# Stationarity

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October 23, 2020

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Based on Probability, Random Variables and Random Signal Principles, P.Z. Peebles, Jr. and B. Shi





# 2 Correlation and Covariance Functions

First Order Stationary *N* Gaussian random variables

#### Definition

if the first order density function does not change with a shift in time origin

 $f_{\mathbf{X}}(x_1;t_1)=f_{\mathbf{X}}(x_1;t_1+\Delta)$ 

must be true for any time  $t_1$  and any real number  $\Delta$  if X(t) is to be a first-order stationary

Consequences of stationarity *N* Gaussian random variables

#### Definition

 $f_X(x, t_1)$  is independent of  $t_1$ 

the process mean value is a constant

$$m_X(t) = \overline{X} = constant$$

#### the process mean value *N* Gaussian random variables

$$m_{X}(t) = \overline{X} = constant$$

$$m_{X}(t_{1}) = \int_{-\infty}^{\infty} xf_{X}(x; t_{1})dx$$

$$m_{X}(t_{2}) = \int_{-\infty}^{\infty} xf_{X}(x; t_{2})dx$$
let  $t_{2} = t_{1} + \Delta$ 

$$m_{X}(t_{1}) = m_{X}(t_{1} + \Delta)$$

Second-Order Stationary Process *N* Gaussian random variables

#### Definition

if the second order density function does not change with a shift in time origin

 $f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta)$ 

must be true for any time  $t_1$ ,  $t_2$  and any real number  $\Delta$  if X(t) is to be a second-order stationary

Auto-correlation function

$$R_{XX}(t,t+\tau) = E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

 $N^{th}$ -order Stationary Processes N Gaussian random variables

#### Definition

if the second order density function does not change with a shift in time origin

$$f_X(x_1,\cdots,x_N;t_1,\cdots,t_N)=f_X(x_1,\cdots,x_N;t_1+\Delta,\cdots,t_N+\Delta)$$

must be true for any time  $t_1, ..., t_N$  and any real number  $\Delta$  if X(t) is to be a second-order stationary

Wide Sense Stationary Process *N* Gaussian random variables

#### Definition

 $m_X(t) = \overline{X} = constant$ 

 $E[X(t)X(t+\tau)] = R_{XX}(\tau)$ 

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# The properties of autocorrelation functions (1) *N* Gaussian random variables

#### Definition

 $|R_{XX}(\tau)| \leq R_{XX}(0)$ 

 $R_{XX}(-\tau) = R_{XX}(\tau)$ 

 $R_{XX}(0) = E\left[X^2(t)\right]$ 

$$P[|X(t+\tau) - X(t)| > \varepsilon] = \frac{2}{\varepsilon^2} (R_{XX}(0) - R_{XX}(\tau))$$

# The properties of autocorrelation functions (2) *N* Gaussian random variables

#### Definition

if  $X(t) = \overline{X} + N(t)$  where N(t) is WSS, is zero-mean, and has autocorrelation function  $R_{NN}(\tau) \to 0$  as  $|\tau| \to \infty$ , then

$$\lim_{\tau|\to\infty}R_{XX}(\tau)=\overline{X}^2$$

if X(t) is mean square periodic, i.e, there exists a  $T \neq 0$ such that  $E[(X(t+T) - X(t))^2] = 0$  for all t, then  $R_{XX}(t)$  will have a **periodic** component with the same period  $R_{XX}(\tau)$  cannot have an arbitrary shape

Crosscorrelation functions (1)*N* Gaussian random variables

#### Definition

 $R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$   $R_{XY}(t, t + \tau) = E[X(t)Y(t + \tau)] = R_{XY}(\tau)$ if  $R_{XY}(t, t + \tau) = 0$ 

then X(t) and Y(t) are called **orthogonal processes** 

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Crosscorrelation functions (2) N Gaussian random variables

