Laurent Series and z-Transform - Geometric Series Time Shift A

20180913 Thr

Causal Signal $a(n) \Rightarrow f(z), \quad X(z)$

$(n \geq 0) \qquad (|z| < p) \quad (|z| > p^{-1})$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^1 + \left(\frac{1}{2}\right)^2 z^2 + \cdots = \frac{1}{1 - \left(\frac{z}{2}\right)}$$

$$X(z) = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \cdots = \frac{1}{1 - \left(\frac{1}{2z}\right)}$$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1 - \left(\frac{z}{2}\right)} \rightarrow \frac{2}{2 - z} \qquad (|z| < 2)$$

$$X(z) = \frac{1}{1 - \left(\frac{1}{2z}\right)} \rightarrow \frac{z}{z - 0.5} \qquad (|z| > 0.5)$$

Anti-Causal Signal $a(n) \Rightarrow -f(z), -X(z)$

$(n<0)$ $\quad (|z|>p) \quad (|z|<p^{-1})$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n<0)$$

$$f_2(z) = \left(\frac{1}{2}\right)^{-1} z^{-1} + \left(\frac{1}{2}\right)^{-2} z^{-2} + \left(\frac{1}{2}\right)^{-3} z^{-3} + \cdots = \frac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)}$$

$$X_2(z) = \left(\frac{1}{2}\right)^{-1} z^{1} + \left(\frac{1}{2}\right)^{-2} z^{2} + \left(\frac{1}{2}\right)^{-3} z^{3} + \cdots = \frac{(2z)}{1-(2z)}$$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n<0)$$

$$f_2(z) = \frac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} \rightarrow \frac{2}{z-2} = -f(z) \quad (|z|>2)$$

$$X_2(z) = \frac{(2z)}{1-(2z)} \rightarrow \frac{z}{0.5-z} = -X(z) \quad (|z|<0.5)$$

$$a'_n = -\left(\frac{1}{2}\right)^n \quad (n<0)$$

$$f(z) = \frac{2}{2-z} \rightarrow \frac{-\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} \quad (|z|>2)$$

$$X(z) = \frac{z}{z-0.5} \rightarrow \frac{-(2z)}{1-(2z)} \quad (|z|<0.5)$$

Inverse z

$z \leftarrow z^{-1}$, $ROC(z) \leftarrow ROC(z^{-1})$

Causal $\qquad z^{-1} \qquad$ anti-Causal

$$f(z) = \frac{2}{2-z} \quad (|z| < 2) \qquad \Longleftrightarrow \qquad f(z^{-1}) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5) \qquad \Longleftrightarrow \qquad X(z^{-1}) = \frac{2}{2-z} \quad (|z| < 2)$$

$$f(z^{-1}) = \frac{2}{2-z^{-1}} \quad (|z^{-1}| < 2) \quad \Longrightarrow \quad f(z^{-1}) = X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

$$X(z^{-1}) = \frac{z^{-1}}{z^{-1}-0.5} \quad (|z^{-1}| > 0.5) \quad \Longrightarrow \quad X(z^{-1}) = f(z) = \frac{2}{2-z} \quad (|z| < 2)$$

$f(z^{-1}) = X(z)$ Laurent Series (anti-causal signal)
with the same formula as causal X(z)

$X(z^{-1}) = f(z)$ z-Transform (anti-causal signal)
with the same formula as causal f(z)

Causal $\qquad\qquad$ anti-Causal

$$f(z) = \frac{2}{2-z} \quad (|z| < 2) \qquad\qquad X(z^{-1}) = \frac{2}{2-z} \quad (|z| < 2)$$

$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5) \qquad\qquad f(z^{-1}) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

Inverse z

$f(z^{-1})$, $ROC(z^{-1}) \Longrightarrow a_{-n}$

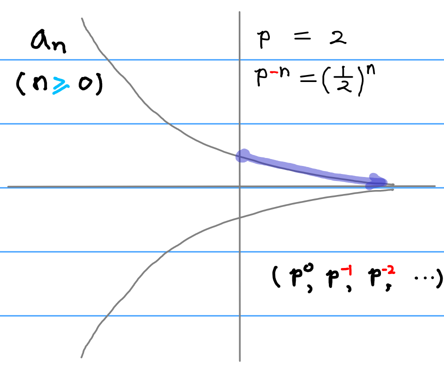
Causal	anti-Causal
$f(z) = \dfrac{2}{2-z}$ $(\lvert z\rvert < 2)$	$f(z^{-1}) = \dfrac{z}{z-0.5}$ $(\lvert z\rvert > 0.5)$
$X(z) = \dfrac{z}{z-0.5}$ $(\lvert z\rvert > 0.5)$	$X(z^{-1}) = \dfrac{2}{2-z}$ $(\lvert z\rvert < 2)$

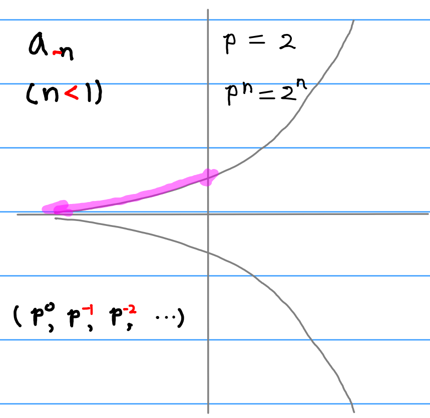
$$f(z) = \frac{2}{2-z} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \qquad n = 0, 1, 2, \cdots$$

$$f(z^{-1}) = \frac{2}{2-z^{-1}} = \frac{1}{1-\left(\frac{1}{2z}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \cdots \qquad a_{-n} = \left(\tfrac{1}{2}\right)^{-n}$$

$$= p^0 z^0 + p^{1} z^{-1} + p^{2} z^{-2} + \cdots \qquad n = 0, -1, -2, \cdots$$





Inverse z $X(z^{-1}),\ ROC(z^{-1}) \Longrightarrow a_{-n}$

Causal	anti-Causal
$f(z) = \dfrac{2}{2-z} \quad (\lvert z\rvert < 2)$	$f(z^{-1}) = \dfrac{z}{z-0.5} \quad (\lvert z\rvert > 0.5)$
$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z\rvert > 0.5)$	$X(z^{-1}) = \dfrac{2}{2-z} \quad (\lvert z\rvert < 2)$

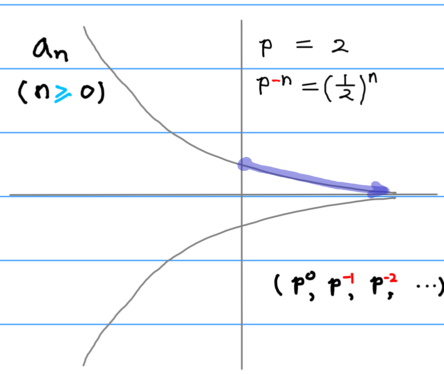
$$X(z) = \frac{z}{z-0.5} = \frac{1}{1-\left(\frac{1}{2z}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

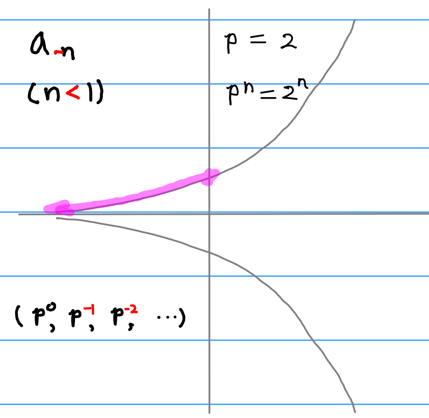
$$= p^0 z^0 + p^{-1} z^{-1} + p^{-2} z^{-2} + \cdots \qquad n = 0, 1, 2, \cdots$$

$$X(z^{-1}) = \frac{z^{-1}}{z^{-1}-0.5} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \qquad a_{-n} = \left(\tfrac{1}{2}\right)^{-n}$$

$$= p^0 z^0 + p^1 z^1 + p^2 z^2 + \cdots \qquad n = 0, -1, -2, \cdots$$

z





Inverse ROC

$ROC(z) \leftarrow ROC(z^{-1})$

Causal $\quad\quad\quad$ anti-Causal

$f(z) = \dfrac{2}{2-z} \quad (|z| < 2) \quad \Rightarrow \quad -f(z) = -\dfrac{2}{2-z} \quad (|z| > 2)$

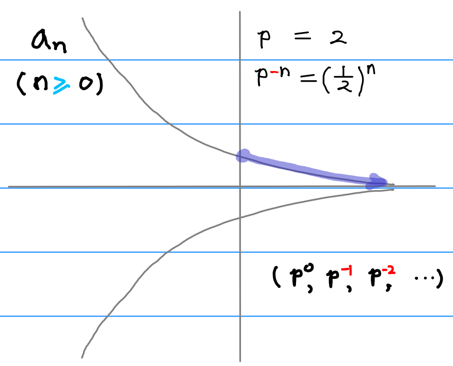
$X(z) = \dfrac{z}{z-0.5} \quad (|z| > 0.5) \quad \Rightarrow \quad -X(z) = -\dfrac{z}{z-0.5} \quad (|z| < 0.5)$

