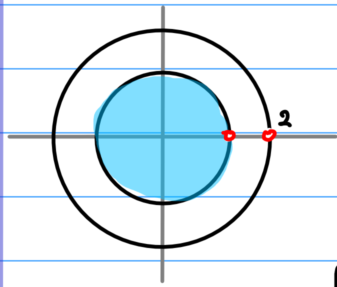
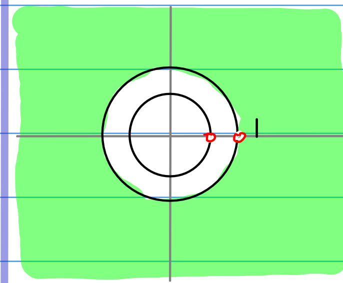
Laurent Series and z-Transform Examples case 1.A

20171019

I





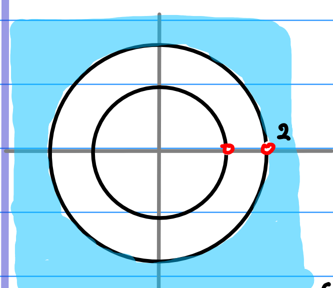
$$a_n = \begin{cases} \left(\frac{1}{2}\right)^{n+1} - 1 & (n \geq 0) \\ 0 & (n < 0) \end{cases}$$

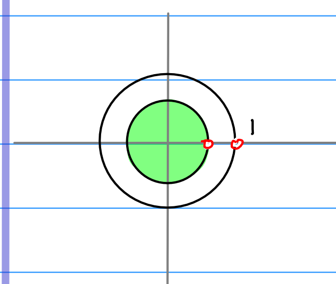
$$x_n = \begin{cases} 0 & (n > 0) \\ \left(\frac{1}{2}\right)^{n+1} - 1 & (n \leq 0) \end{cases}$$

$$f(z) = \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^{n+1} - 1 \right] z^n$$

$$X(z) = \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^{n+1} - 1 \right] z^n$$

II





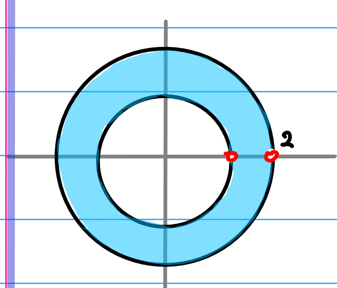
$$a_n = \begin{cases} 0 & (n \geq 0) \\ \left(1 - \left(\frac{1}{2}\right)^{n+1}\right) & (n < 0) \end{cases}$$

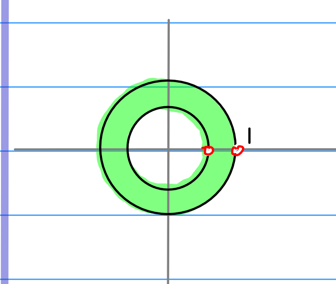
$$x_n = \begin{cases} \left(1 - \left(\frac{1}{2}\right)^{n+1}\right) & (n > 0) \\ 0 & (n \leq 0) \end{cases}$$

$$f(z) = \sum_{n=-1}^{-\infty} \left(1 - \left(\frac{1}{2}\right)^{n+1}\right) z^n$$

$$X(z) = \sum_{n=-1}^{-\infty} \left(1 - \left(\frac{1}{2}\right)^{n+1}\right) z^n$$

III





$$a_n = \begin{cases} \left(\frac{1}{2}\right)^{n+1} & (n \geq 0) \\ 1 & (n < 0) \end{cases}$$

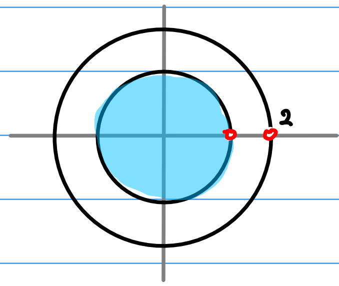
$$x_n = \begin{cases} 1 & (n > 0) \\ \left(\frac{1}{2}\right)^{n+1} & (n \leq 0) \end{cases}$$

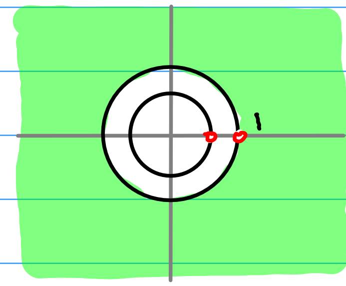
$$f(z) = \sum_{n=-1}^{-\infty} z^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^n$$

$$X(z) = \sum_{n=-1}^{-\infty} z^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^n$$

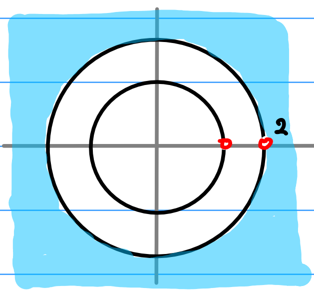
1. A

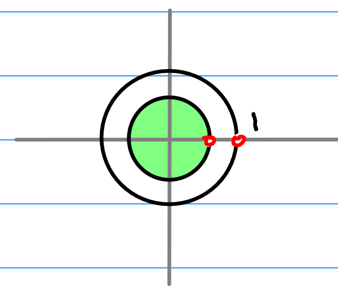
$$f(z) = \frac{-1}{(z-1)(z-2)} \xrightarrow{z^{-1}} X(z) = \frac{-0.5\,z^2}{(z-1)(z-0.5)}$$



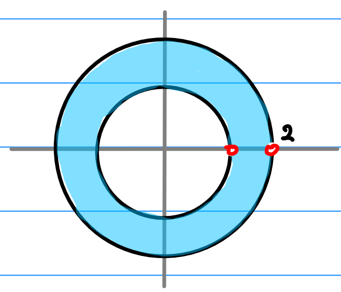


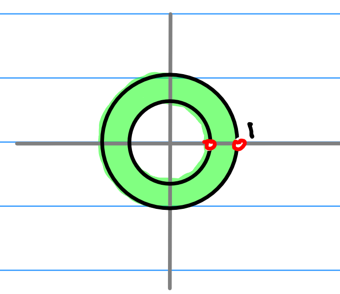
$$\sum_{n=0}^{\infty}\left[\left(\frac{1}{2}\right)^{n+1}-1\right]z^n \quad = \quad \sum_{n=0}^{\infty}\left[\left(\frac{1}{2}\right)^{n+1}-1\right]z^n$$





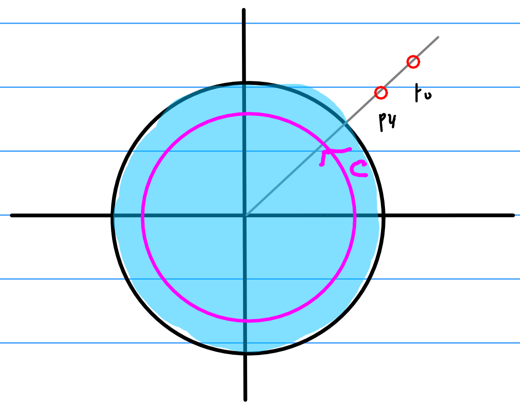
$$\sum_{n=-1}^{-\infty}\left(1-\left(\frac{1}{2}\right)^{n+1}\right)z^n \quad = \quad \sum_{n=-1}^{-\infty}\left(1-\left(\frac{1}{2}\right)^{n+1}\right)z^n$$

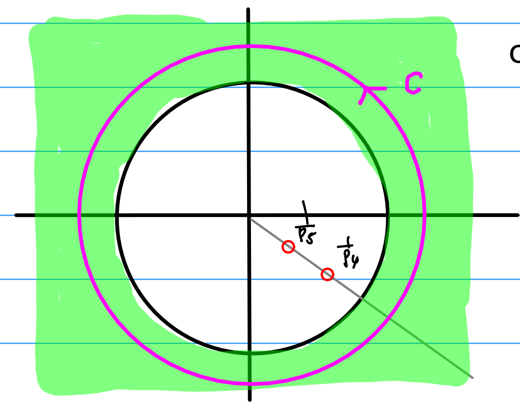




$$\sum_{n=-1}^{-\infty} z^n + \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1} z^n \quad = \quad \sum_{n=-1}^{-\infty} z^n + \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1} z^n$$

