Sampling Basics (1B)

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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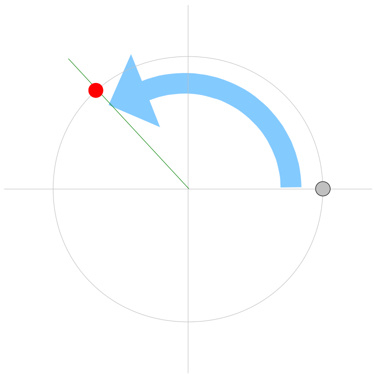
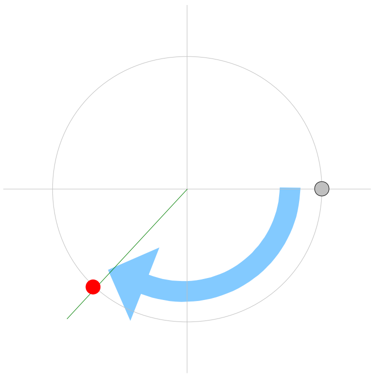
Measuring Rotation Rate

Angular Speed (Frequency)

$$\omega = \frac{2\pi}{T} = 2\pi f$$





$+\omega_0 \; rad \; / \; 1 \; sec$ $-\omega_0 \; rad \; / \; 1 \; sec$

$+\omega_0 \; (rad/sec)$ $-\omega_0 \; (rad/sec)$

RPM

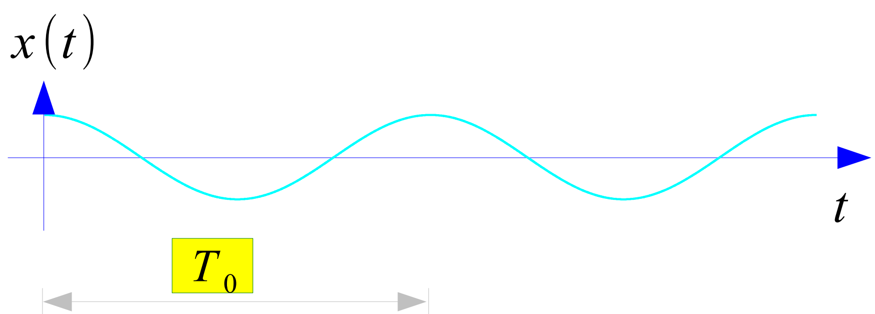
$$rpm = revolutions \; / \; minute$$

$$1 \; rpm = 2\pi \; rad \; / \; 1 \; min$$
$$= 2\pi \; rad \; / \; 60 \; sec$$
$$= \frac{\pi}{30} \; rad/sec$$

