)	: free variable y	: not a sentence
$\forall x$ tall(x)	: no free variable	: a sentence

Finding the truth value

Find the truth values of **all sentences**

1. ¬, ∧, ∨, ⇒, ⇔

2. = symbol

3. ∀x p(x)

4. ∃x p(x)

5. the **operator precedence** is as follows ¬, =, ∧, ∨, ⇒, ⇔

6. the **quantifiers** (∀, ∃) have precedence over the **operators**

7. **parentheses** change the order of the precedence

Truth values of sentences

Propositional Logic

	A	B	
Interpretation I_1 ➡	T	T	
Interpretation I_2 ➡	T	F	
Interpretation I_3 ➡	F	T	
Interpretation I_4 ➡	F	F	

First Order Logic

Sentences

	P1( )	P2( )	...	S1	S2
Interpretation I_1 ➡	T	T			
Interpretation I_2 ➡	T	F			
Interpretation I_3 ➡	F	T			
Interpretation I_4 ➡	F	F			

1. ¬, ∧, ∨, ⇒, ⇔
2. = symbol
3. ∀x p(x)
4. ∃x p(x)
5. **operator precedence**
6. **quantifiers** (∀, ∃) high recedence
7. **parentheses** change the order

An **formula**

- A **atomic formula**
- The operator ¬ followed by a **formula**
- Two formulas separated by ∧, ∨, ⇒, ⇔
- A quantifier following by a variable followed by a formula

A **sentence**

- A **formula** with no free variables

Sentence Examples (1)

Signature

Constant Symbols = {Socrates, Plato, Zeus, Fido}
Predicate Symbols = {human, mortal, legs} all arity one

Model
D: the set of these four particular individuals

Interpretation

(a) a different individual is assigned to each of the constant symbols

(b) the truth value assignment

human(Socrates)=T, human(Plato)=T, human(Zeus)=F, human(Fido)=F
mortal(Socrates)=T, mortal(Plato)=T, mortal(Zeus)=F, mortal(Fido)=T
legs(Socrates)=T, legs(Plato)=T, legs(Zeus)=T, legs(Fido)=T

Sentence Examples (2)

Sentence 1: human(Zeus) ∧ human(Fido) ∨ human(Socrates) = T
F ∧ F ∨ T

Sentence 2: human(Zeus) ∧ (human(Fido) ∨ human(Socrates)) = F
F ∧ (F ∨ T)

Sentence 3: ∀x human(x) = F
human(Zeus)=F, human(Fido)=F

Sentence 4: ∀x mortal(x) = F
mortal(Zeus)=F

Sentence 5: ∀x legs(x) = T
legs(Socrates)=T, legs(Plato)=T, legs(Zeus)=T, legs(Fido)=T

Sentence 6: ∃x human(x) = T
human(Socrates)=T, human(Plato)=T

Sentence 7: ∀x (human(x) ⇒ mortal(x)) = T

Sentence Examples (3)

Sentence 7: $\forall x\ (human(x) \Rightarrow mortal(x)) = T$

human(Socrates)=T,	mortal(Socrates)=T,	$T \Rightarrow T$: T
human(Plato)=T,	mortal(Plato)=T,	$T \Rightarrow T$: T
human(Zeus)=F,	mortal(Zeus)=F,	$F \Rightarrow F$: T
human(Fido)=F	mortal(Fido)=T	$F \Rightarrow T$: T

References

[1] en.wikipedia.org
[2] en.wiktionary.org
[3] U. Endriss, “Lecture Notes : Introduction to Prolog Programming”
[4] http://www.learnprolognow.org/ Learn Prolog Now!
[5] http://www.csupomona.edu/~jrfisher/www/prolog_tutorial
[6] www.cse.unsw.edu.au/~billw/cs9414/notes/prolog/intro.html
[7] www.cse.unsw.edu.au/~billw/dictionaries/prolog/negation.html
[8] http://ilppp.cs.lth.se/, P. Nugues,` An Intro to Lang Processing with Perl and Prolog