L.S at $z=0$ | Z.T.





causal

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n \, z^n}$$

$$\boxed{X(z) = \sum_{k=0}^{\infty} x_k \, z^{-k}}$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} \, dz = \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

$$x_n = \frac{1}{2\pi i} \oint_C X(z) z^{n-1} \, dz = \sum_k \mathrm{Res}\left(X(z) z^{n-1}, z_k\right)$$

Poles z_k

$n \geq 0$ $z_1, z_2, z_3, 0$

$n < 0$ z_1, z_2, z_3

Poles z_k

$n > 0$ z_1, z_2, z_3

$n \leq 0$ $z_1, z_2, z_3, 0$

Z-transform

$\boxed{z_m = 0}$

$$\boxed{\begin{aligned} x[n] &= \frac{1}{2\pi i}\oint_C f(z)\, z^{n-1}\, dz \\ &= \sum_k \mathrm{Res}\left(f(z)\, z^{n-1},\, z_k\right) \end{aligned}}$$

$n > 0$ $\quad z_k : \{\text{poles of } f(z)\}$

$n = 0$ $\quad z_k : \{\text{poles of } f(z)\} + \{z = 0\}$

$z^{n-1} = z^{-1} = \frac{1}{z}$

$x[n]$ includes $u[n]$ $\rightarrow$ $X[z]$ contains z on its numerator

Also, think about modified partial fraction $\frac{X[z]}{z}$

Laurent Expansion

expansion at z_m

$$\boxed{\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_m)^{n+1}}\, dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z - z_m)^{n+1}},\, z_k\right) \end{aligned}}$$

$\boxed{z_m = 0}$

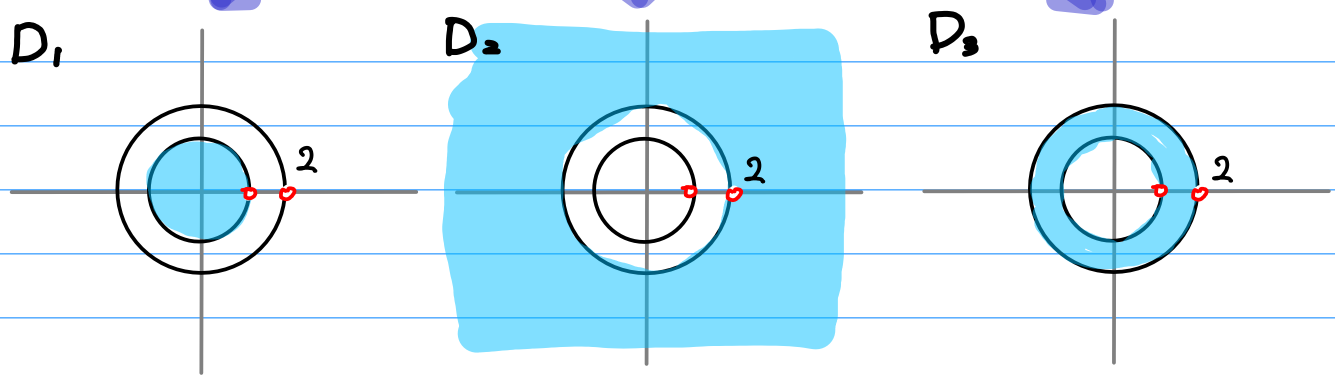
$$\boxed{\begin{aligned} a_n^{\{0\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\, dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}},\, z_k\right) \end{aligned}}$$

$$\begin{aligned} a_{-n}^{\{0\}} &= \frac{1}{2\pi i}\oint_C f(z)\, z^{n-1}\, dz \\ &= \sum_k \mathrm{Res}\left(f(z)\, z^{n-1},\, z_k\right) \end{aligned}$$

$$\begin{aligned} a_{-n}^{\{0\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{-n+1}}\, dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{-n+1}},\, z_k\right) \end{aligned}$$

L.S. first

$$\boxed{f(z) = \frac{-1}{(z-1)(z-2)}}$$



a_n ?	a_n ?	a_n ?
$\parallel$	$\parallel$	$\parallel$
x_n	x_n	x_n
$\updownarrow$	$\updownarrow$	$\updownarrow$
$X(z)$?	$X(z)$?	$X(z)$?

$$\boxed{X(z) = ?}$$

L.S. first

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{(z-1)} - \frac{1}{(z-2)}$$

Complex Variables and Ap
Brown & Churchill

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

D_1 : $|z| < 1$

D_2 : $2 < |z|$

D_3 : $1 < |z| < 2$

I

D_1 $\quad |z| < 1$ $\qquad \left|\frac{z}{1}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-\left(\frac{z}{1}\right)} + \frac{\frac{1}{2}}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \left(\left(\frac{1}{2}\right)^{n+1} - 1\right) z^n \qquad |z|<1$$

II

D_2 $\quad 2 < |z|$ $\qquad \left|\frac{1}{z}\right| < 1, \quad \left|\frac{2}{z}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} - \frac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n} = \sum_{n=-1}^{-\infty} \left(1 - \left(\frac{1}{2}\right)^{n+1}\right) z^n$$

III

D_3 $\quad 1 < |z| < 2$ $\quad \Rightarrow \quad \left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{\frac{1}{z}}{1-\left(\frac{1}{z}\right)} + \frac{\frac{1}{2}}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

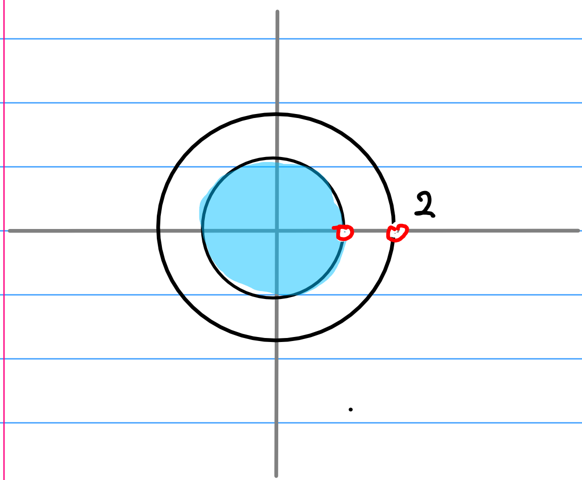
$$= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=-1}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$(n < 1)$ $\qquad (n \geq 0)$

L.S. first

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{(z-1)} - \frac{1}{(z-2)}$$

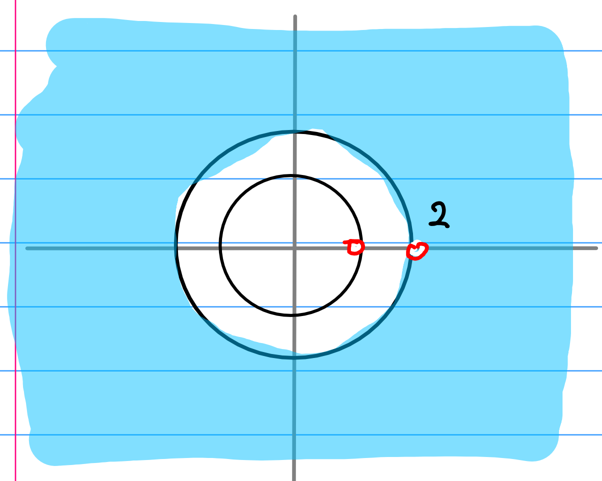
Ⓘ D_1 $|z| < 1$ causal $a_n = 0$ $(n \leq 0)$



$$\boxed{f(z) = \frac{-1}{(z-1)(z-2)}}$$

$$a_n = \begin{cases} \left(\frac{1}{2}\right)^{n+1} - 1 & (n \geq 0) \\ 0 & (n < 0) \end{cases}$$