#### Definition

if X(t) and Y(t) are statistically independent

$$R_{XY}(t,t+\tau) = E[X(t)Y(t+\tau)] = m_X(t)m_Y(t+\tau)$$

if X(t) and Y(t) are stistically independent and are at least WSS,

$$R_{XY}(\tau) = \overline{XY}$$

which is constant

# The properties of crosscorrelation functions (1) *N* Gaussian random variables

#### Definition

$$R_{XY}(\tau) = R_{XY}(-\tau)$$

$$|R_{XY}(\tau)| = \sqrt{R_{XX}(0)R_{YY}(0)}$$

$$|R_{XY}(\tau)| \leq \frac{1}{2} [R_{XX}(0) + R_{YY}(0)]$$

# The properties of crosscorrelation functions (2) *N* Gaussian random variables

#### Definition

$$R_{\mathbf{Y}\mathbf{X}}(-\tau) = E\left[\mathbf{Y}(t)\mathbf{X}(t-\tau)\right] = E\left[\mathbf{Y}(s+\tau)\mathbf{X}(s)\right] = R_{\mathbf{X}\mathbf{Y}}(\tau)$$

$$E\left[\{\mathbf{Y}(t+\tau)+\alpha X(t)\}^2\right]\geq 0$$

the **geometric mean** of two positive numbers cannot exceed their **arithmetic mean** 

# The properties of crosscorrelation functions (3) *N* Gaussian random variables

#### Definition

$$\begin{aligned} |R_{XY}(\tau)| &\leq \frac{1}{2} \left[ R_{XX}(0) + R_{YY}(0) \right] \\ \sqrt{R_{XX}(0)R_{YX}(0)} &\leq \frac{1}{2} \left[ R_{XX}(0) + R_{YY}(0) \right] \end{aligned}$$

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### Covariance Functions *N* Gaussian random variables

# Definition

$$C_{XX}(t,t+\tau) = E\left[\left\{X(t) - m_X(t)\right\} \left\{X(t+\tau) - m_X(t+\tau)\right\}\right]$$
  

$$C_{XY}(t,t+\tau) = E\left[\left\{X(t) - m_X(t)\right\} \left\{Y(t+\tau) - m_Y(t+\tau)\right\}\right]$$
  

$$C_{XX}(t,t+\tau) = R_{XX}(t,t+\tau) - m_X(t)m_X(t+\tau)$$
  

$$C_{XY}(t,t+\tau) = R_{XY}(t,t+\tau) - m_X(t)m_Y(t+\tau)$$
  
at least jointly WSS

$$C_{XX}(\tau) = R_{XX}(\tau) - \overline{X}^2$$

$$C_{XY}(\tau) = R_{XY}(\tau) - \overline{XY}$$

# The properties of covariance functions *N* Gaussian random variables

#### Definition

For a WSS process, variance does not depend on time and if  $\tau=0$ 

$$C_{XX}(0) = R_{XX}(0) - \overline{X}^2$$

$$\sigma_{\mathbf{X}}^{2} = E\left[\left\{\mathbf{X}(t) - E\left[\mathbf{X}(t)\right]\right\}^{2}\right] = C_{\mathbf{X}\mathbf{X}}(0)$$

it the two random processes uncorrelated

$$C_{XY}(t,t+\tau) = R_{XY}(t,t+\tau) - m_X(t)m_Y(t+\tau) = 0$$

$$R_{XY}(t,t+\tau) = m_X(t)m_Y(t+\tau)$$

# Discrete-Time Processes and Sequences (1) *N* Gaussian random variables

#### Definition

 $m_X[n] = \overline{X}, m_Y[n] = \overline{Y}$ 

 $R_{XX}[n, n+k] = R_{XX}[k]$  $R_{YY}[n, n+k] = R_{YY}[k]$ 

$$C_{XX}[n, n+k] = R_{XX}[k] - \overline{X}^{2}$$
$$C_{YY}[n, n+k] = R_{YY}[k] - \overline{Y}^{2}$$

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# Discrete-Time Processes and Sequences (2) *N* Gaussian random variables

#### Definition

 $m_X[n] = \overline{X}, m_Y[n] = \overline{Y}$ 

 $R_{XY}[n, n+k] = R_{XY}[k]$ 

 $R_{\mathbf{Y}\mathbf{X}}[\mathbf{n},\mathbf{n}+\mathbf{k}] = R_{\mathbf{Y}\mathbf{X}}[\mathbf{k}]$ 

 $C_{XY}[n, n+k] = R_{XY}[k] - \overline{XY}$  $C_{YX}[n, n+k] = R_{YX}[k] - \overline{YX}$ 

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