- **Negative Angles**

Angular Frequency and Sinusoid

Time Domain

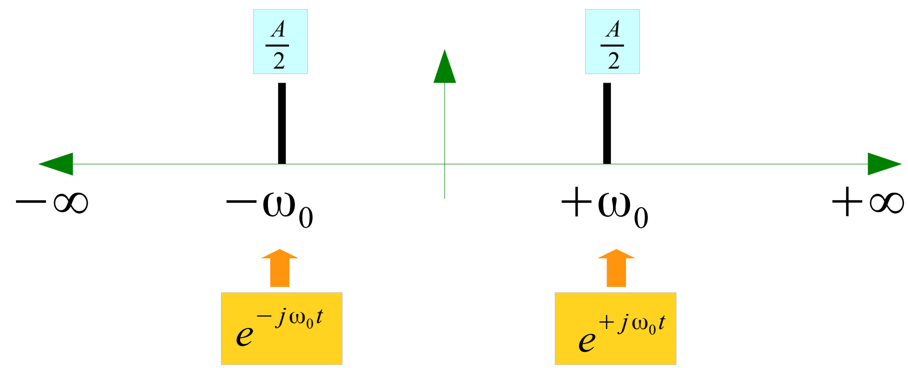


$$\omega_0 = \frac{2\pi}{T_0}$$

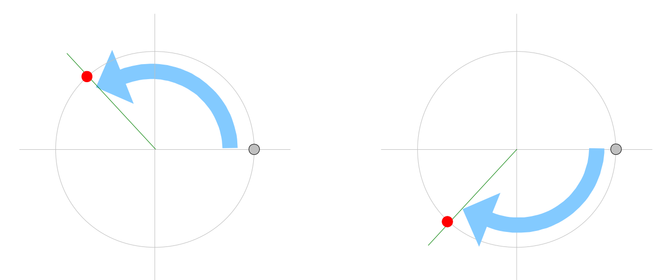
$$x(t) = A\cos(\omega_0 t)$$

$$= \frac{A}{2}e^{+j\omega_0 t} + \frac{A}{2}e^{-j\omega_0 t}$$

Frequency Domain ***spectrum***



For 1 second *For 1 second*



$+\omega_0 \ (rad/sec)$ $-\omega_0 \ (rad/sec)$

Angular Speed and Frequency

$$\omega = \frac{2\pi}{T} = 2\pi f$$

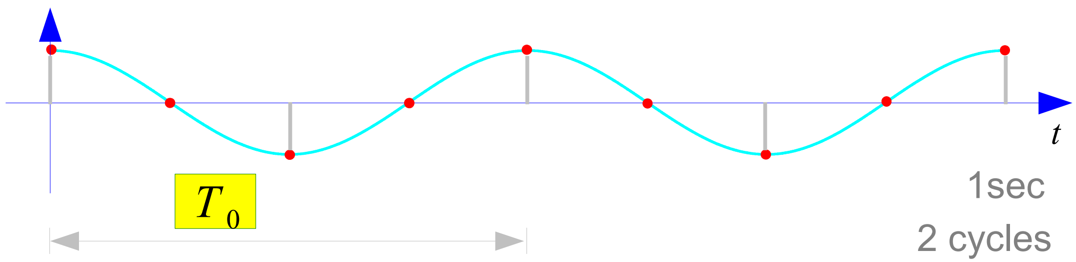
$$\frac{1}{T} = f$$

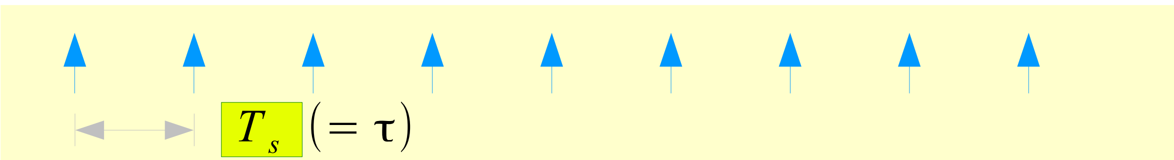
T (sec)	0.01 sec	0.1 sec	1 sec	10 sec	100 sec	
f (Hz)	100 Hz	10 Hz	1 Hz	0.1 Hz	0.01 Hz	**Frequency**
ω (rad/sec)	200π (rad/sec)	20π (rad/sec)	2π (rad/sec)	0.2π (rad/sec)	0.02π (rad/sec)	**Angular Speed / Radian Frequency**
(rad/sec)	= 628	= 62.8	= 6.28	= 0.628	=0.0628	

