

# Example Random Processes

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Based on

Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles,Jr. and B. Shi

# Outline

1 Gaussian Random Processes

2 Poisson Random Process

# Gaussian Random Process

$N$  Gaussian random variables

## Definition

$$f_X(x_1, \dots, x_N; t_1, \dots, t_N) =$$

$$\frac{\exp\left\{-(1/2)[x - \bar{X}]^t [C_x]^{-1} [x - \bar{X}]\right\}}{\sqrt{(2\pi)^N |[C_x]|}}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \quad \bar{X} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_N \end{bmatrix} \quad [x - \bar{X}] = \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \\ \vdots \\ x_N - \bar{x}_N \end{bmatrix}$$

# The Covariance Matrix (1)

$N$  Gaussian random variables

## Definition

$$\bar{X}_i = E[\textcolor{blue}{X}_i] = E[\textcolor{blue}{X}(\textcolor{brown}{t}_i)]$$

$$\bar{X} = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \vdots \\ \bar{X}_N \end{bmatrix} = \begin{bmatrix} E[\textcolor{blue}{X}_1] \\ E[\textcolor{blue}{X}_2] \\ \vdots \\ E[\textcolor{blue}{X}_N] \end{bmatrix} = \begin{bmatrix} E[\textcolor{blue}{X}(\textcolor{brown}{t}_1)] \\ E[\textcolor{blue}{X}(\textcolor{brown}{t}_2)] \\ \vdots \\ E[\textcolor{blue}{X}(\textcolor{brown}{t}_N)] \end{bmatrix}$$

# The Covariance Matrix (2)

$N$  Gaussian random variables

## Definition

$$\begin{aligned} C_{ik} &= C_{\mathbf{X}_i \mathbf{X}_k} = E[(\mathbf{X}_i - \bar{\mathbf{X}}_i)(\mathbf{X}_k - \bar{\mathbf{X}}_k)] \\ &= E[(\mathbf{X}(\mathbf{t}_i) - E[\mathbf{X}(\mathbf{t}_i)])(\mathbf{X}(\mathbf{t}_k) - E[\mathbf{X}(\mathbf{t}_k)])] \end{aligned}$$

$$\begin{aligned} C_{ik} &= C_{\mathbf{X}_i \mathbf{X}_k} = C_{\mathbf{XX}}(\mathbf{t}_i, \mathbf{t}_k) \\ &= R_{\mathbf{XX}}(\mathbf{t}_i, \mathbf{t}_k) - E[\mathbf{X}(\mathbf{t}_i)]E[\mathbf{X}(\mathbf{t}_k)] \end{aligned}$$

# Stationary Gaussian Process

$N$  Gaussian random variables

## Definition

$$\bar{X}_i = E[\textcolor{blue}{X}_i] = E[\textcolor{blue}{X}(\textcolor{brown}{t}_i)] = \bar{X} = \text{const}$$

$$C_{\textcolor{blue}{XX}}(\textcolor{brown}{t}_i, \textcolor{brown}{t}_k) = C_{\textcolor{blue}{XX}}(t_k - t_i)$$

$$R_{\textcolor{blue}{XX}}(\textcolor{brown}{t}_i, \textcolor{brown}{t}_k) = R_{\textcolor{blue}{XX}}(t_k - t_i)$$

# Jointly Gaussian Process

$N$  Gaussian random variables

## Definition

the two random processes  $X(t)$  and  $Y(t)$

are **jointly Gaussian** if the random variables

$X(t_1), \dots, X(t_N)$  at times  $t_1, \dots, t_N$  for  $X(t)$  and

$Y(t'_1), \dots, Y(t'_M)$  at times  $t'_1, \dots, t'_M$  for  $Y(t)$

are **jointly gaussian** for any  $N$ ,  $t_1, \dots, t_N$ , and  $M$ ,  $t'_1, \dots, t'_M$

## Stationary Gaussian Markov Process

$N$  Gaussian random variables

## Definition

$$C_{\text{XX}}(\tau) = \sigma^2 e^{-\beta|\tau|}$$

$$C_{\text{XX}}[k] = \sigma^2 a^{-|k|}$$

$$a = e^{\beta T_S}$$

# Poisson Random Process

$N$  Gaussian random variables

## Definition

$$p[X(t) = k] = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, 2, \dots$$

$$f_X(x) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x - k)$$

# Poisson Random Process - mean and 2nd moment

$N$  Gaussian random variables

## Definition

$$\begin{aligned} E[X(t)] &= \int_{-\infty}^{\infty} xf_X(x)dx = \int_{-\infty}^{\infty} x \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k) dx \\ &= \sum_{k=0}^{\infty} \frac{k(\lambda t)^k e^{-\lambda t}}{k!} = \lambda t \end{aligned}$$

$$\begin{aligned} E[X^2(t)] &= \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_{-\infty}^{\infty} x^2 \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k) dx \\ &= \sum_{k=0}^{\infty} \frac{k^2(\lambda t)^k e^{-\lambda t}}{k!} = \lambda t(1 + \lambda t) \end{aligned}$$



# Poisson Random Process - joint probability density

$N$  Gaussian random variables

## Definition

$$P[X(t_1) = k_1] = \frac{(\lambda t_1)^{k_1} e^{-\lambda t_1}}{k_1!} \quad k_1 = 0, 1, 2, \dots$$

$$P[X(t_2) = k_2 | X(t_1) = k_1] = \frac{[\lambda(t_2 - t_1)]^{k_2 - k_1} e^{-\lambda(t_2 - t_1)}}{(k_2 - k_1)!}$$

$$P(k_1, k_2) = P[X(t_2) = k_2 | X(t_1) = k_1] \cdot P[X(t_1) = k_1]$$

$$= \frac{(\lambda t_1)^{k_1} [\lambda(t_2 - t_1)]^{k_2 - k_1} e^{-\lambda t_2}}{k_1! (k_2 - k_1)!}$$

$$f_{X_1, X_2}(x_1, x_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=k_1}^{\infty} P(k_1, k_2) \delta(x_1 - k_1) \delta(x_2 - k_2)$$



# Bernoulli Random Process

$N$  Gaussian random variables

## Definition

the Bernoulli random process at sample index  $n$  is  $I[n]$   
the number of events that have occurred  
after sample index 0 and up to  $n$

$$X[n] = \sum_{m=1}^n I[m]$$

The binomial counting process is an example of what is called a sum process, since it can be obtained by summing the values of another random process

# Bernoulli Random Process (2)

$N$  Gaussian random variables

## Definition

the density function for  $X[n]$  is represented by  
a binomial density function

$$f_X(x) = \sum_{k=0}^n P(k) \delta(x - k)$$

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

the mean and the variance of the binomial counting process

$$E[X[n]] = np$$

$$\text{Var}[X[n]] = np(1-p)$$



# Binomial Counting Process

$N$  Gaussian random variables

## Definition

$$f_X(x_1, x_2) = \sum_{k_1=0}^{n_1} \sum_{k_2=k_1}^{n_2} P(k_1, k_2) \delta(x_1 - k_1) \delta(x_2 - k_2)$$

$$P(k_1, k_2) = P[x[n_1] = k_1, x[n_2] = k_2]$$

$$= \binom{n_2 - n_1}{k_2 - k_1} \binom{n_1}{k_1} p^{k_2} (1-p)^{n_2 - k_2}$$

$$P(k) = \frac{(np)^k e^{-np}}{k!} = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$



