Complex Functions (1C)

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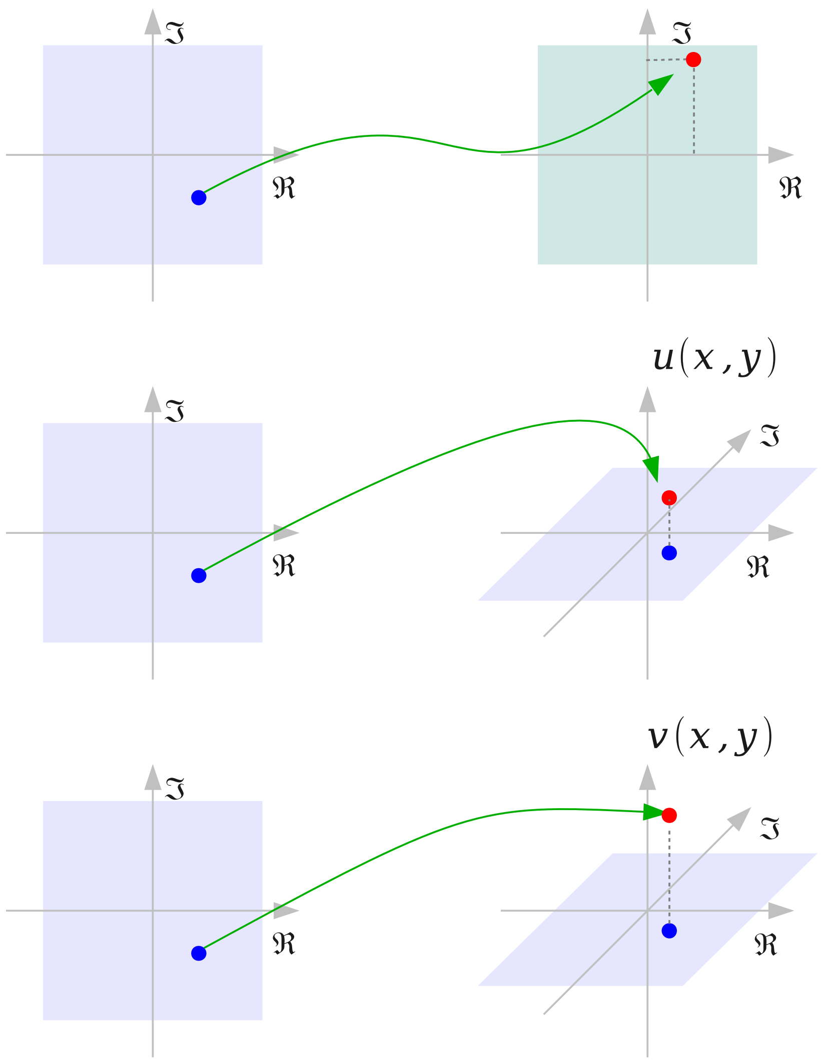
Visualizing Functions of a complex variable

$z = x + i\,y$

$f(z) = u(x,y) + i\,v(u,y)$



Complex Differentiable

z-differentiable
complex-differentiable

$$\frac{df}{dz} = \lim_{\Delta \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

z-derivative
complex-derivative

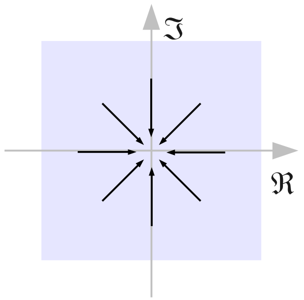
$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

the limist exists
when it has the same value
for any direction of approaching z_0

- linear operator
- the product rule
- the quotient rule
- the chain rule

$f(z)=\bar{z}$ is not complex-differentiable at zero, because the value of $(f(z) - f(0)) / (z - 0)$ is different depending on the approaching direction.

- along the real axis, the limit is 1,
- along the imaginary axis, the limit is −1.
- other directions yield yet other limits.



Complex Differentiable (z-differentiable)

$$f(z) = \Re(z)$$

$$\frac{df}{dz} = \lim_{\Delta \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta \to 0} \frac{\Re(z+\Delta z) - \Re(z)}{\Delta z}$$

$$= \lim_{\Delta \to 0} \frac{\Re(\Delta z)}{\Delta z}$$

real Δz

$\Re(\Delta z) = \Delta z$ $\qquad \frac{df}{dz} = +1$

imaginary Δz

$\Im(\Delta z) = 0$ $\qquad \frac{df}{dz} = +0$

Depends on an approaching path

continuous everywhere
differentiable nowhere

$$f(z) = z^*$$

$$\frac{df}{dz} = \lim_{\Delta \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta \to 0} \frac{(z^* + \Delta z^*) - z^*}{\Delta z^*}$$

$$= \frac{\lim_{\Delta \to 0} \Delta z^*}{\Delta z} = \frac{|a| e^{-i\theta}}{|a| e^{+i\theta}} = e^{-i2\theta}$$

real Δz

$\theta = 0$ $\qquad \frac{df}{dz} = +1$

imaginary Δz

$\theta = \pi/2$ $\qquad \frac{df}{dz} = -1$

Depends on an approaching path

continuous everywhere
differentiable nowhere

Cauchy-Rieman Equations

Cauchy-Rieman Equations: **The necessary and sufficient condition**

if *f(z)* is **differentiable**
in a neighborhood of *z*
(if *df/dz* exists)