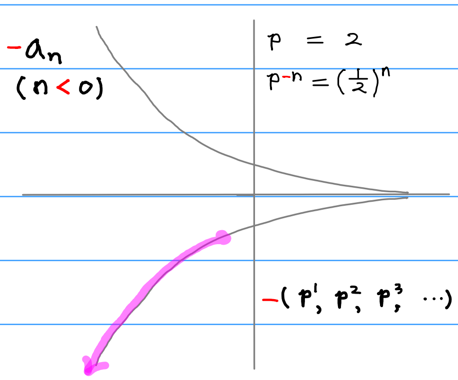
$f(z) = \dfrac{2}{2-z} = \dfrac{1}{1-\left(\frac{z}{2}\right)} = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^1 + \left(\frac{1}{2}\right)^2 z^2 + \cdots \quad a_n = \left(\frac{1}{2}\right)^n$

$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \quad n = 0, 1, 2, \cdots$

$-f(z) = \dfrac{2}{z-2} = \dfrac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} = \left(\frac{1}{2}\right)^{-1} z^{-1} + \left(\frac{1}{2}\right)^{-2} z^{-2} + \left(\frac{1}{2}\right)^{-3} z^{-3} + \cdots \quad a_n = -\left(\frac{1}{2}\right)^n$

$= p^1 z^{-1} + p^2 z^{-2} + p^3 z^{-3} + \cdots \quad n = -1, -2, -3, \cdots$





Inverse ROC $\quad f(z),\ ROC(z^{-1}) \Rightarrow -a_n$

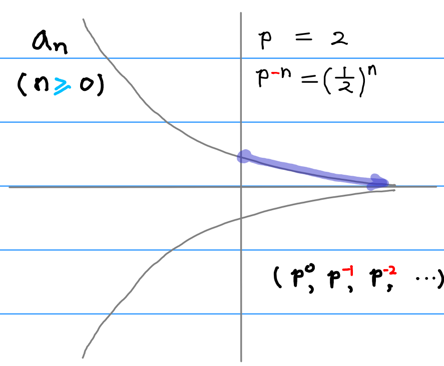
Causal	anti-Causal
$f(z) = \dfrac{2}{2-z} \quad (\lvert z \rvert < 2)$	$f(z) = \dfrac{2}{2-z} \quad (\lvert z \rvert > 0.5)$
$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z \rvert > 0.5)$	$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z \rvert < 2)$

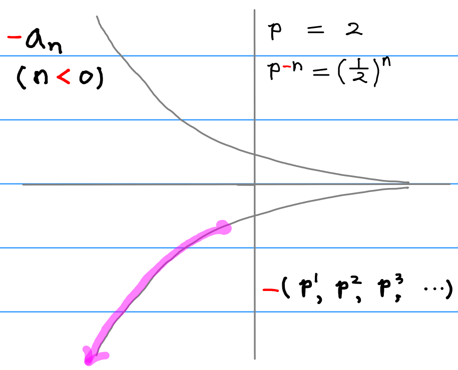
$$f(z) = \frac{2}{2-z} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \qquad \therefore a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \qquad n = 0, 1, 2, \cdots$$

$$f(z) = \frac{2}{2-z} = \frac{-\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} = -\left[(2)^1 z^{-1} + (2)^2 z^{-2} + \left(\tfrac{1}{2}\right)^3 z^{-3} + \cdots\right] \qquad \therefore -a_n = -\left(\tfrac{1}{2}\right)^n$$

$$= -\left[p^1 z^{-1} + p^2 z^{-2} + p^3 z^{-3} + \cdots\right] \qquad n = -1, -2, -3, \cdots$$





Inverse ROC $X(z)$, ROC(z^{-1}) $\Rightarrow$ $-a_n$

Causal **anti-Causal**

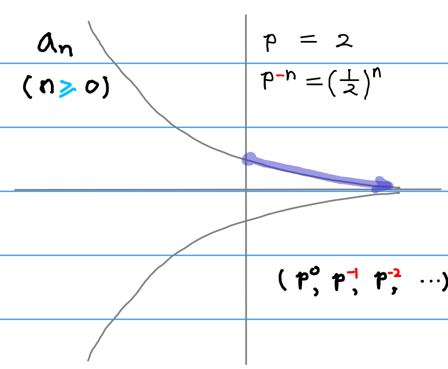
Causal	anti-Causal
$f(z) = \dfrac{2}{2-z} \quad (\lvert z \rvert < 2)$	$f(z) = \dfrac{2}{2-z} \quad (\lvert z \rvert > 0.5)$
$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z \rvert > 0.5)$	$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z \rvert < 2)$

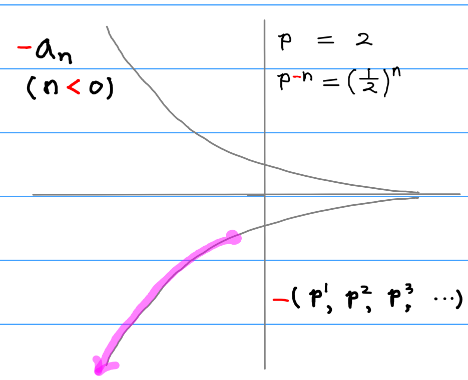
$$X(z) = \frac{z}{z-0.5} = \frac{1}{1-\left(\frac{1}{2z}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^{-1} + p^{-2} z^{-2} + \cdots \qquad n = 0, 1, 2, \cdots$$

$$X(z) = \frac{z}{z-0.5} = \frac{-(2z)}{1-(2z)} = -\left[(2)^1 z^1 + (2)^2 z^2 + \left(\tfrac{1}{2}\right)^3 z^3 + \cdots\right] \qquad -a_n = -\left(\tfrac{1}{2}\right)^n$$

$$= -\left[p^1 z^1 + p^2 z^2 + p^3 z^3 + \cdots\right] \qquad n = -1, -2, -3, \cdots$$





p=2

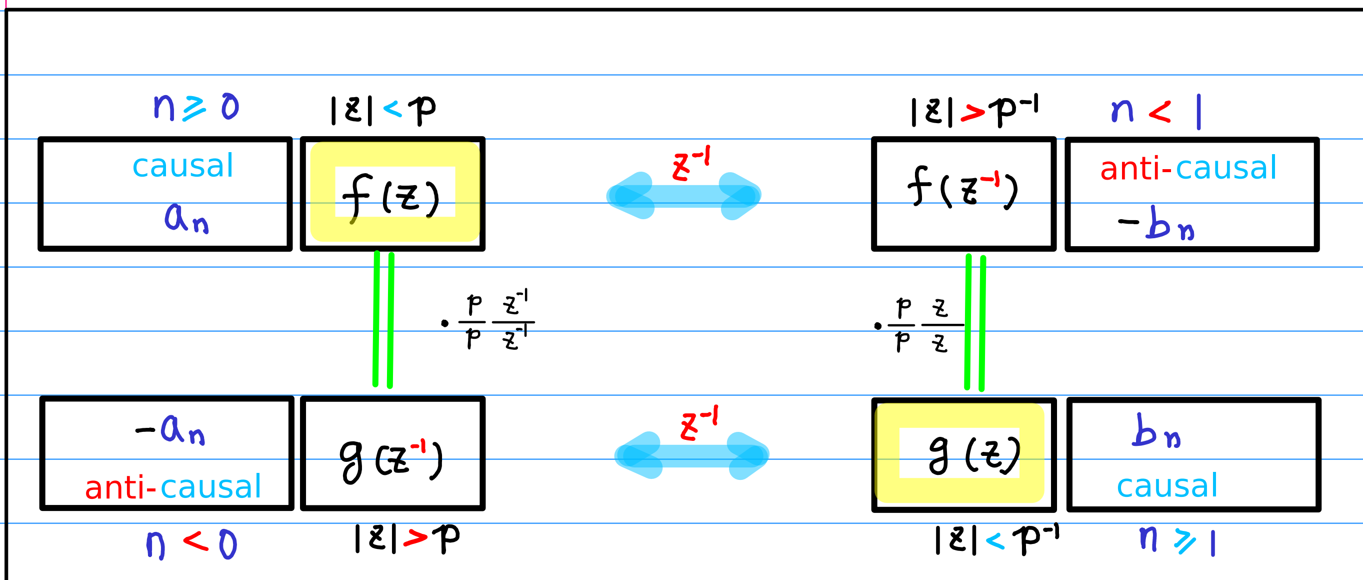
	causal	anti-causal
	$a_n = (\frac{1}{2})^n \quad (n \geq 0)$	$a_n = (\frac{1}{2})^n \quad (n < 0)$
	$f(z) = \frac{1}{1-(\frac{z}{2})} \quad (\lvert z \rvert < 2)$ $X(z) = \frac{1}{1-(\frac{1}{2z})} \quad (\lvert z \rvert > 0.5)$	$f(z) = \frac{(\frac{2}{z})}{1-(\frac{2}{z})} \quad (\lvert z \rvert > 2)$ $X(z) = \frac{(2z)}{1-(2z)} \quad (\lvert z \rvert < 0.5)$
	$f(z) = \frac{2}{2-z} \quad (\lvert z \rvert < 2)$ $X(z) = \frac{z}{z-0.5} \quad (\lvert z \rvert > 0.5)$	$f(z) = \frac{2}{z-2} \quad (\lvert z \rvert > 2)$ $X(z) = \frac{z}{0.5-z} \quad (\lvert z \rvert < 0.5)$