Discrete Time Sequence

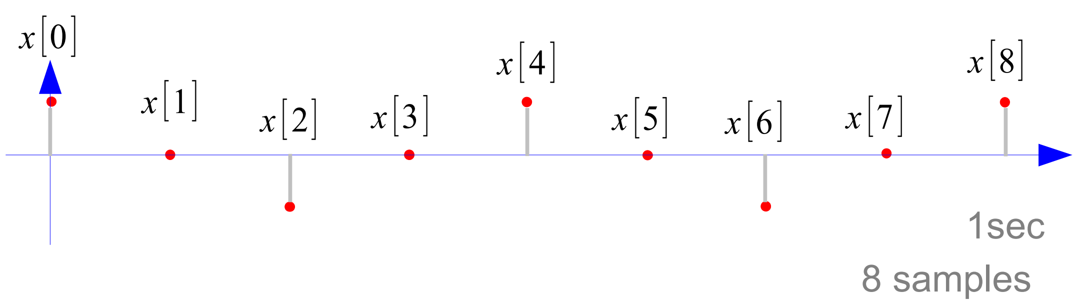
continuous-time signals

$$x(t) = A\cos(\omega_0 t)$$





discrete-time sequence



Sampling Time

$$T_s \; (= \tau)$$

Sequence Time Length

$$T = N \cdot T_s$$

Sampling Frequency

$$f_s = \frac{1}{T_s}$$ ***(samples/sec)***

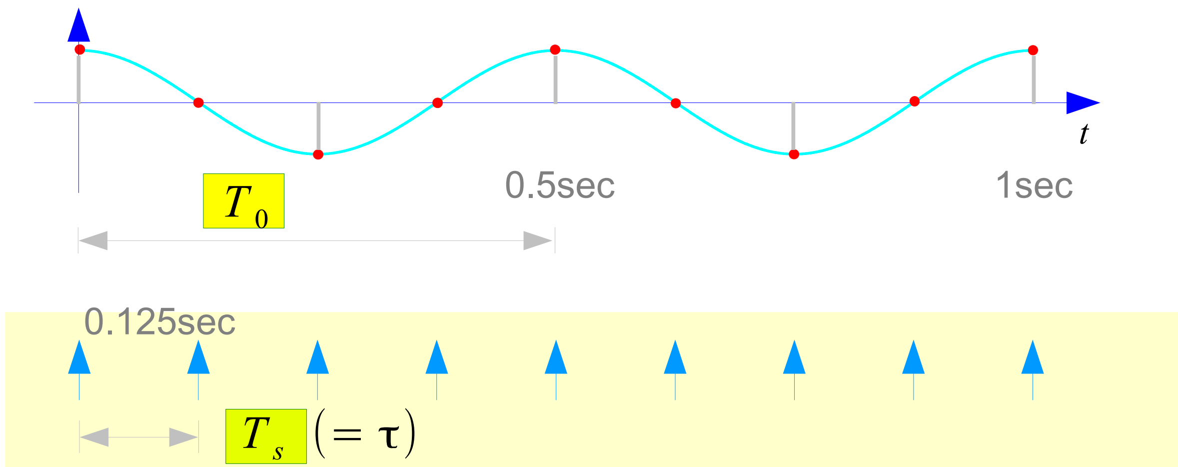
Signal's Frequency

$$f_0 = \frac{1}{T_0}$$ **(cycles/sec)**

Sampling Continuous Time Signal

continuous-time signals

$$x(t) = A\cos(\omega_0 t)$$



For 1 second	
$\dfrac{1}{T_0}$	**_(cycles / sec)_**
For 1 cycle	
1 **_(cycles)_** / T_0 **_(sec)_**	

For 1 second	
$\dfrac{1}{T_s}$	**_(samples / sec)_**
For 1 sample	
1 **_(samples)_** / T_s **_(sec)_**	