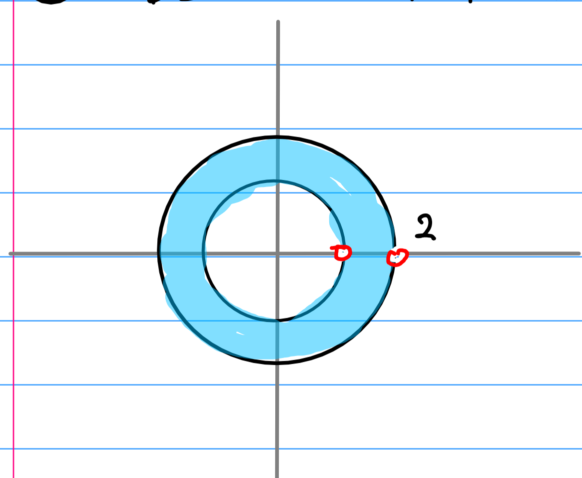
Ⅱ D_3 $|z| > 2$ anti-causal $a_n = 0$ $(n \geq 0)$



$$\boxed{f(z) = \frac{-1}{(z-1)(z-2)}}$$

$$a_n = \begin{cases} 0 & (n \geq 0) \\ 1 - \left(\frac{1}{2}\right)^{n+1} & (n < 0) \end{cases}$$

Ⅲ D_2 $1 < |z| < 2$ two-sided



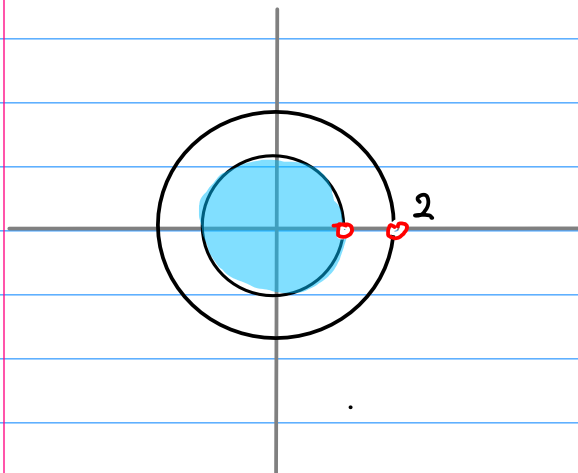
$$\boxed{f(z) = \frac{-1}{(z-1)(z-2)}}$$

$$a_n = \begin{cases} \left(\frac{1}{2}\right)^{n+1} & (n \geq 0) \\ 1 & (n < 0) \end{cases}$$

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{(z-1)} - \frac{1}{(z-2)}$$

L.S. first

Ⅰ D_1 $|z| < 1$ causal $a_n = 0 \ (n \le 0)$



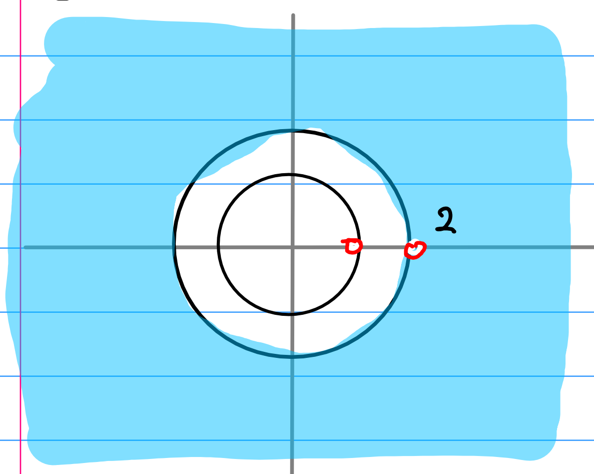
$\left|\frac{z}{1}\right| < 1 \qquad \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{-1}{1-\left(\frac{z}{1}\right)} + \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} (1)^n z^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^n$$

$$= \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^{n+1} - 1\right] z^n$$

Ⅱ D_3 $|z| > 2$ anti-causal $a_n = 0 \ (n \ge 0)$



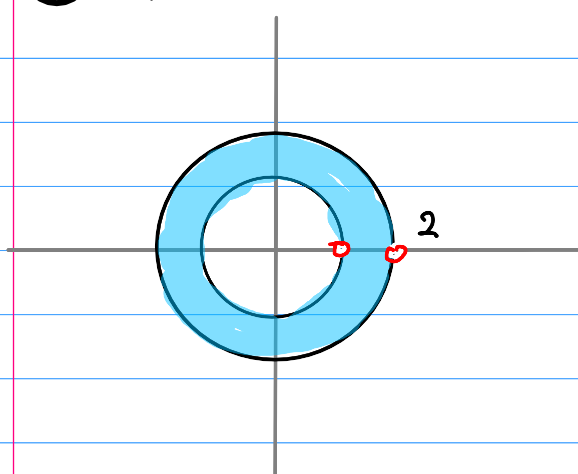
$\left|\frac{1}{z}\right| < 1 \qquad \left|\frac{2}{z}\right| < 1$

$$f(z) = \frac{\frac{1}{z}}{1-\left(\frac{1}{z}\right)} - \frac{\frac{2}{z}}{1-\left(\frac{2}{z}\right)}$$

$$= \sum_{n=0}^{\infty} 1^n z^{-n-1} - \sum_{n=0}^{\infty} 2^n z^{-n-1}$$

$$= \sum_{n=-1}^{-\infty} \left(1 - \left(\frac{1}{2}\right)^{n+1}\right) z^n$$

Ⅲ D_2 $1 < |z| < 2$ two-sided



$\left|\frac{1}{z}\right| < 1 \qquad \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{\frac{1}{z}}{1-\left(\frac{1}{z}\right)} + \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} 1^n z^{-n-1} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^n$$

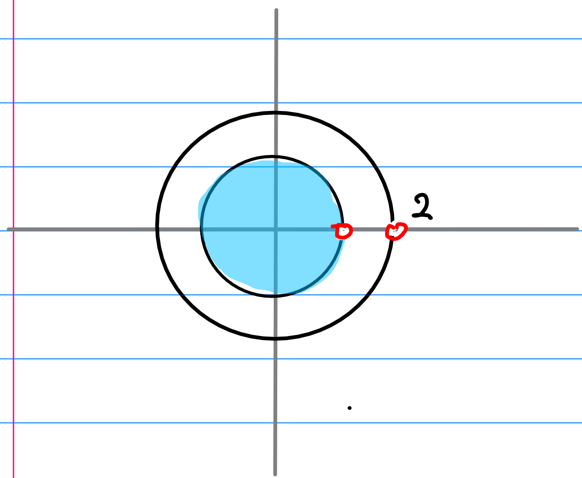
$$= \sum_{n=-1}^{-\infty} z^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} \cdot z^n$$

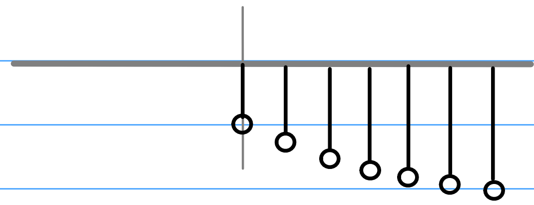
L.S. first

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{(z-1)} - \frac{1}{(z-2)}$$

Ⅰ D_1 $|z| < 1$ causal $a_n = 0$ $(n \le 0)$

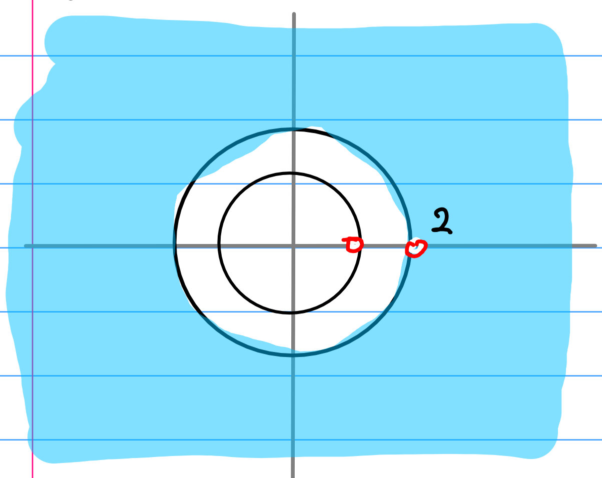
$$f(z) = \sum_{n=0}^{\infty} \left[\left(\tfrac{1}{2}\right)^{n+1} - 1 \right] z^n$$