p=1/2

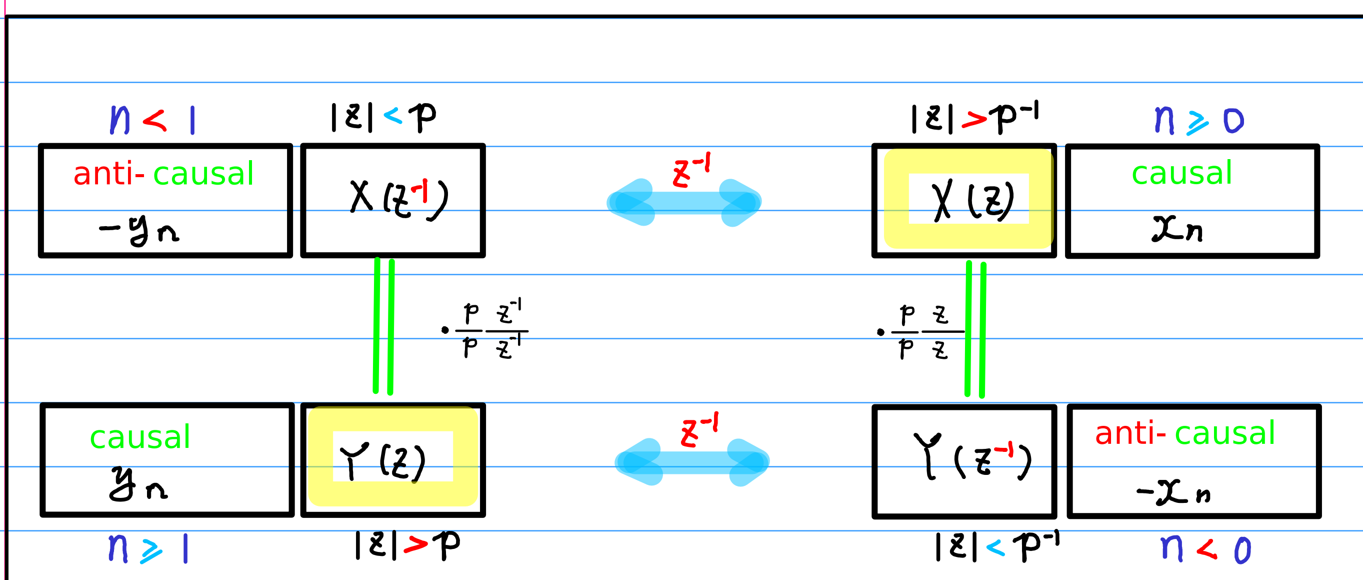
	causal	anti-causal
	$a_n = (2)^n \quad (n \geq 0)$	$a_n = (2)^n \quad (n < 0)$
	$f(z) = \frac{1}{1-(2z)} \quad (\lvert z \rvert < 0.5)$ $X(z) = \frac{1}{1-(\frac{2}{z})} \quad (\lvert z \rvert > 2)$	$f(z) = \frac{(\frac{1}{2z})}{1-(\frac{1}{2z})} \quad (\lvert z \rvert > 0.5)$ $X(z) = \frac{(\frac{z}{2})}{1-(\frac{z}{2})} \quad (\lvert z \rvert < 2)$
	$f(z) = \frac{0.5}{0.5-z} \quad (\lvert z \rvert < 0.5)$ $X(z) = \frac{z}{z-2} \quad (\lvert z \rvert > 2)$	$f(z) = \frac{0.5}{z-0.5} \quad (\lvert z \rvert > 0.5)$ $X(z) = \frac{z}{2-z} \quad (\lvert z \rvert < 2)$

$f(z)$, $g(z^{-1})$	$f(z^{-1})$, $g(z)$	$X(z^{-1})$, $Y(z)$	$X(z)$, $Y(z^{-1})$

Laurent Series $a_n \leftrightarrow f(z)$ $b_n \leftrightarrow g(z)$



Z-Transform $X(z) \leftrightarrow x_n$ $Y(z) \leftrightarrow y_n$



$f(z)$ $g(z^{-1})$	$f(z^{-1})$ $g(z)$	$X(z^{-1})$ $Y(z)$	$X(z)$ $Y(z^{-1})$

2

Laurent Series $\quad a_n \leftrightarrow f(z) \quad -a_{-n} = b_n \leftrightarrow g(z)$

causal a_n	$\xleftrightarrow{z^{-1}}$	anti-causal a_{-n}
$-a_n$ anti-causal	$\xleftrightarrow{z^{-1}}$	$-a_{-n}$ causal

Z- Transform $\quad X(z) \leftrightarrow x_n \quad Y(z) \leftrightarrow y_n = -x_{-n}$

anti- causal x_{-n}	$\xleftrightarrow{z^{-1}}$	causal x_n
causal $-x_{-n}$	$\xleftrightarrow{z^{-1}}$	anti- causal $-x_n$

f(z), X(z) $= \dfrac{2}{2-z}, \dfrac{z}{z-0.5}$

f(z) Laurent Series

$n \in [0, \infty)$

causal

$a_n = 0.5^n$

$|z| < 2$

$\dfrac{2}{2-z}$

z^{-1}

$|z| > 0.5$

$\dfrac{z}{z-0.5}$

$n \in (-\infty, 0]$

anti-causal

$a_{-n} = 0.5^{-n}$

$\cdot \dfrac{p}{p} \dfrac{z^{-1}}{z^{-1}}$

$\cdot \dfrac{p}{p} \dfrac{z}{z}$

$-a_n = -0.5^n$

anti-causal

$n \in (-\infty, -1]$

$\dfrac{2}{2-z}$

$|z| > 2$

z^{-1}

$\dfrac{z}{z-0.5}$

$|z| < 0.5$

$-a_{-n} = -0.5^{-n}$

causal

$n \in [1, \infty)$

X(z) z-Transform

$n \in (-\infty, 0]$

anti-causal X(z)

$a_{-n} = 0.5^{-n}$

$|z| < p$

$\dfrac{2}{2-z}$

z^{-1}

$|z| > p^{-1}$

$\dfrac{z}{z-0.5}$

$n \in [0, \infty)$

causal X(z)

$a_n = 0.5^n$

$\cdot \dfrac{p}{p} \dfrac{z^{-1}}{z^{-1}}$

$\cdot \dfrac{p}{p} \dfrac{z}{z}$

$-a_{-n} = -0.5^{-n}$

causal X(z)

$n \in [1, \infty)$

$\dfrac{2}{2-z}$

$|z| > p$

z^{-1}

$\dfrac{z}{z-0.5}$

$|z| < p^{-1}$

$-a_n = -0.5^n$

anti-causal X(z)

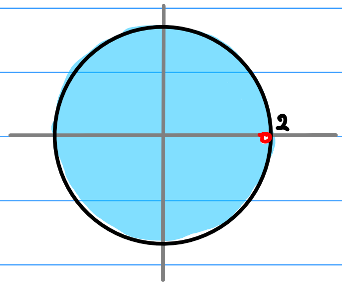
$n \in (-\infty, -1]$

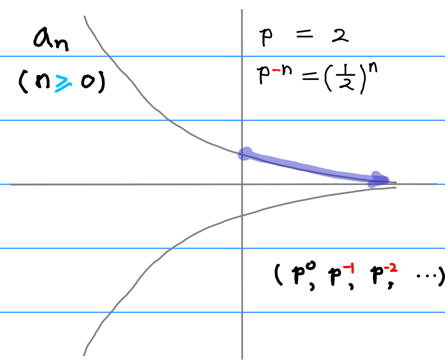
$$f(z) = \frac{z}{2-z}, \quad \frac{z}{z-0.5}$$

0.5^{n}	0.5^{-n}
-0.5^{n}	-0.5^{-n}

$n \in [0, \infty)$ $\quad |z| < 2$

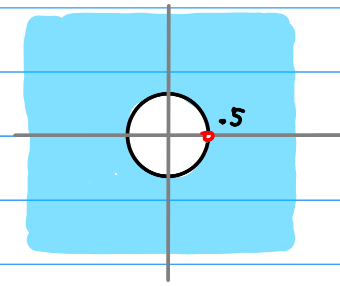
causal $\quad a_n = 0.5^{n}$ $\quad \boxed{\frac{z}{2-z}}$