⇒

The necessary condition

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

conjugate functions:
u(x,y), v(x,y)
Re *and* ***Im*** *part of a*
differentiable *function f(z)*

⇐

The sufficient condition

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

if C-R eq holds
in a neighborhood of $z = x + iy$

C-R Eq : the necessary condition

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$\Delta z = \Delta x + i \cdot 0$$

$$\Re\left(\frac{df}{dz}\right) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$$

$$\Im\left(\frac{df}{dz}\right) = \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$\frac{df}{dz} = \Re\left(\frac{df}{dz}\right) + i \cdot \Im\left(\frac{df}{dz}\right) = \frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x}$$

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$\Delta z = 0 + i \cdot \Delta y$$

$$\Re\left(\frac{df}{dz}\right) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y}$$

$$\Im\left(\frac{df}{dz}\right) = \lim_{\Delta x \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

$$\frac{df}{dz} = \Re\left(\frac{df}{dz}\right) + i \cdot \Im\left(\frac{df}{dz}\right) = -i\frac{\partial u}{\partial y} + \cdot \frac{\partial v}{\partial y}$$

the **necessary** conditions on *u(x,y)* and *v(x,y)*
if *f(z)* is **differentiable** in a neighborhood of z

if *df/dz* exists,
then this C-R equations necessarily hold.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

C-R Eq : the sufficient condition (1)

$$\Delta z = \alpha + i\beta \qquad \Delta x = \alpha \qquad \Delta y = \beta$$

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$f(z+\Delta z) = u(x+\alpha, y+\beta) + iv(x+\alpha, y+\beta)$$

$$-f(z) = -u(x,y) - iv(x,y)$$

$$\Re\left(\frac{df}{dz}\right) = \lim_{\Delta z \to 0} \frac{u(x+\alpha, y+\beta) - u(x,y)}{\alpha + i\beta}$$

$$\Im\left(\frac{df}{dz}\right) = \lim_{\Delta z \to 0} \frac{v(x+\alpha, y+\beta) - v(x,y)}{\alpha + i\beta}$$

$$f(z+\Delta z) - f(z)$$

$$= [u(x+\alpha, y+\beta) - u(x,y)]$$

$$+ i[v(x+\alpha, y+\beta) - v(x,y)]$$

$$f(z+\Delta z) - f(z) =$$

$$= \left[\alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} + o(\Delta z)\right] + i\left[\alpha\frac{\partial v}{\partial x} + \beta\frac{\partial v}{\partial y} + o(\Delta z)\right]$$

$$= \alpha\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] + \beta\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right] + o(\Delta z)$$

$$u(x+\alpha, y+\beta) - u(x,y) = \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} + o(\Delta z)$$

$$v(x+\alpha, y+\beta) - v(x,y) = \alpha\frac{\partial v}{\partial x} + \beta\frac{\partial v}{\partial y} + o(\Delta z)$$

o(a) : a remainder that goes to zero faster than a (little oh of a)

O(a) : a remainder that goes to zero as fast as a (big Oh of a)

C-R Eq : the sufficient condition (2)

$$f(z+\Delta z) - f(z) =$$

$$= \alpha\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] + \beta\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right] + o(\Delta z)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \Rightarrow \quad \textit{differentiable}$$

$$= \alpha\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] + i\beta\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] + o(\Delta z)$$

$$= [\alpha + i\beta]\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] + o(\Delta z)$$

$$\frac{df}{dz} = \lim_{\Delta z \to 0}\left[\frac{\alpha + i\beta}{\alpha + i\beta}\right]\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]$$

$$f(z+\Delta z) - f(z) =$$

$$= \alpha\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] + \beta\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right] + o(\Delta z)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \Rightarrow \quad \textit{differentiable}$$

$$= \alpha\left[\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right] + i\beta\left[-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right] + o(\Delta z)$$

$$= [\alpha + i\beta]\left[\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right] + o(\Delta z)$$

$$\frac{df}{dz} = \lim_{\Delta z \to 0}\left[\frac{\alpha + i\beta}{\alpha + i\beta}\right]\left[\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right] = \left[\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right]$$

conjugate functions: u(x,y), v(x,y)
***Re** and **Im** part of*
*a **differentiable** function f(z)*

$$\Leftarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

the limit is independent of a path

A Holomorphic Function

a complex-valued function of one or more complex variables that is ***complex differentiable*** in a neighborhood of **every point** in its domain.

sometimes referred to as **regular functions** or as **conformal maps**.

from the Greek ὅλος (holos) meaning "**entire**", and μορφή (morphē) meaning "form" or "appearance".