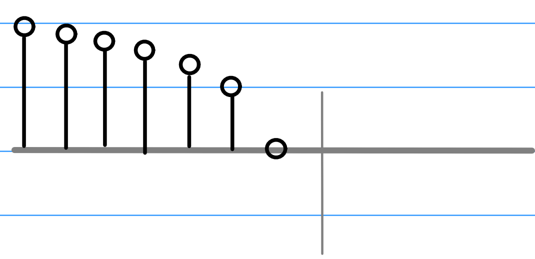




Ⅱ D_3 $|z| > 2$ anti-causal $a_n = 0$ $(n \ge 0)$

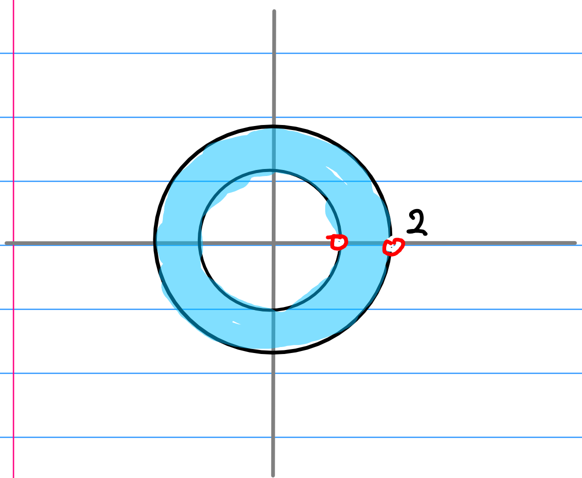
$$f(z) = \sum_{n=-1}^{-\infty} \left[1 - \left(\tfrac{1}{2}\right)^{n+1} \right] z^n$$

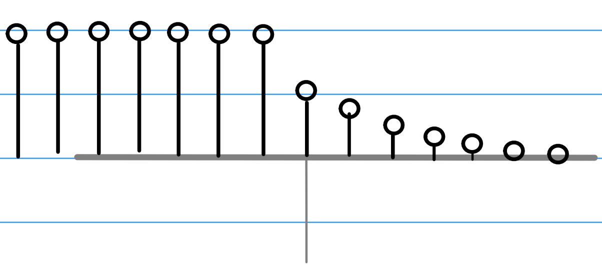




Ⅲ D_2 $1 < |z| < 2$ two-sided

$$f(z) = \sum_{n=-1}^{-\infty} z^n + \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^{n+1} \cdot z^n$$



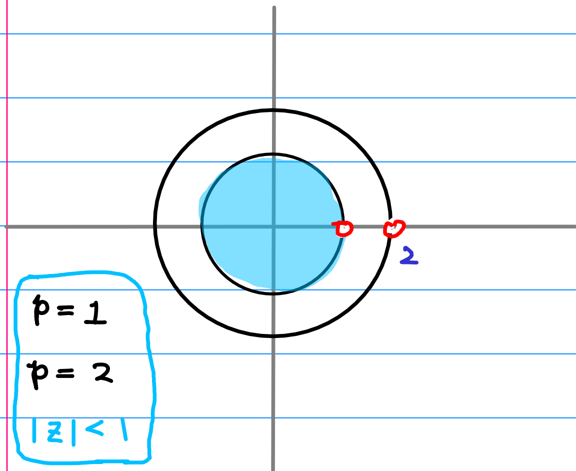


$$f(z) = \frac{-1}{(z-1)(z-2)} \qquad X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

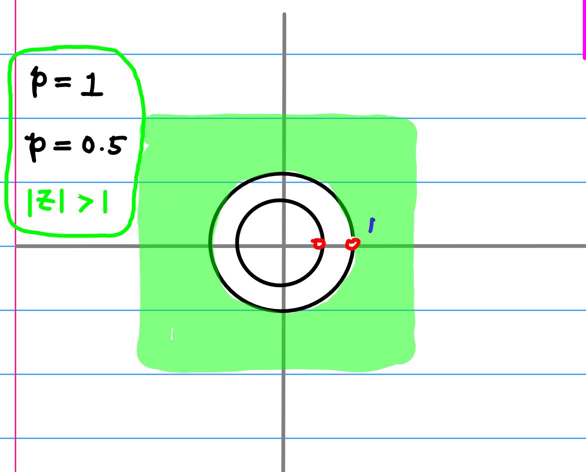
L.S. first

Ⅰ − 1

Ⅰ D_1 $|z|<1$ $\left[\left|\frac{z}{1}\right|<1, \ \left|\frac{z}{2}\right|<1 \right]$



$$\begin{aligned} f(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\ &= \frac{-1}{1-\left(\frac{z}{1}\right)} + \frac{\frac{1}{2}}{1-\left(\frac{z}{2}\right)} \\ &= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \\ &= \sum_{n=0}^{\infty} \left[2^{-n-1} - 1 \right] z^n \qquad |z|<1 \\ &= \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^{n+1} - 1 \right] z^n \qquad |z|<1 \end{aligned}$$



$$\begin{aligned} \boxed{X(z) = f(z^{-1})} &= \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^{n+1} - 1 \right] z^{-n} \qquad |z|>2 \\ &= \sum_{n=0}^{\infty} 2^{-n-1} z^{-n} - \sum_{n=0}^{\infty} 1 \cdot z^{-n} \\ &= \frac{\frac{1}{2}}{1-\left(\frac{1}{2z}\right)} - \frac{1}{1-\left(\frac{1}{z}\right)} \\ &= \frac{0.5z}{z-0.5} - \frac{z}{z-1} \\ &= \frac{0.5z - 0.5 - z + 0.5}{(z-0.5)(z-1)} \cdot z \\ &= \frac{-0.5z^2}{(z-1)(z-0.5)} \end{aligned}$$

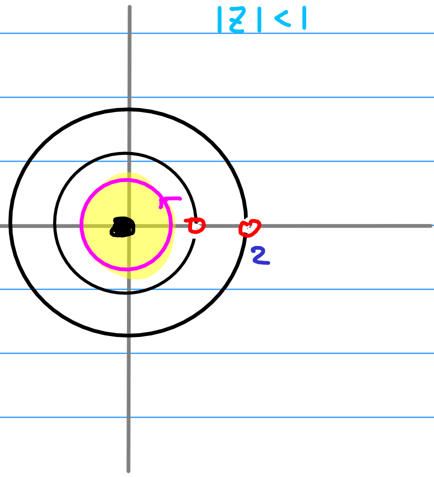
$$f(z) = \frac{-1}{(z-1)(z-2)}$$

$$f(z^{-1}) = \frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \frac{-z^2}{(1-z)(1-2z)} = \frac{-0.5z^2}{(z-1)(z-0.5)} = X(z)$$

L.S. first

Ⓘ -2

$$a_n \Leftarrow f(z) = \frac{-1}{(z-1)(z-2)}$$



$p=1$

$p=2$

$|z|<1$

$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

$$= \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$\frac{f(z)}{z^{n+1}} = \left(\frac{1}{z-1} - \frac{1}{z-2}\right)\frac{1}{z^{n+1}}$$