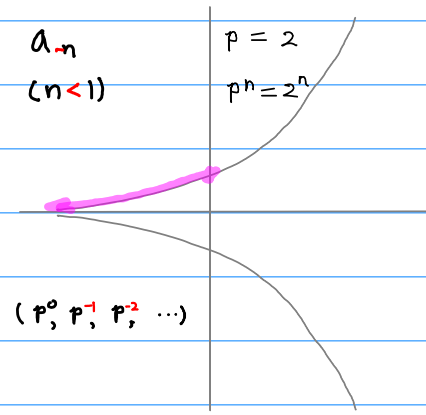




$|z| > 0.5$ $\quad n \in (-\infty, 0]$

$\boxed{\frac{z}{z-0.5}}$ $\quad$ anti-causal $\quad a_{-n} = 0.5^{-n}$

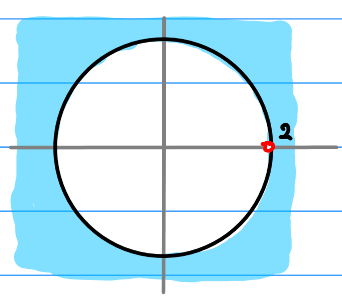


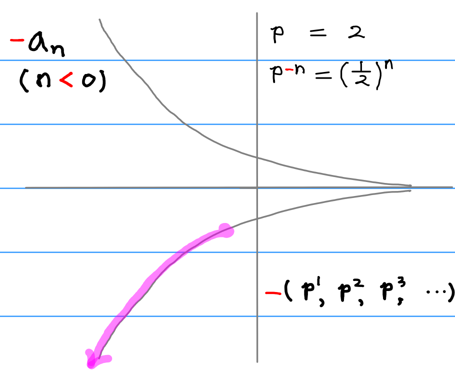


$-a_n = -0.5^{n}$ $\quad \boxed{\frac{z}{2-z}}$

anti-causal

$n \in (-\infty, -1]$ $\quad |z| > 2$

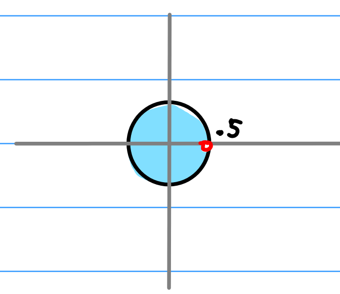


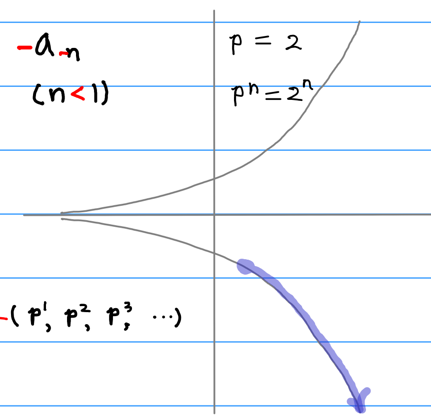


$\boxed{\frac{z}{z-0.5}}$ $\quad -a_{-n} = -0.5^{-n}$

causal

$|z| < 0.5$ $\quad n \in [1, \infty)$





$$X(z) = \frac{2}{2-z}, \quad \frac{z}{z-0.5}$$

0.5^{-n}	0.5^{n}
-0.5^{-n}	-0.5^{n}

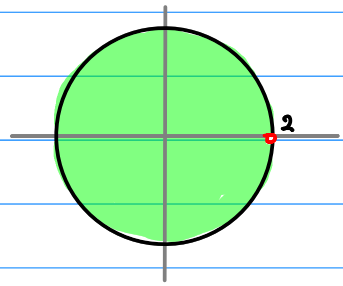
$n < 1$ $\quad |z| < 2$

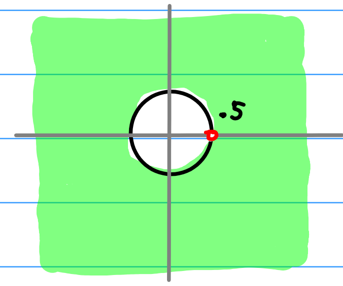
x_{-n} anti-causal $x_{-n} = 0.5^{-n}$ $\quad -\frac{0.5}{1-0.5z}$

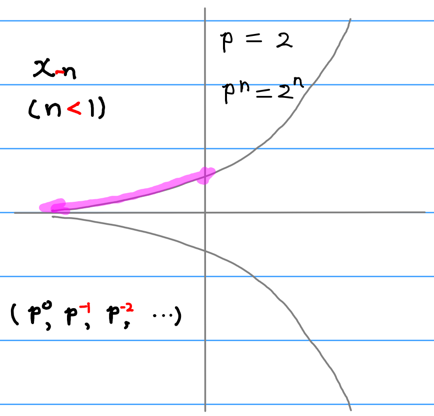
$\xleftrightarrow{z^{-1}}$

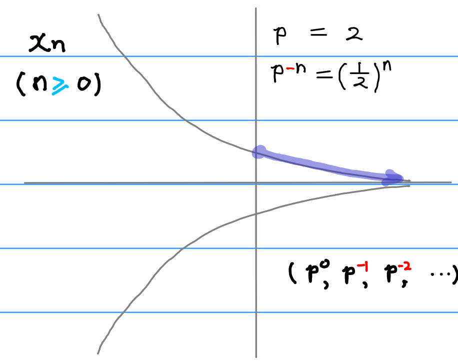
$|z| > 0.5$ $\quad n \geq 0$

$-\frac{0.5}{1-0.5z^{-1}}$ causal $x_n = 0.5^{n}$ $\quad x_n$









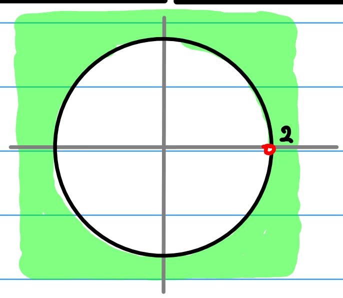
$n \geq 1$ $\quad |z| > 2$

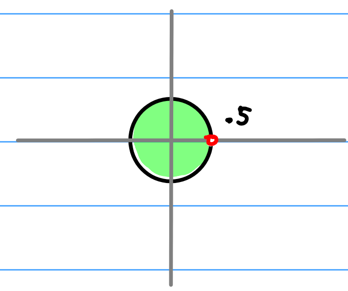
y_n $\quad -x_{-n} = -0.5^{-n}$ causal $\quad \frac{z^{-1}}{1-2z^{-1}}$

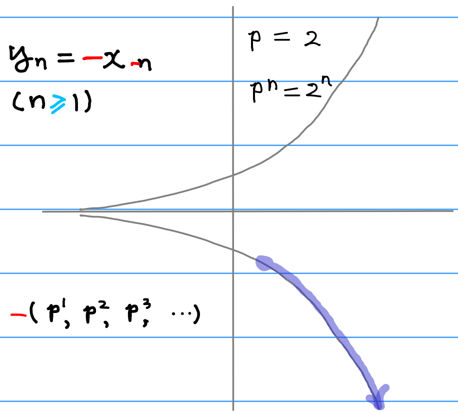
$\xleftrightarrow{z^{-1}}$

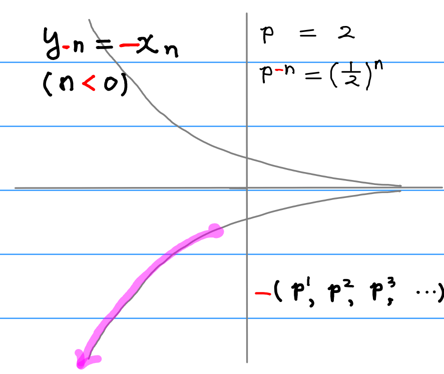
$|z| < 0.5$ $\quad n < 0$

$\frac{z}{1-2z}$ $\quad -x_n = -0.5^{n}$ anti-causal $\quad y_{-n}$









$n \in [0, \infty)$ $\quad$ $n \in (-\infty, 0]$

$\boxed{0.5^{n}}$ $\quad$ $\boxed{0.5^{-n}}$

$\boxed{-0.5^{n}}$ $\quad$ $\boxed{-0.5^{-n}}$

$n \in (-\infty, -1]$ $\quad$ $n \in [1, \infty)$

$$\frac{1}{1 - 0.5z} \qquad \frac{1}{1 - 2z^{-1}}$$

$$\frac{-0.5^{-1}z^{-1}}{1 - 0.5^{-1}z^{-1}} \qquad \frac{-2z}{1 - 2z}$$

$$\frac{2}{2 - z} \qquad \frac{0.5}{0.5 - z^{-1}}$$

$$\frac{z^{-1}}{z^{-1} - 2} \qquad \frac{z}{z - 0.5}$$

2 formulas (A)

$$\boxed{\frac{1}{z-p}} \qquad \boxed{\frac{1}{z^{-1}-p}}$$

2 representations each

$$\begin{tikzcd}[column sep=large]
 & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z}} \\
\boxed{\dfrac{1}{z-p}} \arrow[ur, dash] \arrow[dr, dash] & \\
 & \boxed{\dfrac{z^{-1}}{1-pz^{-1}}}
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=large]
 & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}} \\
\boxed{\dfrac{1}{z^{-1}-p}} \arrow[ur, dash] \arrow[dr, dash] & \\
 & \boxed{\dfrac{z}{1-pz}}
\end{tikzcd}$$

2 formulas (B)

$$\boxed{\frac{p}{z-p}} \qquad \boxed{\frac{p}{z^{-1}-p}}$$

2 representations each

$$\begin{tikzcd}[column sep=large]
 & \boxed{-\dfrac{1}{1-p^{-1}z}} \\
\boxed{\dfrac{p}{z-p}} \arrow[ur, dash] \arrow[dr, dash] & \\
 & \boxed{\dfrac{pz^{-1}}{1-pz^{-1}}}
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=large]
 & \boxed{-\dfrac{1}{1-p^{-1}z^{-1}}} \\
\boxed{\dfrac{p}{z^{-1}-p}} \arrow[ur, dash] \arrow[dr, dash] & \\
 & \boxed{\dfrac{pz}{1-pz}}
\end{tikzcd}$$

$$\xrightarrow{\;p\;}$$
$$\xleftarrow[\;1/p\;]{}$$

Ⓐ $f(z) \longrightarrow a_n$

2 formulas Ⓐ

$$\boxed{\frac{1}{z-p}} \qquad \boxed{\frac{1}{z^{-1}-p}}$$

2 representations each

- $\boxed{\dfrac{1}{z-p}}$
  - $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z}} \triangleq f(z)$ causal $\quad -(p^{-1})^{n+1} \quad (n \geq 0)$
  - $\|$
  - $\boxed{\dfrac{z^{-1}}{1-pz^{-1}}} \triangleq g(z^{-1})$ anti- causal $\quad (p^{-1})^{n+1} \quad (n < 0)$