Sampling Time

$$T_s \; (= \tau)$$

Sequence Time Length

$$T = N \cdot T_s$$

Sampling Frequency

$$f_s = \frac{1}{T_s} \quad \textbf{(samples / sec)}$$

Signal's Frequency

$$f_0 = \frac{1}{T_0} \quad \textbf{(cycles / sec)}$$

Young Won Lim
10/5/13

Angular Frequencies in Sampling

continuous-time signals

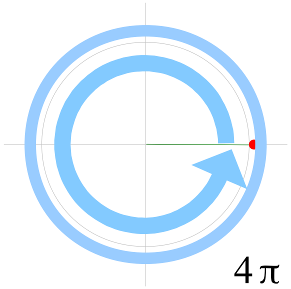
For 1 second

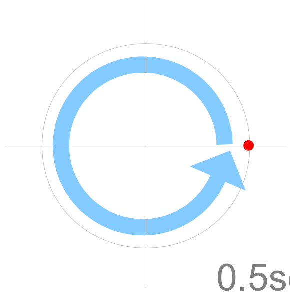
$\omega_0 = 2\pi f_0 \ (rad/sec)$

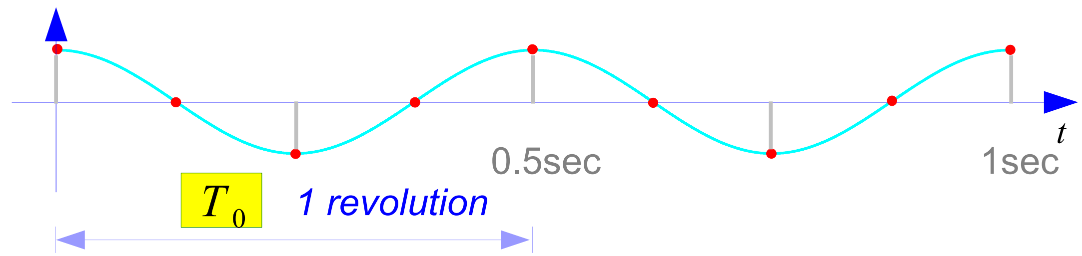
For 1 revolution

$2\pi\ (rad)\ /\ \boxed{T_0\ (sec)}$

$x(t) = A\cos(\omega_0 t)$







2 cycles, 2 revolutions

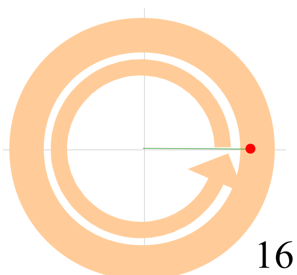
sampling sequence

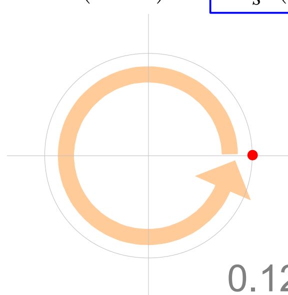
For 1 second

$\omega_s = 2\pi f_s \ (rad/sec)$

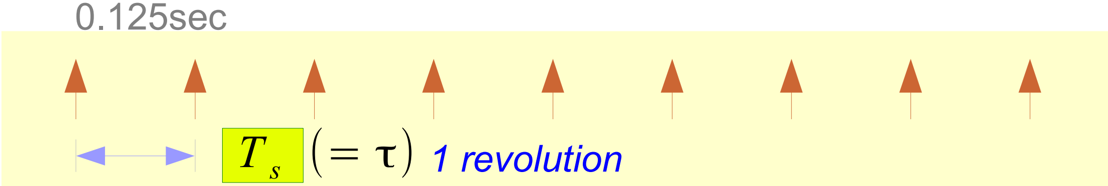
For 1 revolution

$2\pi\ (rad)\ /\ \boxed{T_s\ (sec)}$





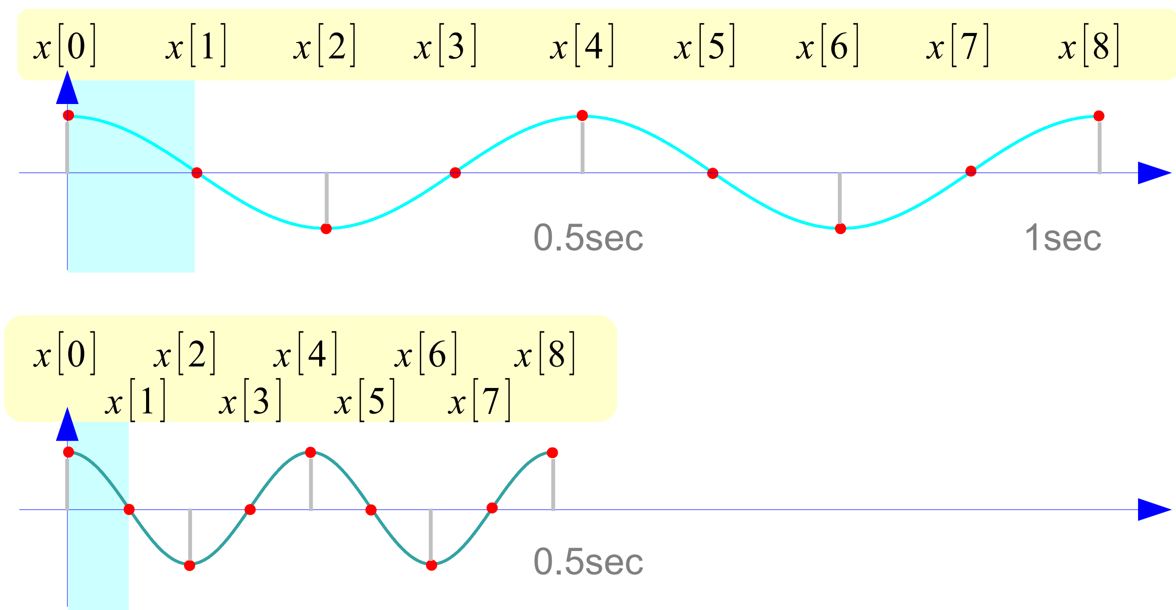
0.125sec



8 cycles, 8 revolutions, 8 samples

Dimensionless Sequence

$$x[n] \Rightarrow \cdots, \ x[0], \ x[1], \ x[2], \ x[3], \ x[4], \ x[5], \ x[6], \ x[7], \ x[8], \ \cdots$$



0.25 cycle / sample

the same normalized frequency

the same sequence

0.25 cycle / sample

<u>Infinite number</u> of continuous time signals

Sampling →

The <u>same</u> discrete-time sequence

Sampling of Sinusoid Functions

$$x(t) = A\cos(\omega t + \phi)$$

$$t \rightarrow n\,T_s$$

$$x[n] = x(n\,T_s)$$

$$= A\cos(\omega \cdot n\,T_s + \phi)$$

$$= A\cos(\omega \cdot T_s\, n + \phi)$$

$$= A\cos(\hat{\omega} \cdot n + \phi)$$

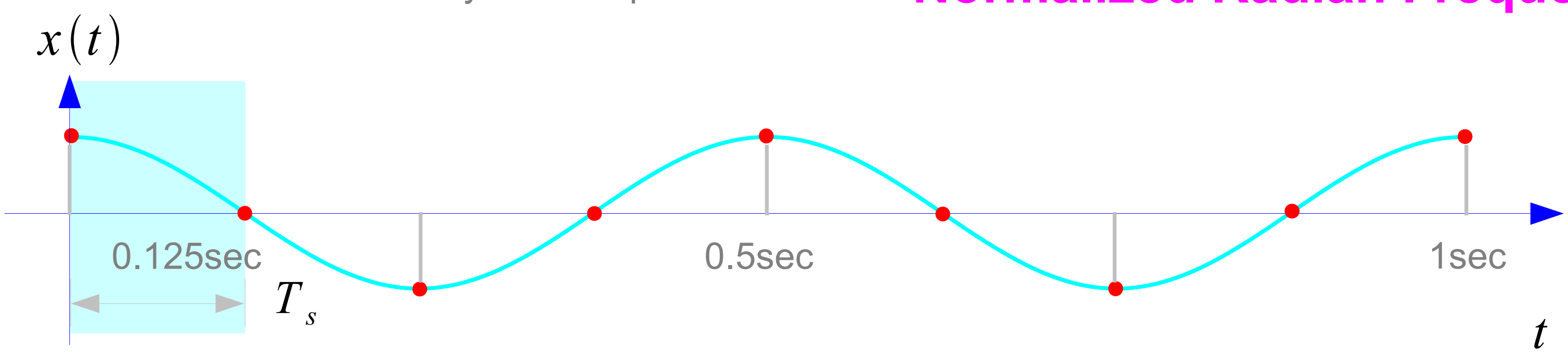
$$\hat{\omega} = \omega \cdot T_s = \frac{\omega}{1/T_s}$$