$n \geq 0$ then the pole $z=0$

$$\boxed{\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \quad (\text{order } n)}$$

$$\frac{d}{dz}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)\left((z-1)^{-2} - (z-2)^{-2}\right)$$

$$\frac{d^2}{dz^2}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)(-2)\left((z-1)^{-3} - (z-2)^{-3}\right)$$

$$\frac{d^3}{dz^3}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)(-2)(-3)\left((z-1)^{-4} - (z-2)^{-4}\right)$$

$$\frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n\, n!\left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

$$\frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n} z^{n+1}\left((z-1)^{-1} - (z-2)^{-1}\right)\frac{1}{z^{n+1}} = \frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right)$$

$$= \frac{1}{n!} \lim_{z \to 0} (-1)^n\, n!\left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

$$= (-1)^n \lim_{z \to 0}\left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

$$= (-1)^n\left((-1)^{-n-1} - (-2)^{-n-1}\right)$$

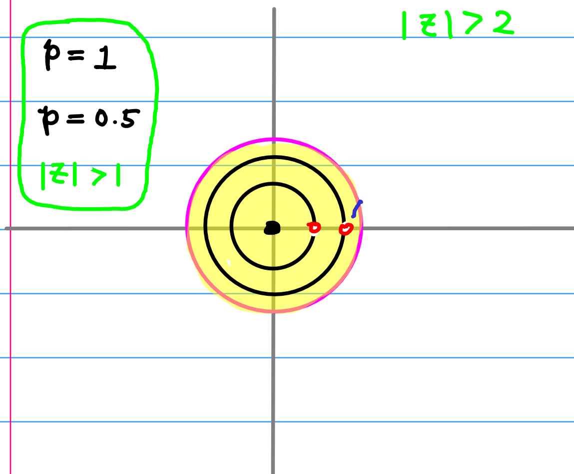
$$= -1 + 2^{-n-1}$$

$$a_n = -1 + 2^{-n-1} \qquad (n \geq 0)$$

$$f(z) = \sum_{n=n_1}^{\infty} a_n z^n = \sum_{n=0}^{\infty}\left(\left(\frac{1}{2}\right)^{n+1} - 1\right) z^n$$

L.S. first Ⓘ – 3

$$x_n \Leftarrow X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$



$$x[n] = \frac{1}{2\pi i}\int_C \boxed{X(z)\, z^{n-1}}\, dz$$

$$= \sum_{j=1}^{k} \mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, z_j\right)$$

$$X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

$$X(z)\, z^{n-1} = \frac{-0.5z^2}{(z-1)(z-0.5)}\, z^{n-1}$$

$$\mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, 1\right) = (z-1)\frac{-0.5z^{n+1}}{(z-1)(z-0.5)}\Bigg|_{z=1} = -1$$

$$\mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, \frac{1}{2}\right) = (z-0.5)\frac{-0.5z^{n+1}}{(z-1)(z-0.5)}\Bigg|_{z=0.5} = +\left(\frac{1}{2}\right)^{n+1}$$

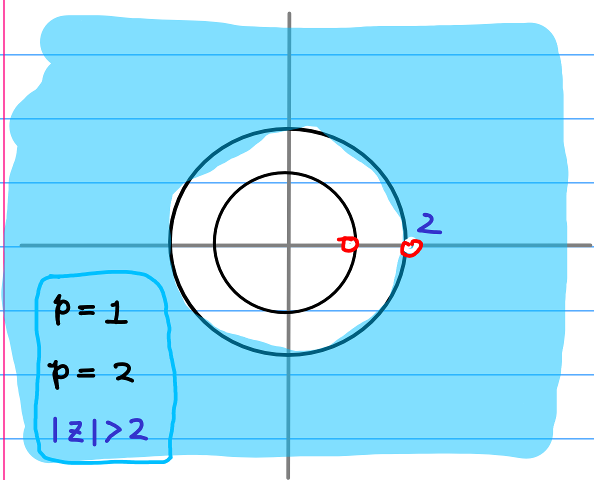
$$x[n] = \left(\frac{1}{2}\right)^{n+1} - 1$$

$$f(z) = \frac{-1}{(z-1)(z-2)} \qquad X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

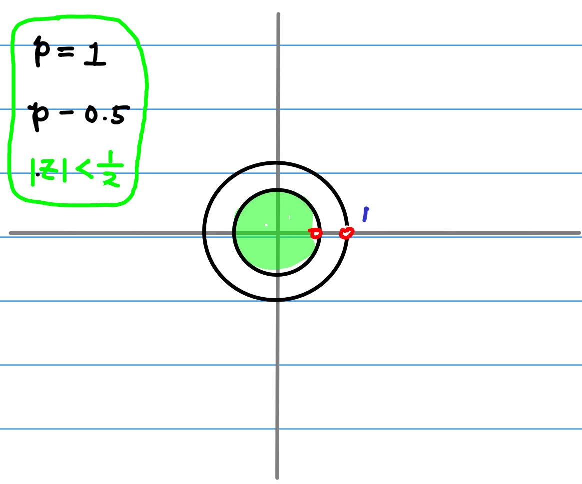
L.S. first

II - 1

$$\boxed{\text{II} \quad D_2 \quad |z| > 2 \quad \left[\,\left|\tfrac{1}{z}\right| < 1, \quad \left|\tfrac{2}{z}\right| < 1\,\right]}$$



$$\begin{aligned}
f(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\
&= \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} - \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{2}{z}\right)} \\
&= \sum_{n=0}^{\infty} \left(\tfrac{1}{z}\right)\left(\tfrac{1}{z}\right)^n - \sum_{n=0}^{\infty} \left(\tfrac{1}{z}\right)\left(\tfrac{2}{z}\right)^n \\
&= \sum_{n=0}^{\infty} z^{-n-1} - \sum_{n=0}^{\infty} 2^n z^{-n-1} \\
&= \sum_{n=1}^{\infty} z^{-n} - \sum_{n=1}^{\infty} 2^{n-1} z^{-n} \\
&= \sum_{n=-1}^{\infty} \left[1 - 2^{-n-1}\right] z^n = \sum_{n=-1}^{\infty} \left[1 - \left(\tfrac{1}{2}\right)^{n+1}\right] z^n
\end{aligned}$$



$$\begin{aligned}
\boxed{X(z) = f(z^{-1})} &= \sum_{n=-1}^{\infty} \left[1 - 2^{-n-1}\right] z^{-n} \\
&= \sum_{n=1}^{\infty} \left[1 - 2^{n-1}\right] z^n \\
&= \sum_{n=1}^{\infty} 1 \cdot z^n - \sum_{n=1}^{\infty} 2^{n-1} z^n \\
&= \frac{z}{1-z} - \frac{z}{1-2z} \\
&= -\frac{z}{z-1} + \frac{0.5z}{z-0.5} \\
&= \frac{-z + 0.5 + 0.5z - 0.5}{(z-1)(z-0.5)}\, z \\
&= \frac{-0.5z^2}{(z-1)(z-0.5)}
\end{aligned}$$