- $\boxed{\dfrac{1}{z^{-1}-p}}$
  - $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}} \triangleq f(z^{-1})$ anti- causal $\quad -(p^{-1})^{-n+1} \quad (n < 1)$
  - $\|$
  - $\boxed{\dfrac{z}{1-pz}} \triangleq g(z)$ causal $\quad (p^{-1})^{-n+1} \quad (n \geq 1)$

- $\boxed{-\dfrac{1}{2-z}}$
  - $\boxed{-\dfrac{0.5}{1-0.5z}}$ causal $\quad -0.5^{n+1} \quad (n \geq 0)$
  - $\boxed{\dfrac{z^{-1}}{1-2z^{-1}}}$ anti $\quad 0.5^{n+1} \quad (n < 0)$

- $\boxed{-\dfrac{0.5z}{z-0.5}}$
  - $\boxed{-\dfrac{0.5}{1-0.5z^{-1}}}$ anti $\quad -0.5^{-n+1} \quad (n < 1)$
  - $\boxed{\dfrac{z}{1-2z}}$ causal $\quad 0.5^{-n+1} \quad (n \geq 1)$

Ⓐ $f(z) \longrightarrow a_n$ method 1

p^{-1} $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z}}$ $-(p^{-1}) \Rightarrow -(p^{-1})^{n+1} \Rightarrow -(p^{-1})^{n+1}$

$n = 0, 1, 2, \cdots$

$-(p^{-1}z^{0} + p^{-2}z^{1} + p^{-3}z^{2} + \cdots)$

p $\boxed{\dfrac{z^{-1}}{1-pz^{-1}}}$ $(p) \Rightarrow (p)^{-n-1} \Rightarrow (p^{-1})^{n+1}$

$n = -1, -2, -3, \cdots$

$(p^{0}z^{-1} + p^{1}z^{-2} + p^{2}z^{-3} + \cdots)$

p^{-1} $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}$ $-(p^{-1}) \Rightarrow -(p^{-1})^{-n+1} \Rightarrow -(p^{-1})^{-n+1}$

$n = 0, -1, -2, \cdots$

$-(p^{-1}z^{0} + p^{-2}z^{-1} + p^{-3}z^{-2} + \cdots)$

p $\boxed{\dfrac{z}{1-pz}}$ $(p) \Rightarrow (p)^{n-1} \Rightarrow (p^{-1})^{-n+1}$

$n = 1, 2, 3, \cdots$

$(p^{0}z^{1} + p^{1}z^{2} + p^{2}z^{3} + \cdots)$

Ⓐ $f(z) \longrightarrow a_n$ method 2

p^{-1} $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z}}$ $\dfrac{-(p^{-1})}{z} \rightarrow n=0,1,2,\cdots \rightarrow \dfrac{-(p^{-1})^{n+1}}{(p)^{-n-1}}$

p $\boxed{\dfrac{z^{-1}}{1-pz^{-1}}}$ $\dfrac{(p)}{z^{-1}/z^{-1}} \rightarrow n=-1,-2,-3,\cdots \rightarrow \dfrac{(p^{-1})^{n+1}}{(p)^{-n-1}}$

p $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}$ $\dfrac{-(p^{-1})}{z^{-1}} \rightarrow n=0,-1,-2,\cdots \rightarrow \dfrac{-(p^{-1})^{-n+1}}{(p)^{n-1}}$

p^{-1} $\boxed{\dfrac{z}{1-pz}}$ $\dfrac{(p)}{z/z} \rightarrow n=1,2,3,\cdots \rightarrow \dfrac{(p^{-1})^{-n+1}}{(p)^{n-1}}$

(B) $f(z) \longrightarrow a_n$

2 formulas

$$\boxed{\frac{p}{z-p}} \qquad \boxed{\frac{p}{z^{-1}-p}}$$

2 representations each

$\boxed{\dfrac{p}{z-p}}$

- $\boxed{-\dfrac{1}{1-p^{-1}z}} \triangleq f(z)$ causal $\quad -(p^{-1})^n \quad (n \geq 0)$
- $\|$
- $\boxed{\dfrac{pz^{-1}}{1-pz^{-1}}} \triangleq g'(z^{-1})$ anti- causal $\quad (p^{-1})^n \quad (n<0)$

$\boxed{\dfrac{p}{z^{-1}-p}}$

- $\boxed{-\dfrac{1}{1-p^{-1}z^{-1}}} \triangleq f'(z^{-1})$ anti- causal $\quad -(p^{-1})^{-n} \quad (n<1)$
- $\|$
- $\boxed{\dfrac{pz}{1-pz}} \triangleq g'(z)$ causal $\quad (p^{-1})^{-n} \quad (n \geq 1)$

$\boxed{\dfrac{2}{2-z}}$

- $\boxed{\dfrac{1}{1-0.5z}}$ causal $\quad 0.5^n \ (n \geq 0)$
- $\boxed{\dfrac{-2z^{-1}}{1-2z^{-1}}}$ anti $\quad -0.5^n \ (n<0)$

$\boxed{\dfrac{z}{z-0.5}}$

- $\boxed{\dfrac{1}{1-0.5z^{-1}}}$ anti $\quad 0.5^{-n} \ (n<1)$
- $\boxed{\dfrac{-2z}{1-2z}}$ causal $\quad -0.5^{-n} \ (n \geq 1)$

(B) $f(z) \longrightarrow a_n$ method 1

p^{-1} $\boxed{-\frac{1}{1-p^{-1}z}}$ $-(p^{-1}) \Rightarrow -(p^{-1})^n \Rightarrow -(p^{-1})^n$

$n = 0, 1, 2, \cdots$

$-(p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots)$

p $\boxed{\frac{pz^{-1}}{1-pz^{-1}}}$ $(p) \Rightarrow (p)^{-n} \Rightarrow (p^{-1})^n$

$n = -1, -2, -3, \cdots$

$(p^1 z^{-1} + p^2 z^{-2} + p^3 z^{-3} + \cdots)$

p $\boxed{-\frac{1}{1-p^{-1}z^{-1}}}$ $-(p^{-1}) \Rightarrow -(p^{-1})^{-n} \Rightarrow -(p^{-1})^{-n}$

$n = 0, -1, -2, \cdots$

$-(p^0 z^0 + p^{-1} z^{-1} + p^{-2} z^{-2} + \cdots)$

p^{-1} $\boxed{\frac{pz}{1-pz}}$ $(p) \Rightarrow (p)^n \Rightarrow (p^{-1})^{-n}$

$n = 1, 2, 3, \cdots$

$(p^1 z^1 + p^2 z^2 + p^3 z^3 + \cdots)$

Ⓑ $f(z) \longrightarrow a_n$ method 2

p^{-1} $\boxed{-\dfrac{1}{1-p^{-1}z}}$ $\begin{array}{l} -(p^{-1}) \\ z \end{array} \rightarrow n = 0, 1, 2, \cdots \rightarrow \begin{array}{l} -p^{-n} \\ (p)^{-n} \end{array}$

p $\boxed{\dfrac{pz^{-1}}{1-pz^{-1}}}$ $\begin{array}{l} (p) \\ z^{-1}/z^{-1} \end{array} \rightarrow n = -1, -2, -3, \cdots \rightarrow \begin{array}{l} p^{-n} \\ (p)^{-n} \end{array}$

p $\boxed{-\dfrac{1}{1-p^{-1}z^{-1}}}$ $\begin{array}{l} -(p^{-1}) \\ z^{-1} \end{array} \rightarrow n = 0, -1, -2, \cdots \rightarrow \begin{array}{l} -(p^{-1})^{-n} \\ (p)^{n} \end{array}$

p^{-1} $\boxed{\dfrac{pz}{1-pz}}$ $\begin{array}{l} (p) \\ z/z \end{array} \rightarrow n = 1, 2, 3, \cdots \rightarrow \begin{array}{l} (p^{-1})^{-n} \\ (p)^{n} \end{array}$