A holomorphic function whose domain is the whole complex plane is called an **entire function**.

Cauchy's Formula

f(z) is **Holomorphic** → The existence of a complex derivative in a neighborhood → **infinitely differentiable** and equal to its own Taylor series.

Complex Analytic Function

Holomorphic

holomorphic on U ≡ **complex differentiable** **at every point z0** in an open set U

holomorphic at z0 ≡ not just **complex differentiable** at z0 but **complex differentiable** **everywhere** within some neighborhood of z0 in the complex plane.

f(z) is **holomorphic** ⟺ u and v have first partial derivatives with respect to x and y, and satisfy the Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Holomorphic and Analytic

holomorphic functions

a complex-valued function of one or more complex variables that is ***complex differentiable*** in a neighborhood of **every point** in its domain.

The term analytic function is often used interchangeably with "holomorphic function",

analytic function

any function (real, complex, or of more general type) that can be written as a **convergent power series** in a neighborhood of **every point** in its domain.

a function that is locally given by a convergent power series.

analytic ≡ its **Taylor series** about x0 **converges** to the function in a neighborhood of **every** point x0 in its domain.

infinitely differentiable

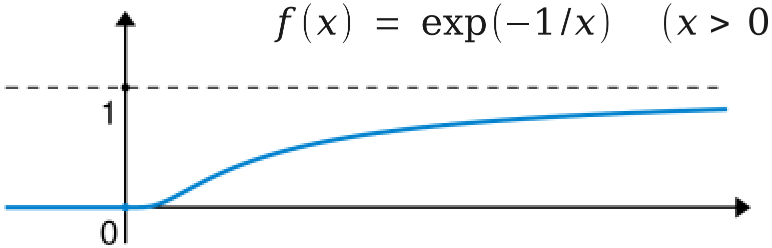
Real & Complex Analytic Functions

analytic ≡ its **Taylor series** about x0 **converges** to the function in a neighborhood of **every** point x0 in its domain.

infinitely differentiable

smooth but not *analytic*

real function $f(x) = 0 \quad (x \le 0)$ $f(x) = \exp(-1/x) \quad (x > 0)$



infinitely differentiable, but

Taylor series not equal at x=0 and x>o

one of the most dramatic differences between real-variable and complex-variable **analysis**.

This cannot occur with complex differentiable functions

all holomorphic functions are analytic

Complex Holomorphic & Analytic Functions

a **complex**-valued function ƒ of a **complex** variable z:

is said to be **holomorphic** at a point **a**
if it is **complex differentiable** at every point
within some open disk centered at **a**, and

is said to be **analytic** at **a**
if in some open disk centered at **a**
it can be expanded as a **convergent power series**

all holomorphic functions are **analytic**

complex **holomorphic** functions	⟺	**complex** **analytic** functions

Complex Analytic Functions

f(z) is **Analytic** (Holomorphic) ⟺ *f(z)* is differentiable in a neighborhood of z (if *df/dz* exists) ⟺ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Cauchy's Integration Formula I: *f(z)* is differentiable in a neighborhood of z (if *df/dz* exists) ⇒ then *f(z)* can be represented by a contour integral

Cauchy's Integration Formula II ⇒ then *f(z)* is **infinitely** differentiable in that neighborhood.

Generalized Cauchy's Formula ⇒ *f(z)* can be expanded in a Taylor series

Cauchy's Integral Formula

Cauchy's Integration Formula **I**

Suppose U is an open subset of the complex plane C,
f : U → C is a holomorphic function and
the closed disk D = { z : | z − z0| ≤ r}
is completely contained in U.
Let Y be the circle forming the boundary of D.
Then for every a in the interior of D:

$$f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)} dz$$

The proof of this formula uses
the **Cauchy integral theorem** and
similarly only requires f to be complex differentiable.

Cauchy's Integration Formula **II**

Since the reciprocal of the denominator of the integrand
in Cauchy's integral formula can be expanded
as a power series in the variable (a − z0),
it follows that holomorphic functions are analytic.
In particular f is actually infinitely differentiable

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{C} \frac{f(z)}{(z-a)^{n+1}} dz$$

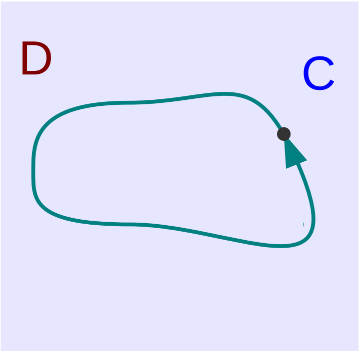
Cauchy's Integral Theorem

If two different paths connect the same two points,
and a function is holomorphic everywhere
"in between" the two paths,
then the two path integrals of the function
will be the same.