$$\hat{\omega} = \frac{\omega}{f_s} = 2\pi \frac{f}{f_s}$$

Normalized to f_s

Normalized Radian Frequency

0.25 cycle / sample



Normalized Radian Frequency

continuous-time signals	Sampling	*discrete-time sequence*
$x(t)$	$t \rightarrow n\,T_s$	$x[n] = x(nT_s)$

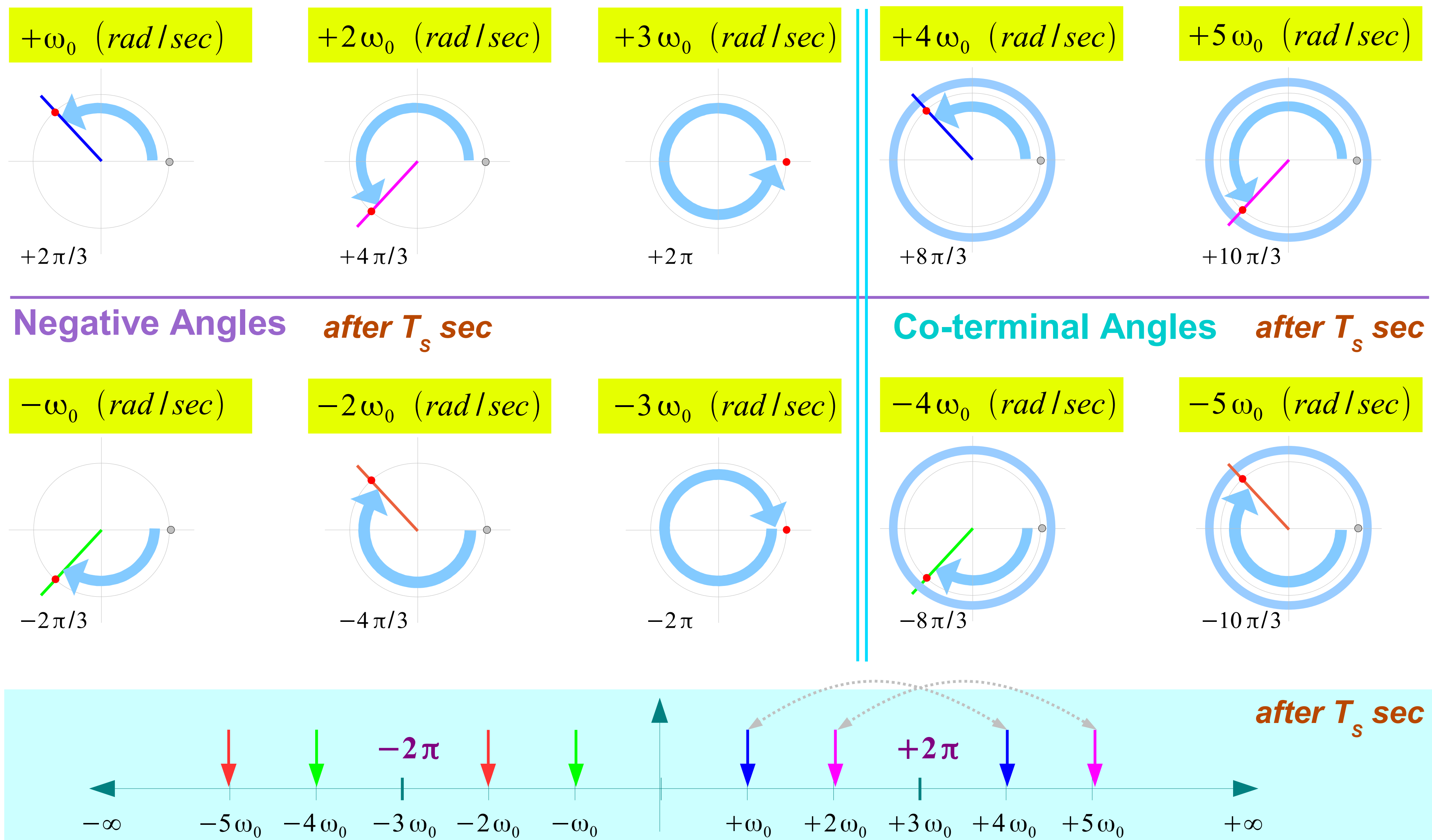
Angular Frequency	Sampling	Normalized Radian Frequency
$\omega \quad (rad/sec)$	$\times\, T_s$	$\hat{\omega} = \omega \cdot T_s \quad (rad/sample)$