2 formulas

$$\boxed{\frac{p}{z-p}} \qquad \boxed{\frac{p}{z^{-1}-p}}$$

2 representations each

$$\boxed{\frac{p}{z-p}} \begin{cases} \boxed{-\dfrac{1}{1-p^{-1}z}} \triangleq \overset{\text{causal}}{f'(z)} = \overset{\text{anti- causal}}{X'(z^{-1})} \\ \qquad\qquad\qquad\quad \| \qquad\qquad\quad \| \\ \boxed{\dfrac{pz^{-1}}{1-pz^{-1}}} \triangleq \underset{\text{causal}}{Y'(z)} = \underset{\text{anti- causal}}{g'(z^{-1})} \end{cases}$$

$$\boxed{\frac{p}{z^{-1}-p}} \begin{cases} \boxed{-\dfrac{1}{1-p^{-1}z^{-1}}} \triangleq \overset{\text{causal}}{X'(z)} = \overset{\text{anti- causal}}{f'(z^{-1})} \\ \qquad\qquad\qquad\quad \| \qquad\qquad\quad \| \\ \boxed{\dfrac{pz}{1-pz}} \triangleq \underset{\text{causal}}{g'(z)} = \underset{\text{anti- causal}}{Y'(z^{-1})} \end{cases}$$

$$
\begin{tikzcd}[column sep=small]
\begin{array}{c} n \in [0,\infty) \\ \text{causal} \\ a_n = 0.5^{n} \end{array} & \begin{array}{c} |z| < 2 \\ \dfrac{2}{2-z} \end{array} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, dash, "\cdot \frac{p}{p}\frac{z^{-1}}{z^{-1}}"] & \begin{array}{c} |z| > 0.5 \\ \dfrac{z}{z-0.5} \end{array} \arrow[d, dash, "\cdot \frac{p}{p}\frac{z}{z}"'] & \begin{array}{c} n \in (-\infty, 0] \\ \text{anti-causal} \\ a_{-n} = 0.5^{-n} \end{array} \\
\begin{array}{c} -a_n = -0.5^{n} \\ \text{anti-causal} \\ n \in (-\infty,-1] \end{array} & \begin{array}{c} \dfrac{2}{2-z} \\ |z| > 2 \end{array} \arrow[r, leftrightarrow, "z^{-1}"] & \begin{array}{c} \dfrac{z}{z-0.5} \\ |z| < 0.5 \end{array} & \begin{array}{c} -a_{-n} = -0.5^{-n} \\ \text{causal} \\ n \in [1,\infty) \end{array}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\dfrac{2}{2-z} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, "-\frac{1}{2}"] & \dfrac{z}{z-0.5} \arrow[d, "-\frac{1}{2}"] \\
-\dfrac{1}{2-z} \arrow[r, leftrightarrow, "z^{-1}"] & -\dfrac{0.5z}{z-0.5}
\end{tikzcd}
$$

$$0.5^{-n+1} = \left(\frac{1}{2}\right)^{-n+1}$$

$$= 2^{+n-1}$$

$\frac{2}{2-z}$ $\quad$ $\frac{0.5}{0.5-z^{-1}}$

$\frac{z^{-1}}{z^{-1}-2}$ $\quad$ $\frac{z}{z-0.5}$

$\frac{2}{2-z}$

$\frac{1}{1-0.5z}$ causal 0.5^{n} $(n \geq 0)$

$\frac{-2z^{-1}}{1-2z^{-1}}$ anti -0.5^{n} $(n<0)$

$\frac{z}{z-0.5}$

$\frac{1}{1-0.5z^{-1}}$ anti 0.5^{-n} $(n<1)$

$\frac{-2z}{1-2z}$ causal -0.5^{-n} $(n \geq 1)$

$-\frac{1}{2-z}$

$\frac{-0.5}{1-0.5z}$ causal -0.5^{n+1} $(n \geq 0)$

$\frac{z^{-1}}{1-2z^{-1}}$ anti 0.5^{n+1} $(n<0)$

$-\frac{0.5z}{z-0.5}$

$\frac{-0.5}{1-0.5z^{-1}}$ anti -0.5^{-n+1} $(n<1)$

$\frac{z}{1-2z}$ causal 0.5^{-n+1} $(n \geq 1)$

$\frac{2}{2-z}, \frac{z}{z-0.5}$ vs. $-\frac{0.5}{1-0.5z}, \frac{z}{1-2z}$

$n \in [0,\infty)$	$\lvert z\rvert < 2$		$\lvert z\rvert > 0.5$	$n \in (-\infty, 0]$
causal $a_n = 0.5^n$	$\boxed{\frac{2}{2-z}}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{z-0.5}$	anti-causal $a_{-n} = 0.5^{-n}$
	$\cdot \frac{p}{p} \frac{z^{-1}}{z^{-1}}$		$\cdot \frac{p}{p} \frac{z}{z}$	
$-a_n = -0.5^n$ anti-causal	$\frac{2}{2-z}$	$\xleftrightarrow{z^{-1}}$	$\boxed{\frac{z}{z-0.5}}$	$-a_{-n} = -0.5^{-n}$ causal
$n \in (-\infty, -1]$	$\lvert z\rvert > 2$		$\lvert z\rvert < 0.5$	$n \in [1, \infty)$

	$n \in [0,\infty)$	$\lvert z\rvert < 2$		$\lvert z\rvert > 0.5$	$n \in (-\infty, 0]$
a_n	causal -0.5^{n+1}	$\boxed{-\frac{0.5}{1-0.5z}}$	$\xleftrightarrow{z^{-1}}$	$-\frac{0.5}{1-0.5z^{-1}}$	anti-causal -0.5^{-n+1}
					-2^{+n-1}
					2^{+n-1}
b_n	0.5^{n+1} anti-causal	$\frac{z^{-1}}{1-2z^{-1}}$	$\xleftrightarrow{z^{-1}}$	$\boxed{\frac{z}{1-2z}}$	0.5^{-n+1} causal
	$n \in (-\infty, -1]$	$\lvert z\rvert > 2$		$\lvert z\rvert < 0.5$	$n \in [1, \infty)$

$\frac{2}{2-z}$

$\frac{1}{1-0.5z}$ causal 0.5^{n} $(n \geq 0)$

$\frac{-2z^{-1}}{1-2z^{-1}}$ anti -2^{-n} $(n<0)$

$\frac{z}{z-0.5}$

$\frac{1}{1-0.5z^{-1}}$ anti 0.5^{-n} $(n<1)$

$\frac{-2z}{1-2z}$ causal -0.5^{-n} $(n \geq 1)$

$n \in [0, \infty)$ $|z| < 2$ $|z| > 0.5$ $n \in (-\infty, 0]$

causal $a_n = 0.5^{n}$	$\frac{2}{2-z}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{z-0.5}$	anti-causal $a_{-n} = 0.5^{-n}$
	$\cdot \frac{p}{p} \frac{z^{-1}}{z^{-1}}$		$\cdot \frac{p}{p} \frac{z}{z}$	
$-a_n = -0.5^{n}$ anti-causal	$\frac{2}{2-z}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{z-0.5}$	$-a_{-n} = -0.5^{-n}$ causal

$n \in (-\infty, -1]$ $|z| > 2$ $|z| < 0.5$ $n \in [1, \infty)$

$$-\frac{1}{2-z} \;\rightarrow\; \begin{cases} \dfrac{-0.5}{1-0.5z} & \text{causal: } -0.5^{n+1}\ (n \geq 0) \\[2ex] \dfrac{z^{-1}}{1-2z^{-1}} & \text{anti: } 0.5^{n+1}\ (n < 0) \end{cases}$$

$$-\frac{0.5z}{z-0.5} \;\rightarrow\; \begin{cases} \dfrac{-0.5}{1-0.5z^{-1}} & \text{anti: } -0.5^{-n+1}\ (n < 1) \\[2ex] \dfrac{z}{1-2z} & \text{causal: } 0.5^{-n+1}\ (n \geq 1) \end{cases}$$

	$n \in [0, \infty)$	$\lvert z\rvert < 2$		$\lvert z\rvert > 0.5$	$n \in (-\infty, 0]$
a_n	causal -0.5^{n+1}	$-\frac{0.5}{1-0.5z}$	$\xleftrightarrow{z^{-1}}$	$-\frac{0.5}{1-0.5z^{-1}}$	anti-causal -0.5^{-n+1}

	$n \in (-\infty, -1]$	$\lvert z\rvert > 2$		$\lvert z\rvert < 0.5$	$n \in [1, \infty)$
b_n	0.5^{n+1} anti-causal	$\frac{z^{-1}}{1-2z^{-1}}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{1-2z}$	0.5^{-n+1} causal

	$n \in [0, \infty)$	$\lvert z\rvert < 2$		$\lvert z\rvert > 0.5$	$n \in (-\infty, 0]$
a_n	causal -0.5^{n+1}	$-\frac{0.5}{1-0.5z}$	$\xleftrightarrow{z^{-1}}$	$-\frac{0.5}{1-0.5z^{-1}}$	anti-causal -2^{+n-1}

	$n \in (-\infty, -1]$	$\lvert z\rvert > 2$		$\lvert z\rvert < 0.5$	$n \in [1, \infty)$
b_n	0.5^{n+1} anti-causal	$\frac{z^{-1}}{1-2z^{-1}}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{1-2z}$	2^{+n-1} causal

Time Shift

$P = 2$

① $(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = \frac{2}{2-z}$ $X(z) = \frac{z}{z-0.5}$

③ $(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = -\frac{2}{2-z}$ $X(z) = -\frac{z}{z-0.5}$

⑤ $(n \geq 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = \frac{2z}{2-z}$ $X(z) = \frac{1}{z-0.5}$

⑦ $(n < 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = -\frac{2z}{2-z}$ $X(z) = -\frac{1}{z-0.5}$

⑨ $(n \geq -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = \frac{2}{(2-z)z}$ $X(z) = \frac{z^2}{z-0.5}$