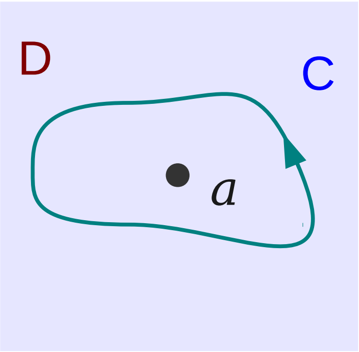
The theorem is usually formulated
for closed paths as follows:
let U be an open subset of C which is simply connected,
let f : U → C be a holomorphic function, and
let Y be a rectifiable path in U
whose start point is equal to its end point.

$$\oint_C f(z)\, dz = 0$$

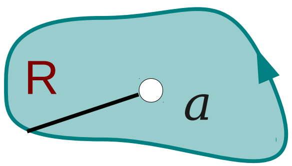
$f(z)$



$f(z)$



$\dfrac{f(z)}{(z-a)}$



Laplace's Equation

f(z) is **Analytic** (Holomorphic) ⟺ *f(z)* is differentiable in a neighborhood of z (if *df/dz* exists) ⟺ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Harmonic

a **harmonic function** is
a twice continuously differentiable function
f : U → R (where U is an open subset of R^n)
which satisfies **Laplace's equation**, i.e.

The solutions of **Laplace's equation** are
the **harmonic functions**,

The general theory of solutions to Laplace's equation is known as potential theory.

the study of oscillations of more general physical objects than strings leads to partial differential equations that are variations of the wave equation.

Harmonic functions then encode the relative amplitudes at different places in the object for each mode of vibration.

fromhttp://math.stackexchange.com/questions/123620/why-are-harmonic-functions-called-harmonic-functions

Laplace's equation

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots \frac{\partial^2 f}{\partial x_n^2} = 0$$

$$\nabla^2 f = 0$$

Poisson's equation

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots \frac{\partial^2 f}{\partial x_n^2} = g(x_{1,} x_{2,} \cdots, x_n)$$

$$\nabla^2 f = g$$

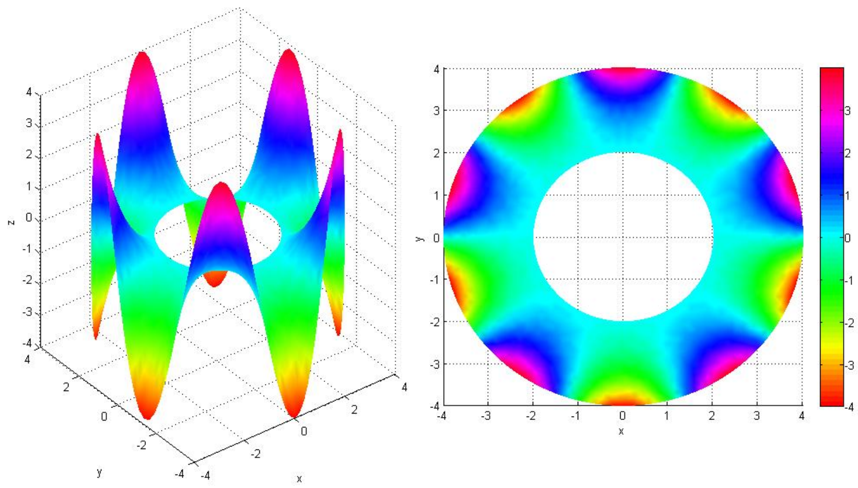
Harmonic

Solutions of Laplace's equation are called
harmonic functions;
they are all analytic within the domain where the equation is satisfied.

If any two functions are solutions to Laplace's equation (or any linear homogeneous differential equation),
their sum (or any linear combination) is also a solution.

the **principle of superposition**

solutions to complex problems
can be constructed by summing simple solutions



Laplace's Equation on an annulus
(inner radius r=2 and outer radius R=4)
with Dirichlet Boundary Conditions:
u(r=2)=0 and u(R=4)=4sin(5*θ)

Cauchy-Rieman Equation

f(z) is **Holomorphic** ⟺ u(x,y), v(x,y) continuous and differentiable and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

f(z) is **Holomorphic** ⟹ *f(z)* is **continuous**

f(z) & *g(z)* are **Holomorphic** ⟹ $c_1 f(z) + c_2 g(z)$, $f(z)g(z)$, $f(z) \circ g(z)$ are **continuous**

f(z) is **Holomorphic** & real value ⟹ *f(z)* is **constant**

f(z) is **Holomorphic** & |*f(z)*| const ⟹ *f(z)* is **constant**

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