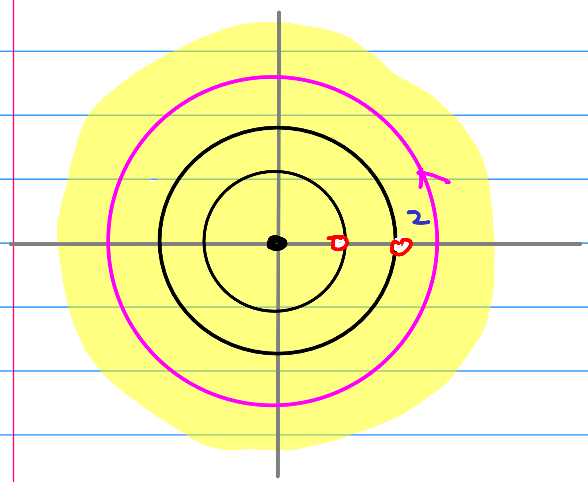
$$f(z) = \frac{-1}{(z-1)(z-2)}$$

$$f(z^{-1}) = \frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \frac{-z^2}{(1-z)(1-2z)} = \frac{-0.5z^2}{(z-1)(z-0.5)} = X(z)$$

$$a_n \Leftarrow f(z) = \frac{-1}{(z-1)(z-2)}$$

L.S. first

Ⅱ -2



$$a_n = \sum_{k=1}^{M} \text{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

$$= \text{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$+ \text{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$

$$+ \text{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 2\right)$$

$$\text{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) = -1 + 2^{-n-1} \quad (n \geq 0)$$

$$\text{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) = \lim_{z \to 1} (z-1)\frac{-1}{(z-1)(z-2)z^{n+1}} = 1$$

$$\text{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 2\right) = \lim_{z \to 2} (z-2)\frac{-1}{(z-1)(z-2)z^{n+1}} = -\frac{1}{2^{n+1}}$$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
0	0	0	$-1+2^{-1}$	$-1+2^{-2}$	$-1+2^{-3}$	$\text{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$
1	1	1	1	1	1	$\text{Res}\left(\frac{f(z)}{z^{n+1}}, 1\right)$
-2^2	-2^1	-2^0	-2^{-1}	-2^{-2}	-2^{-3}	$\text{Res}\left(\frac{f(z)}{z^{n+1}}, 2\right)$
$1-2^2$	$1-2$	0	0	0	0	
$-(-3)-1$	$-(-2)-1$	$-(-1)-1$				

$$a_n = 1 - 2^{-n-1} \quad n < 0$$

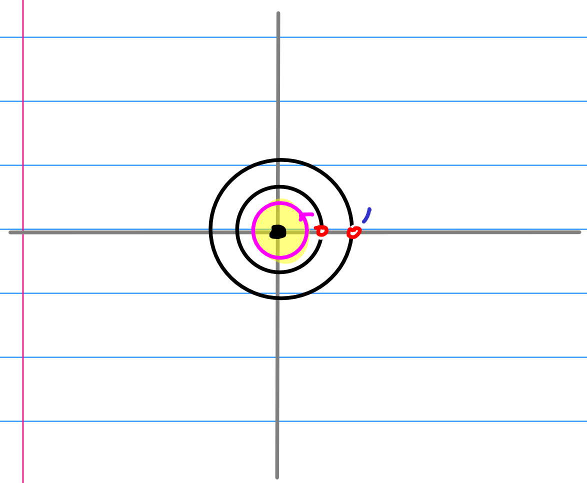
$$f(z) = \sum_{n=-1}^{-\infty} (1-2^{-n-1}) z^n = \sum_{n=-1}^{-\infty} \left(1 - \left(\tfrac{1}{2}\right)^{n+1}\right) z^n$$

$$= \sum_{n=1}^{\infty} (1-2^{n-1}) z^{-n}$$

L.S. first

Ⅱ - 3

$$x_n \leftarrow X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$



$$x[n] = \sum_{j=1}^{k} \mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, z_j\right)$$

$$= \mathrm{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)}, 0\right)$$

$$= \mathrm{Res}\left(z^{n+1}\left(\frac{1}{z-0.5} - \frac{1}{z-1}\right), 0\right)$$

$$\boxed{\frac{1}{n!} \lim_{z \to z_0} \frac{d^n}{dz^n} (z-z_0)^{n+1} f(z) \quad (n+1)\text{th order}}\ \text{pole at } z=0$$

$n=-1$ $\qquad (-2^0+1)$

$n=-2$ $\quad z^1 z^{-1}\left((z-0.5)^{-1}-(z-1)^{-1}\right) = \left((z-0.5)^{-1}-(z-1)^{-1}\right) \qquad 0!\,(-2^1+1)$

$n=-3$ $\quad \frac{d}{dz} z^2 z^{-2}\left((z-0.5)^{-1}-(z-1)^{-1}\right) = (-1)\left((z-0.5)^{-2}-(z-1)^{-2}\right) \qquad 1!\,(-2^2+1)$

$n=-4$ $\quad \frac{d^2}{dz^2} z^3 z^{-3}\left((z-0.5)^{-1}-(z-1)^{-1}\right) = (-1)(-2)\left((z-0.5)^{-3}-(z-1)^{-3}\right) \qquad 2!\,(-2^3+1)$

$n=-k$ $\quad \frac{d^{k-2}}{dz^{k-2}} z^{k-1} z^{-k+1}\left((z-0.5)^{-1}-(z-1)^{-1}\right) = (-1)(-2)\cdots(-(k-2))\left((z-0.5)^{-k+1}-(z-1)^{-k+1}\right) \qquad (k-2)!\,(-2^{k-1}+1)$

$$x_{-k} = \frac{1}{(k-2)!} \lim_{z\to 0} \frac{d^{k-2}}{dz^{k-2}} z^{k-1} z^{-k+1}\left((z-0.5)^{-1}-(z-1)^{-1}\right) = (-2^{k-1}+1) \quad (k \geq 1) \quad (n \leq -1)$$

$$x_n = (-2^{-n-1}+1) \quad (n<0)$$

$$f(z) = \sum_{n=-1}^{-\infty} (1-2^{-n-1})\, z^n = \sum_{n=-1}^{-\infty} \left(1-\left(\tfrac{1}{2}\right)^{n+1}\right) z^n$$

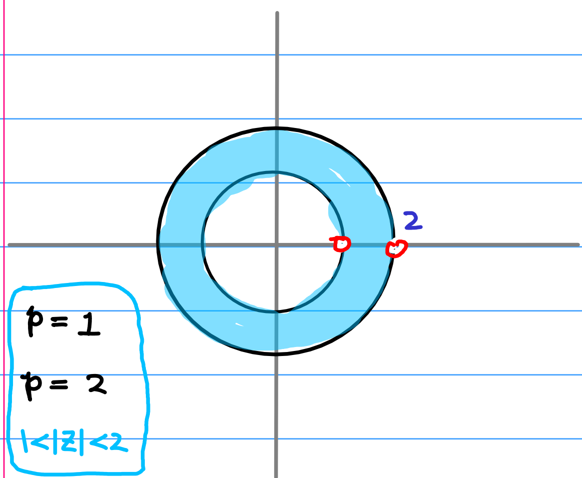
$$= \sum_{n=1}^{\infty} (1-2^{n-1})\, z^{-n}$$

$$f(z) = \frac{-1}{(z-1)(z-2)} \qquad X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

L.S. first

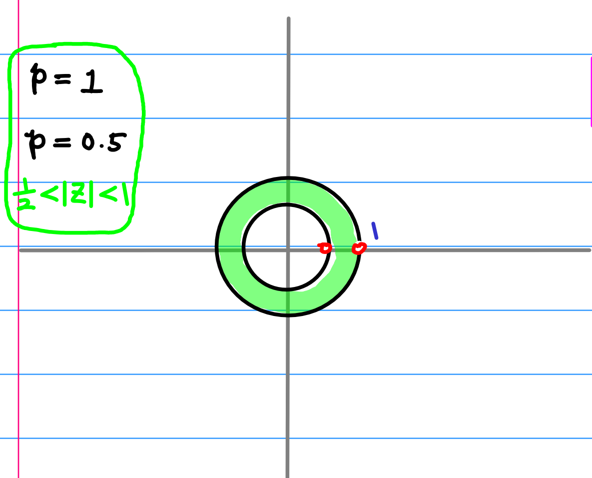
III -1

III D_3 $\quad 1 < |z| < 2 \quad \left[\left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1 \right]$