⑪ $(n < -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = -\frac{2}{(2-z)z}$ $X(z) = -\frac{z^2}{z-0.5}$

Time Shift

$p = \frac{1}{2}$

② $(n \geq 0)$ $a_n = (2)^n$ $f(z) = \frac{0.5}{0.5 - z}$ $X(z) = \frac{z}{z-2}$

④ $(n < 0)$ $a_n = (2)^n$ $f(z) = -\frac{0.5}{0.5 - z}$ $X(z) = -\frac{z}{z-2}$

⑥ $(n \geq 1)$ $a_{n-1} = (2)^{n-1}$ $f(z) = \frac{0.5z}{0.5 - z}$ $X(z) = \frac{1}{z-2}$

⑧ $(n < 1)$ $a_{n-1} = (2)^{n-1}$ $f(z) = -\frac{0.5z}{0.5 - z}$ $X(z) = -\frac{1}{z-2}$

⑩ $(n \geq -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = \frac{0.5}{(0.5 - z)z}$ $X(z) = \frac{z^2}{z-2}$

⑫ $(n < -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = -\frac{0.5}{(0.5 - z)z}$ $X(z) = -\frac{z^2}{z-2}$

Time Shift

$2 \leftrightarrow \frac{1}{2}$

1. $(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = \frac{2}{2-z}$ $X(z) = \frac{z}{z-0.5}$
2. $(n \geq 0)$ $a_n = (2)^n$ $f(z) = \frac{0.5}{0.5-z}$ $X(z) = \frac{z}{z-2}$
3. $(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = -\frac{2}{2-z}$ $X(z) = -\frac{z}{z-0.5}$
4. $(n < 0)$ $a_n = (2)^n$ $f(z) = -\frac{0.5}{0.5-z}$ $X(z) = -\frac{z}{z-2}$
5. $(n \geq 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = \frac{2z}{2-z}$ $X(z) = \frac{1}{z-0.5}$
6. $(n \geq 1)$ $a_{n-1} = (2)^{n-1}$ $f(z) = \frac{0.5z}{0.5-z}$ $X(z) = \frac{1}{z-2}$
7. $(n < 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = -\frac{2z}{2-z}$ $X(z) = -\frac{1}{z-0.5}$
8. $(n < 1)$ $a_{n-1} = (2)^{n-1}$ $f(z) = -\frac{0.5z}{0.5-z}$ $X(z) = -\frac{1}{z-2}$
9. $(n \geq -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = \frac{2}{(2-z)z}$ $X(z) = \frac{z^2}{z-0.5}$
10. $(n \geq -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = \frac{0.5}{(0.5-z)z}$ $X(z) = \frac{z^2}{z-2}$
11. $(n < -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = -\frac{2}{(2-z)z}$ $X(z) = -\frac{z^2}{z-0.5}$
12. $(n < -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = -\frac{0.5}{(0.5-z)z}$ $X(z) = -\frac{z^2}{z-2}$

Shift to the right

Shift to the right →		$\times z$	$\times z^{-1}$
delete a_0			
① $(n \geq 0)$	$a_n = \left(\frac{1}{2}\right)^n$	$f(z) = \frac{2}{2-z}$	$X(z) = \frac{z}{z-0.5}$
⑤ $(n \geq 1)$	$a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	$f(z) = \frac{2z}{2-z}$	$X(z) = \frac{1}{z-0.5}$
② $(n \geq 0)$	$a_n = (2)^n$	$f(z) = \frac{0.5}{0.5-z}$	$X(z) = \frac{z}{z-2}$
⑥ $(n \geq 1)$	$a_{n-1} = (2)^{n-1}$	$f(z) = \frac{0.5z}{0.5-z}$	$X(z) = \frac{1}{z-2}$

Shift to the right →		$\times z$	$\times z^{-1}$
insert a_0			
③ $(n < 0)$	$a_n = \left(\frac{1}{2}\right)^n$	$f(z) = -\frac{2}{2-z}$	$X(z) = -\frac{z}{z-0.5}$
⑦ $(n < 1)$	$a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	$f(z) = -\frac{2z}{2-z}$	$X(z) = -\frac{1}{z-0.5}$
④ $(n < 0)$	$a_n = (2)^n$	$f(z) = -\frac{0.5}{0.5-z}$	$X(z) = -\frac{z}{z-2}$
⑧ $(n < 1)$	$a_{n-1} = (2)^{n-1}$	$f(z) = -\frac{0.5z}{0.5-z}$	$X(z) = -\frac{1}{z-2}$

Shift to the left

Shift to the left ← $\times z^{-1}$ $\times z$

delete a_0

① $(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = \frac{2}{2-z}$ $X(z) = \frac{z}{z-0.5}$

⑨ $(n \geq -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = \frac{2}{(2-z)z}$ $X(z) = \frac{z}{z-0.5}$

② $(n \geq 0)$ $a_n = (2)^n$ $f(z) = \frac{0.5}{0.5-z}$ $X(z) = \frac{z}{z-2}$

⑩ $(n \geq -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = \frac{0.5}{(0.5-z)z}$ $X(z) = \frac{z}{z-2}$

Shift to the left ← $\times z^{-1}$ $\times z$

insert a_0

③ $(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = -\frac{2}{2-z}$ $X(z) = -\frac{z}{z-0.5}$

⑪ $(n < -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = -\frac{2}{(2-z)z}$ $X(z) = -\frac{z}{z-0.5}$

④ $(n < 0)$ $a_n = (2)^n$ $f(z) = -\frac{0.5}{0.5-z}$ $X(z) = -\frac{z}{z-2}$

⑫ $(n < -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = -\frac{0.5}{(0.5-z)z}$ $X(z) = -\frac{z}{z-2}$

$n=-4$	$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
b^{-3}	b^{-2}	b^{-1}	b^0	b^1	b^2	b^3	

b^{n+1} $n=-2,-3,-4,\cdots$

b^{n+1} $n=-1,0,1,\cdots$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$n=3$	
b^{-3}	b^{-2}	b^{-1}	b^0	b^1	b^2	b^3	

b^n $n=-1,-2,-3,\cdots$

b^n $n=0,1,2,\cdots$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$n=3$		
	b^{-3}	b^{-2}	b^{-1}	b^0	b^1	b^2	b^3	

b^{n-1} $n=0,-1,-2,\cdots$

b^{n-1} $n=1,2,3,\cdots$

$$1 \longleftrightarrow \frac{1}{1}$$

1 $(n \geq 0)$ $a_n = (1)^n$ $f(z) = \frac{1}{1-z}$ $X(z) = \frac{z}{z-1}$

2 $(n \geq 0)$ $a_n = (1^{-1})^n$ $f(z) = \frac{1}{1-z}$ $X(z) = \frac{z}{z-1}$

3 $(n < 0)$ $a_n = (1)^n$ $f(z) = -\frac{1}{1-z}$ $X(z) = -\frac{z}{z-1}$

4 $(n < 0)$ $a_n = (1^{-1})^n$ $f(z) = -\frac{1}{1-z}$ $X(z) = -\frac{z}{z-1}$

5 $(n \geq 1)$ $a_{n-1} = (1)^{n-1}$ $f(z) = \frac{z}{1-z}$ $X(z) = \frac{1}{z-1}$

6 $(n \geq 1)$ $a_{n-1} = (1^{-1})^{n-1}$ $f(z) = \frac{z}{1-z}$ $X(z) = \frac{1}{z-1}$

7 $(n < 1)$ $a_{n-1} = (1)^{n-1}$ $f(z) = -\frac{z}{1-z}$ $X(z) = -\frac{1}{z-1}$

8 $(n < 1)$ $a_{n-1} = (1^{-1})^{n-1}$ $f(z) = -\frac{z}{1-z}$ $X(z) = -\frac{1}{z-1}$

9 $(n \geq -1)$ $a_{n+1} = (1)^{n+1}$ $f(z) = \frac{1}{(1-z)z}$ $X(z) = \frac{z}{z-1}$

10 $(n \geq -1)$ $a_{n+1} = (1^{-1})^{n+1}$ $f(z) = \frac{1}{(1-z)z}$ $X(z) = \frac{z}{z-1}$

11 $(n < -1)$ $a_{n+1} = (1)^{n+1}$ $f(z) = -\frac{1}{(1-z)z}$ $X(z) = -\frac{z}{z-1}$

12 $(n < -1)$ $a_{n+1} = (1^{-1})^{n+1}$ $f(z) = -\frac{1}{(1-z)z}$ $X(z) = -\frac{z}{z-1}$

1 $(n \geq 0)\quad a_n = (1)^n \qquad f(z) = \frac{1}{1-z} \qquad X(z) = \frac{z}{z-1}$

2 $(n \geq 0)\quad a_n = (1^{-1})^n \qquad f(z) = \frac{1}{1-z} \qquad X(z) = \frac{z}{z-1}$

shift to the right $\longrightarrow$ $\qquad \times z \qquad \times z^{-1}$

delete a_0

5 $(n \geq 1)\quad a_{n-1} = (1)^{n-1} \qquad f(z) = \frac{z}{1-z} \qquad X(z) = \frac{1}{z-1}$

6 $(n \geq 1)\quad a_{n-1} = (1^{-1})^{n-1} \qquad f(z) = \frac{z}{1-z} \qquad X(z) = \frac{1}{z-1}$