Angular Speed X Sampling Time

Normalized Radian Frequency
can be viewed as
"the <u>angular displacement</u> of a signal
during the period of its <u>sample time</u> T_s*"*

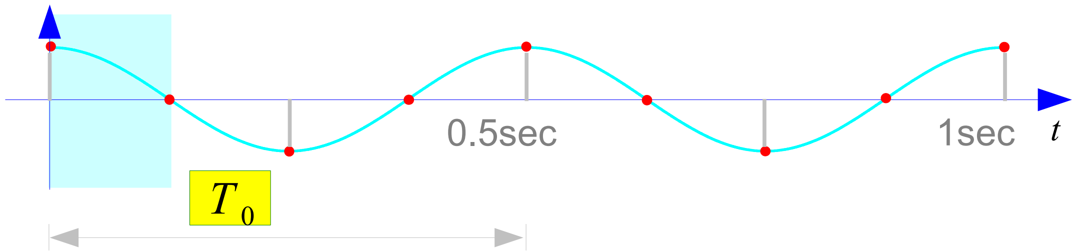
- Negative Angles
  $\rightarrow$ folding
- Co-terminal Angles
  $\rightarrow$ periodic

Co-terminal Angles



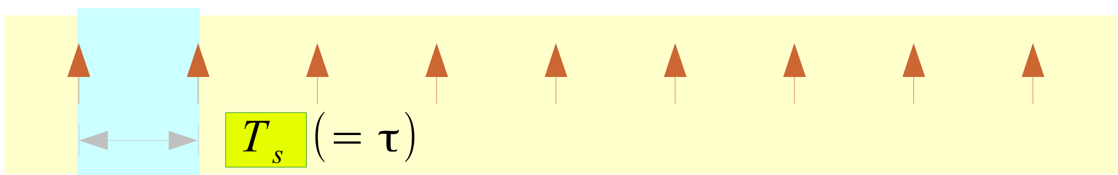
Normalized Radian Frequency Example

$$x(t) = A\cos(\omega_0 t)$$



$$T_0$$

$$\omega_0 = 2\pi f_0 \qquad f_0 = \frac{1}{T_0}$$



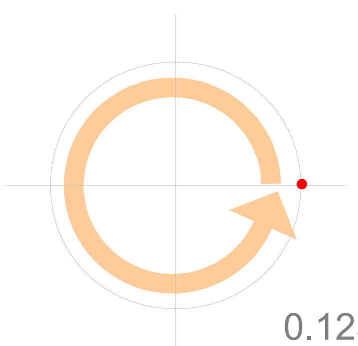
$$T_s \; (= \tau)$$

$$\omega_s = 2\pi f_s \qquad f_s = \frac{1}{T_s}$$

sampling sequence

For 1 sample

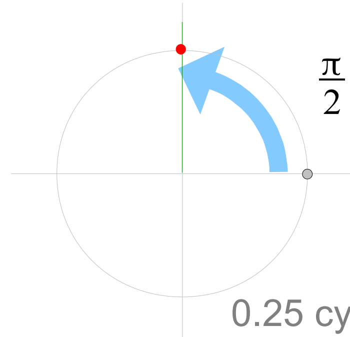
$$2\pi\,(rad) \;/\; T_s\,(sec)$$



continuous-time signals

For T_s second

$$\hat{\omega} = \omega_0 \cdot T_s \quad (rad/sample)$$



$$\hat{\omega} = \omega\, T_s$$

$$\hat{\omega} = \frac{\omega}{f_s}$$

Signal's relative angle position after each of T_s second

Normalized Frequency

Normalized Radian Frequency

$$2\pi \frac{(rad)}{(cycle)} \cdot \frac{f_0}{f_s} \frac{(cycle/sec)}{(sample/sec)} \Rightarrow \frac{\omega_0}{f_s} \; (rad\ /\ sample)$$