$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$
$$= \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} + \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{z}{2}\right)}$$
$$= \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)\left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{z}{2}\right)^n$$
$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$
$$= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$
$$= \sum_{n=-\infty}^{-1} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$[\, 1 \ (n<0), \quad 2^{-n-1} \ (n \geq 0) \,]$



$$\boxed{X(z) = f(z^{-1})} = \sum_{n=-\infty}^{-1} z^{-n} + \sum_{n=0}^{\infty} \frac{z^{-n}}{2^{n+1}}$$
$$= \boxed{\sum_{n=1}^{\infty} z^n} + \sum_{n=0}^{\infty} \frac{z^{-n}}{2^{n+1}}$$
$$= \frac{z}{1-z} - \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2z}\right)}$$
$$= \frac{-z}{z-1} + \frac{0.5z}{z-0.5}$$
$$= \frac{-z+0.5+0.5z-0.5}{(z-1)(z-0.5)} \cdot z$$
$$= \frac{-0.5z^2}{(z-1)(z-0.5)}$$

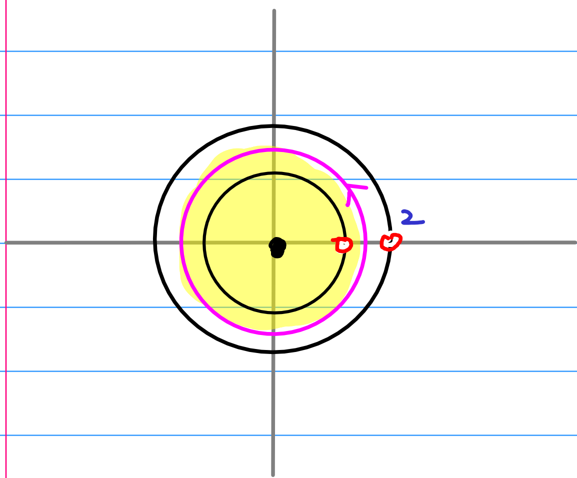
$$f(z) = \frac{-1}{(z-1)(z-2)}$$

$$f(z^{-1}) = \frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \frac{-z^2}{(1-z)(1-2z)} = \frac{-0.5z^2}{(z-1)(z-0.5)} = X(z)$$

$$a_n \leftarrow f(z) = \frac{-1}{(z-1)(z-2)}$$

L.S. first

Ⅲ -2



$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

$$= \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$+ \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$

$$\boxed{\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \quad (\text{order } n)}$$

$$\frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n \lim_{z \to 0} \left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

$$= (-1)^n \left((-1)^{-n-1} - (-2)^{-n-1}\right)$$

$$= -1 + 2^{-n-1}$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) = -1 + 2^{-n-1} \quad (n \geq 0)$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) = \lim_{z \to 1} (z-1) \frac{-1}{(z-1)(z-2)z^{n+1}} = 1$$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
0	0	0	$-1+2^{-1}$	$-1+2^{-2}$	$-1+2^{-3}$	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$
1	1	1	1	1	1	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 1\right)$
1	1	1	2^{-1}	2^{-2}	2^{-3}	

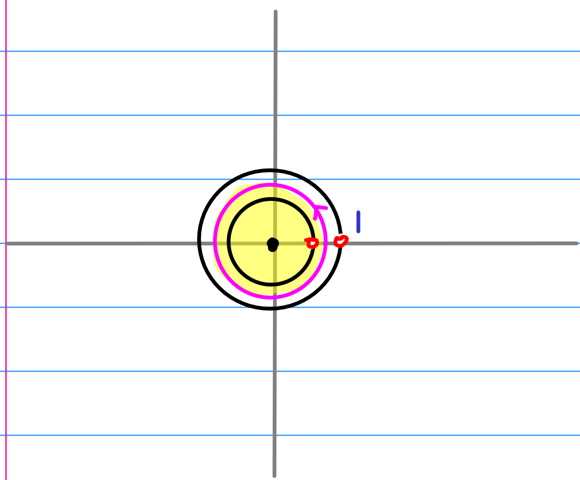
$$\begin{cases} a_n = 2^{-n-1} & n \geq 0 \\ a_n = 1 & n < 0 \end{cases} \qquad \begin{cases} 2^{-n-1} z^n \\ z^{-n} \end{cases}$$

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

L.S. first

Ⅲ-3

$$x_n \leftarrow X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$



$$x[n] = \sum_{j=1}^{k} \mathrm{Res}\left(\boxed{X(z)z^{n-1}}, z_j\right)$$

$$= \mathrm{Res}\left(\frac{-0.5z^{n+1}}{(z-1)(z-0.5)}, 0\right)$$

$$+ \mathrm{Res}\left(\frac{-0.5z^{n+1}}{(z-1)(z-0.5)}, \frac{1}{2}\right)$$

$$\mathrm{Res}\left(\frac{-0.5z^{n+1}}{(z-1)(z-0.5)}, \frac{1}{2}\right) = \lim_{z\to 0.5} \frac{-0.5z^{n+1}}{(z-1)} = \left(\frac{1}{2}\right)^{n+1} = 2^{-n-1}$$

$$\mathrm{Res}\left(\frac{-0.5z^{n+1}}{(z-1)(z-0.5)}, 0\right) = (-2^{-n-1}+1) \qquad (n<0)$$

$(n<0)$ $\quad x_n = (-2^{-n-1}+1) - 2^{-n-1} = 1$

$(n\geq 0)$ $\quad x_n = 2^{-n-1}$

$(n<0)$ $\quad a_n = 1$

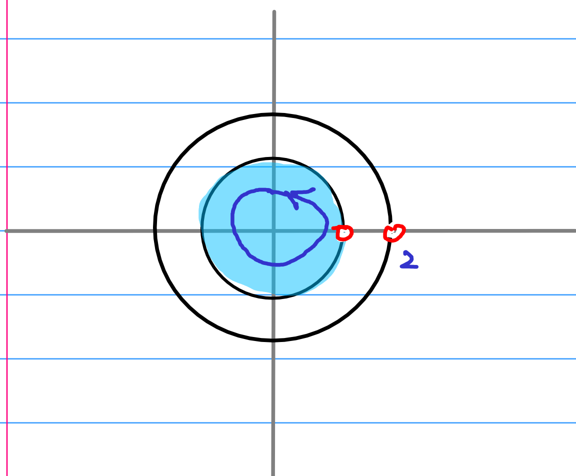
$(n\geq 0)$ $\quad a_n = 2^{-n-1}$

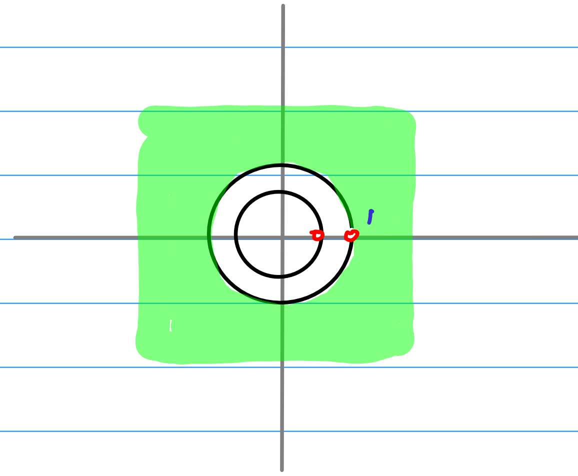
L.S. first

L.S. first

a_n, x_n using Res ← $f(z) = \frac{-1}{(z-1)(z-2)}$

Ⓘ D_1 $|z| < 1$ $\left[\, |\frac{z}{1}| < 1 ,\ |\frac{z}{2}| < 1 \,\right]$





$p = 0$	$\longrightarrow$	∞
$p = 1$	$\longrightarrow$	$p = 1$
$p = 2$	$\longrightarrow$	$p = 0.5$
∞	$\longrightarrow$	$p = 0$

$$a_n = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$x_n = \mathrm{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)}, 0\right)$$