3 $(n < 0)\quad a_n = (1)^n \qquad f(z) = -\frac{1}{1-z} \qquad X(z) = -\frac{z}{z-1}$

4 $(n < 0)\quad a_n = (1^{-1})^n \qquad f(z) = -\frac{1}{1-z} \qquad X(z) = -\frac{z}{z-1}$

shift to the right $\longrightarrow$ $\qquad \times z \qquad \times z^{-1}$

insert a_0

7 $(n < 1)\quad a_{n-1} = (1)^{n-1} \qquad f(z) = -\frac{z}{1-z} \qquad X(z) = -\frac{1}{z-1}$

8 $(n < 1)\quad a_{n-1} = (1^{-1})^{n-1} \qquad f(z) = -\frac{z}{1-z} \qquad X(z) = -\frac{1}{z-1}$

① $(n \geq 0)$ $a_n = (1)^n$ $f(z) = \frac{1}{1-z}$ $X(z) = \frac{z}{z-1}$

② $(n \geq 0)$ $a_n = (1^{-1})^n$ $f(z) = \frac{1}{1-z}$ $X(z) = \frac{z}{z-1}$

shift to the left ← $\times z^{-1}$ $\times z$

delete a_0

⑨ $(n \geq -1)$ $a_{n+1} = (1)^{n+1}$ $f(z) = \frac{1}{(1-z)z}$ $X(z) = \frac{z}{z-1}$

⑩ $(n \geq -1)$ $a_{n+1} = (1^{-1})^{n+1}$ $f(z) = \frac{1}{(1-z)z}$ $X(z) = \frac{z}{z-1}$

③ $(n < 0)$ $a_n = (1)^n$ $f(z) = -\frac{1}{1-z}$ $X(z) = -\frac{z}{z-1}$

④ $(n < 0)$ $a_n = (1^{-1})^n$ $f(z) = -\frac{1}{1-z}$ $X(z) = -\frac{z}{z-1}$

shift to the left ← $\times z^{-1}$ $\times z$

insert a_0

⑪ $(n < -1)$ $a_{n+1} = (1)^{n+1}$ $f(z) = -\frac{1}{(1-z)z}$ $X(z) = -\frac{z}{z-1}$

⑫ $(n < -1)$ $a_{n+1} = (1^{-1})^{n+1}$ $f(z) = -\frac{1}{(1-z)z}$ $X(z) = -\frac{z}{z-1}$

Causality

$$f(z)\ (|z|<p) \leftrightarrow a_n\ (n \geq 0) \quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$$

$$X(z^{-1})\ (|z|<p) \leftrightarrow x_{-n}\ (n < 1) \quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$$

$$f(z^{-1})\ (|z|>p^{-1}) \leftrightarrow a_{-n}\ (n < 1) \quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$$

$$X(z)\ (|z|>p^{-1}) \leftrightarrow x_n\ (n \geq 0) \quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$$

$$f(z)\ (|z|>p) \leftrightarrow -a_n\ (n < 0) \quad (p^{0}, p^{1}, p^{2}, \cdots)$$

$$X(z^{-1})\ (|z|>p) \leftrightarrow -x_{-n}\ (n \geq 1) \quad (p^{0}, p^{1}, p^{2}, \cdots)$$

$$f(z^{-1})\ (|z|<p^{-1}) \leftrightarrow -a_{-n}\ (n \geq 1) \quad (p^{0}, p^{1}, p^{2}, \cdots)$$

$$X(z)\ (|z|<p^{-1}) \leftrightarrow -x_n\ (n < 0) \quad (p^{0}, p^{1}, p^{2}, \cdots)$$

$$\begin{array}{cc} f(z) & f(z^{-1}) \\ g(z^{-1}) & g(z) \end{array} \qquad \begin{array}{cc} X(z^{-1}) & X(z) \\ Y(z) & Y(z^{-1}) \end{array} \qquad \begin{array}{cc} a_n & a_{-n} \\ b_{-n} & b_n \end{array} \qquad \begin{array}{cc} x_{-n} & x_n \\ y_n & y_{-n} \end{array}$$

$$\begin{array}{cc} f(z) & f(z^{-1}) \\ f(z) & f(z^{-1}) \end{array} \qquad \begin{array}{cc} X(z^{-1}) & X(z) \\ X(z^{-1}) & X(z) \end{array}$$

$$\begin{array}{cc} -(p^{-1}, p^{-2}, p^{-3}, \dots) & -(p^{-1}, p^{-2}, p^{-3}, \dots) \\ (p^{0}, p^{1}, p^{2}, \dots) & (p^{0}, p^{1}, p^{2}, \dots) \end{array} \qquad \begin{array}{cc} -(p^{-1}, p^{-2}, p^{-3}, \dots) & -(p^{-1}, p^{-2}, p^{-3}, \dots) \\ (p^{0}, p^{1}, p^{2}, \dots) & (p^{0}, p^{1}, p^{2}, \dots) \end{array}$$

$$\begin{array}{cc} -\dfrac{p^{-1}}{1-p^{-1}z} & -\dfrac{p^{-1}}{1-p^{-1}z^{-1}} \\ \dfrac{z^{-1}}{1-pz^{-1}} & \dfrac{z}{1-pz} \end{array} \qquad \begin{array}{cc} -\dfrac{p^{-1}}{1-p^{-1}z} & -\dfrac{p^{-1}}{1-p^{-1}z^{-1}} \\ \dfrac{z^{-1}}{1-pz^{-1}} & \dfrac{z}{1-pz} \end{array}$$

$f(z)$	$g(z)$
$f(z)$	$g(z)$

$Y(z)$	$X(z)$
$Y(z)$	$X(z)$

a_n	a_{-n}
$-a_n$	$-a_{-n}$

x_{-n}	x_n
$-x_{-n}$	$-x_n$

$\lvert z\rvert < p$	$\lvert z\rvert > p^{-1}$
$\lvert z\rvert > p$	$\lvert z\rvert < p^{-1}$

$\lvert z\rvert < p$	$\lvert z\rvert > p^{-1}$
$\lvert z\rvert > p$	$\lvert z\rvert < p^{-1}$

$[0,\infty)$	$(-\infty, 0]$
$(-\infty,-1]$	$[1,\infty)$

$(-\infty, 0]$	$[0,\infty)$
$[1,\infty)$	$(-\infty,-1]$

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$
$(p^{0}, p^{1}, p^{2}, \cdots)$

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$
$(p^{0}, p^{1}, p^{2}, \cdots)$

$$\boxed{\begin{array}{cc} a_n & a_{-n} \\ -a_n & -a_{-n} \end{array}} \qquad \boxed{\begin{array}{cc} 2^{-n} & 2^{n} \\ -2^{-n} & -2^{n} \end{array}} \qquad a_n = -2^{-n}$$

$$\boxed{\begin{array}{cc} x_{-n} & x_n \\ -x_{-n} & -x_n \end{array}} \qquad \boxed{\begin{array}{cc} 2^{-n} & 2^{n} \\ -2^{-n} & -2^{n} \end{array}} \qquad x_n = -2^{n}$$

$$\boxed{\begin{array}{cc} -(p^{-1}, p^{-2}, p^{-3}, \cdots) & -(p^{-1}, p^{-2}, p^{-3}, \cdots) \\ (p^{0}, p^{1}, p^{2}, \cdots) & (p^{0}, p^{1}, p^{2}, \cdots) \end{array}} \qquad \boxed{\begin{array}{cc} -(-2^{-1}, -2^{-2}, -2^{-3}, \cdots) & -(-2^{-1}, -2^{-2}, -2^{-3}, \cdots) \\ (-2^{0}, -2^{1}, -2^{2}, \cdots) & (-2^{0}, -2^{1}, -2^{2}, \cdots) \end{array}}$$

$$\boxed{\begin{array}{cc} -\dfrac{p^{-1}}{1-p^{-1}z} & -\dfrac{p^{-1}}{1-p^{-1}z^{-1}} \\ \dfrac{z^{-1}}{1-pz^{-1}} & \dfrac{z}{1-pz} \end{array}} \qquad \boxed{\begin{array}{cc} \dfrac{2^{-1}}{1-2^{-1}z} & \dfrac{2^{-1}}{1-2^{-1}z^{-1}} \\ -\dfrac{z^{-1}}{1-2z^{-1}} & -\dfrac{z}{1-2z} \end{array}} \qquad \begin{array}{cc} \dfrac{\left(\frac{1}{2}\right)}{1-\left(\frac{z}{2}\right)} & \dfrac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2z}\right)} \\ -\dfrac{\left(\frac{1}{z}\right)}{1-\left(\frac{2}{z}\right)} & -\dfrac{z}{1-2z} \end{array}$$

$$\boxed{\begin{array}{cc} |z| < p & |z| > p^{-1} \\ |z| > p & |z| < p^{-1} \end{array}} \qquad \boxed{\begin{array}{cc} |z| < 2 & |z| > 2^{-1} \\ |z| > 2 & |z| < 2^{-1} \end{array}}$$

$$\boxed{\begin{array}{cc} [0, \infty) & (-\infty, 0] \\ (-\infty, -1] & [1, \infty) \end{array}} \qquad \boxed{\begin{array}{cc} [0, \infty) & (-\infty, 0] \\ (-\infty, -1] & [1, \infty) \end{array}}$$