$$\hat{\omega} = \frac{\omega}{f_s} = 2\pi \frac{f}{f_s}$$

$$\hat{\omega} = \omega \cdot T_s = \frac{\omega}{1/T_s}$$

Normalized Frequency

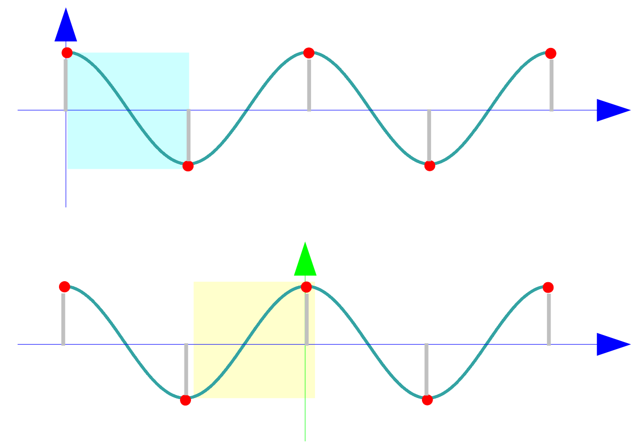
$$\frac{f_0}{f_s} \frac{(cycle\ /\ sec)}{(sample\ /\ sec)} \Rightarrow \frac{f_0}{f_s} \; (cycle\ /\ sample)$$

Normalized Radian Frequency (4)

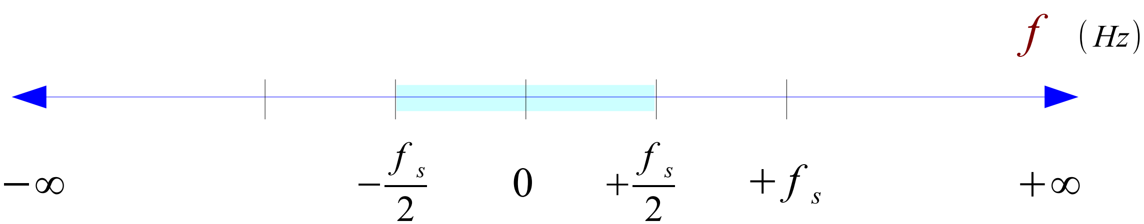
$$Consider \quad f \in \left[-\frac{f_s}{2}, \quad +\frac{f_s}{2} \right)$$

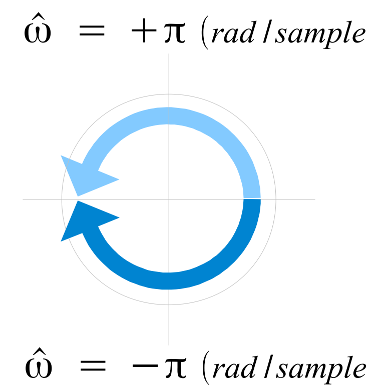
$$\Rightarrow \quad \frac{f}{f_s} \in \left[-\frac{1}{2}, \quad +\frac{1}{2} \right)$$

$$\Rightarrow \quad \hat{\omega} \in \left[-\pi, \quad +\pi \right)$$

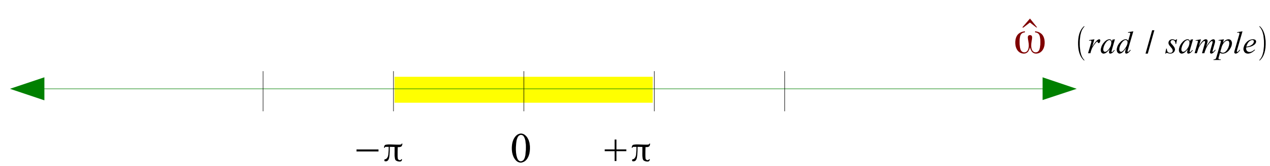


Linear Frequency





Normalized Radian Frequency



Negative Angular Speed Example

$$\omega_s = 2\pi f_s \ (rad/sec)$$

$$2\pi\,(rad)\ /\ T_s\,(sec)$$

$$\hat{\omega}_i = \omega_i \cdot T_s \ (rad/sample)$$

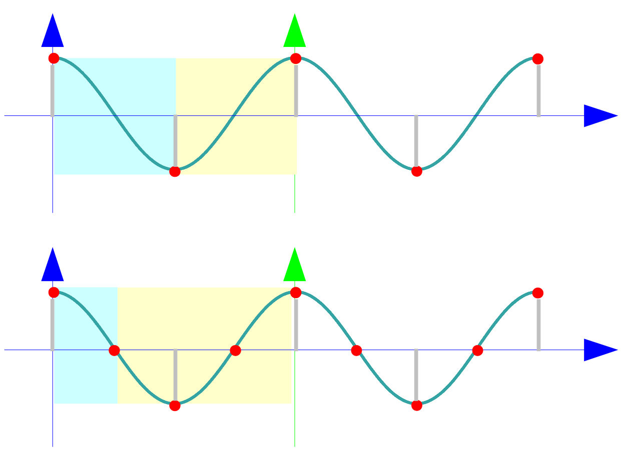
$A\cos(\omega_1 t) = A\cos(+\frac{\omega_s}{2}t) \Rightarrow A\cos(+\pi n)$ $\qquad \hat{\omega}_1 = +\pi\ (rad)$