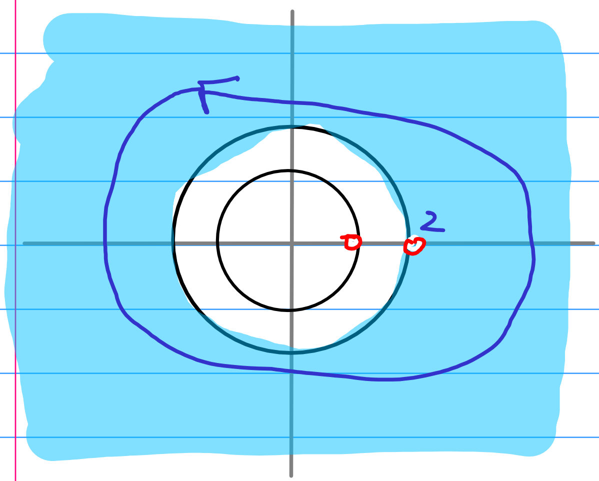
$$= \mathrm{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)}, 1\right)$$

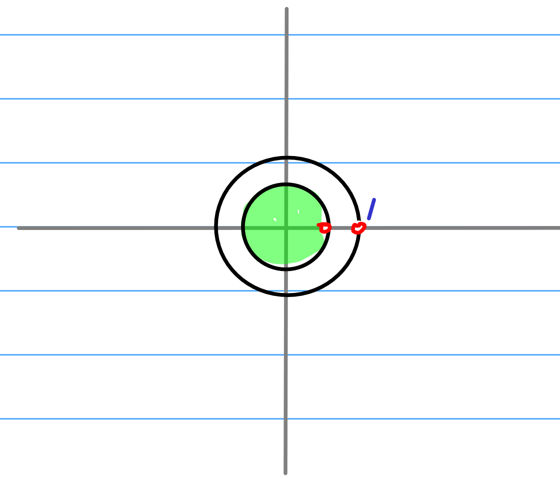
$$+ \mathrm{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)}, \frac{1}{2}\right)$$

L.S. first

a_n, x_n using Res ← $f(z) = \dfrac{-1}{(z-1)(z-2)}$

$\boxed{\text{(II)} \quad D_2 \quad |z| > 2 \quad \left[\,\left|\tfrac{1}{z}\right| < 1, \quad \left|\tfrac{2}{z}\right| < 1\,\right]}$





$p = 0$	$\rightarrow$	∞
$p = 1$	$\rightarrow$	$p = 1$
$p = 2$	$\rightarrow$	$p = 0.5$
∞	$\rightarrow$	$p = 0$

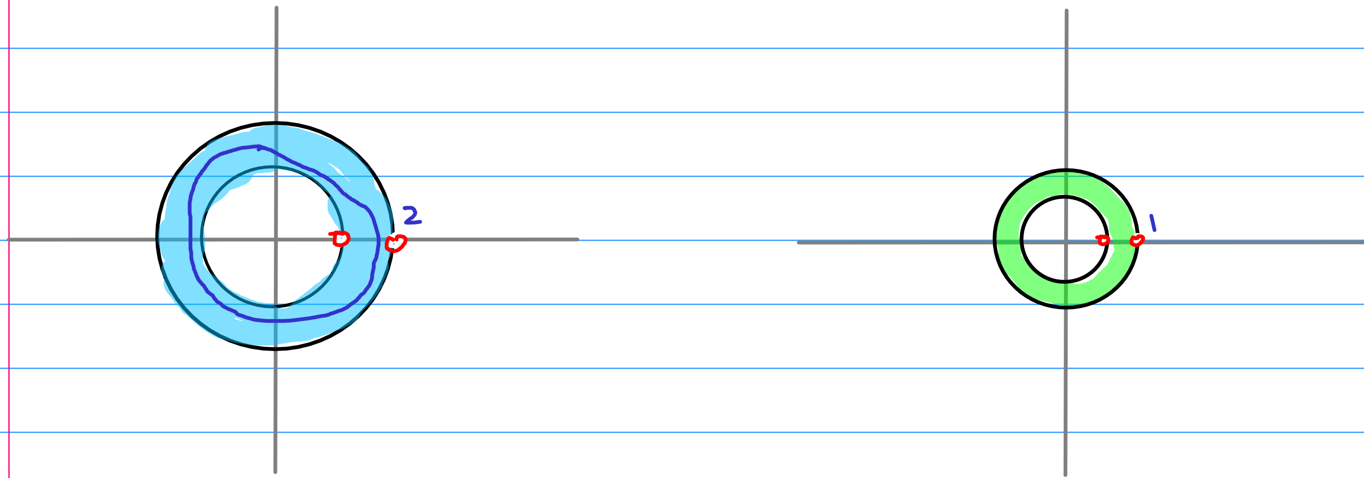
$$a_n = \operatorname{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) + \operatorname{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) + \operatorname{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 2\right)$$

$$x_n = \operatorname{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)}, 0\right)$$

L.S. first

a_n, x_n using **Res** $\Leftarrow$ $f(z) = \dfrac{-1}{(z-1)(z-2)}$

III	D_3 $\quad 1 < \lvert z \rvert < 2$ $\quad \left[\left\lvert \frac{1}{z} \right\rvert < 1 , \quad \left\lvert \frac{z}{2} \right\rvert < 1 \right]$



$p=0$	$\longrightarrow$	∞
$p=1$	$\longrightarrow$	$p=1$
$p=2$	$\longrightarrow$	$p=0.5$
∞	$\longrightarrow$	$p=0$

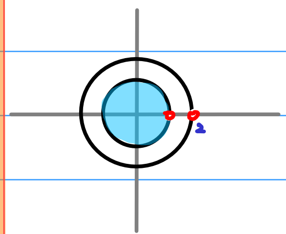
$$a_n = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}} , 0 \right) + \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}} , 1 \right)$$

$$x_n = \mathrm{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)} , 0 \right) + \mathrm{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)} , \frac{1}{2} \right)$$

L.S. first

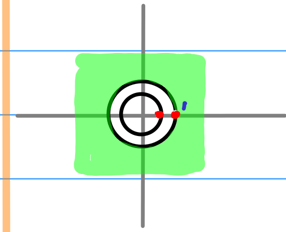
a_n, x_n using Res ← $f(z) = \dfrac{-1}{(z-1)(z-2)}$

(n ≥ 0)



$a_n = -1 + 2^{-n-1}$

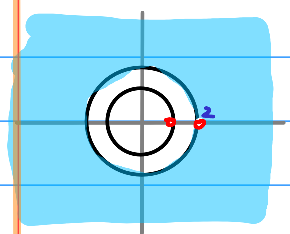
$$a_n = \text{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$



$x_n = -1 + 2^{-n-1}$

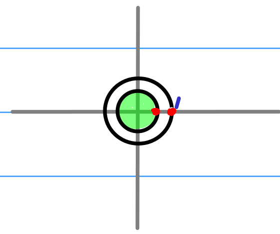
$$x_n = \text{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)}, 1\right) + \text{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)}, \frac{1}{2}\right)$$

(n < 0)



$1 - 2^{-n-1}$

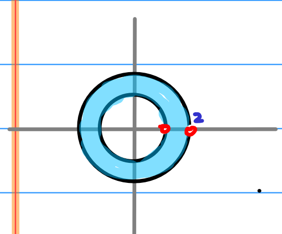
$$a_n = \text{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) + \text{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) + \text{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 2\right)$$



$1 - 2^{-n-1}$

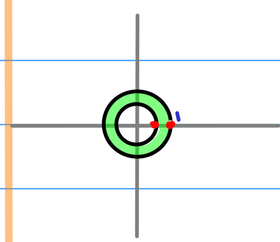
$$x_n = \text{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)}, 0\right)$$

(n ≥ 0)
(n < 0)



$a_n = 2^{-n-1}$
$a_n = 1$

$$a_n = \text{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) + \text{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$



$x_n = 2^{-n-1}$
$x_n = 1$

$$x_n = \text{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)}, 0\right) + \text{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)}, \frac{1}{2}\right)$$

L.S. first