$A\cos(\omega_2 t) = A\cos(-\frac{\omega_s}{2}t) \Rightarrow A\cos(-\pi n)$ $\qquad \hat{\omega}_2 = -\pi\ (rad)$

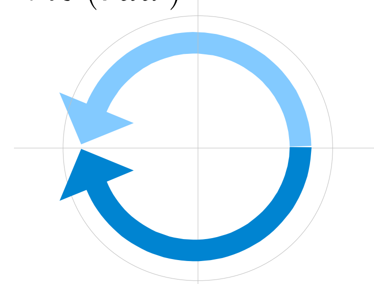
$A\cos(\omega_3 t) = A\cos(+\frac{\omega_s}{4}t) \Rightarrow A\cos(+\frac{\pi}{2}n)$ $\qquad \hat{\omega}_3 = +\frac{\pi}{2}\ (rad)$

$A\cos(\omega_4 t) = A\cos(-\frac{3\omega_s}{4}t) \Rightarrow A\cos(\frac{-3\pi}{2}n)$ $\qquad \hat{\omega}_4 = -\frac{3\pi}{4}\ (rad)$

Negative Angles

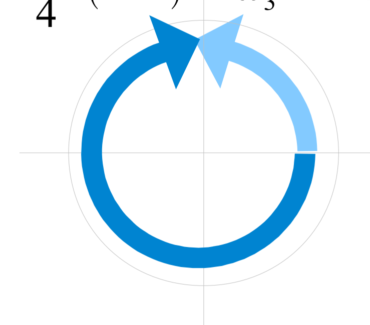


$\hat{\omega}_1 = +\pi\ (rad)$



$\hat{\omega}_2 = -\pi\ (rad)$

$\hat{\omega}_4 = -\frac{3\pi}{4}\ (rad)$ $\qquad \hat{\omega}_3 = +\frac{\pi}{2}\ (rad)$



Co-terminal Angular Speed Example

$$\omega_s = 2\pi f_s \quad (rad/sec)$$

$$2\pi\,(rad) \;/\; T_s\,(sec)$$

$$\hat{\omega}_i = \omega_i \cdot T_s \quad (rad/sample)$$

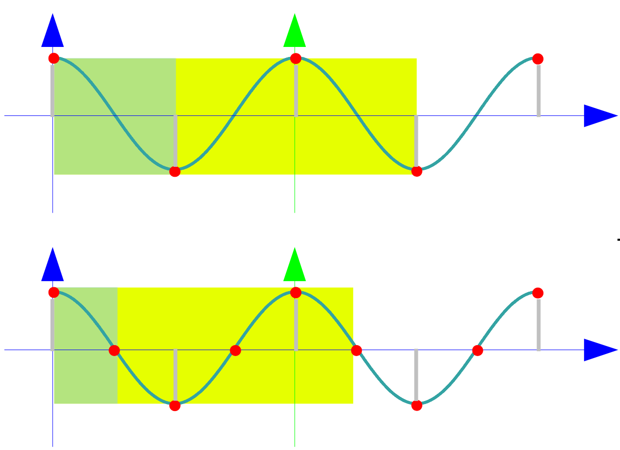
$$A\cos(\omega_1 t) = A\cos\left(+\frac{\omega_s}{2}t\right) \Rightarrow A\cos(+\pi n) \qquad \hat{\omega}_1 = +\pi\,(rad)$$

$$A\cos(\omega_2 t) = A\cos\left(+\frac{3\omega_s}{2}t\right) \Rightarrow A\cos(+\pi n) \qquad \hat{\omega}_2 = +\pi\,(rad)$$

$$A\cos(\omega_3 t) = A\cos\left(+\frac{\omega_s}{4}t\right) \Rightarrow A\cos\left(+\frac{\pi}{2}n\right) \qquad \hat{\omega}_3 = +\frac{\pi}{2}\,(rad)$$

$$A\cos(\omega_4 t) = A\cos\left(+\frac{5\omega_s}{4}t\right) \Rightarrow A\cos\left(+\frac{\pi}{2}n\right) \qquad \hat{\omega}_4 = +\frac{\pi}{2}\,(rad)$$

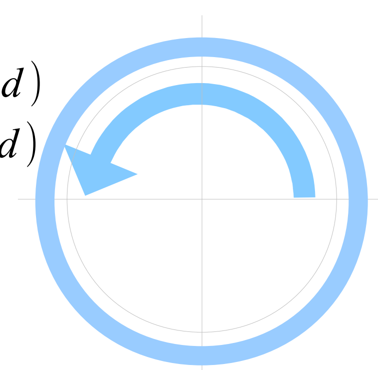
Co-terminal Angles



$$+\frac{3\omega_s}{2} = +\frac{\omega_s}{2} + \omega_s$$

$$+\pi\,(rad)$$

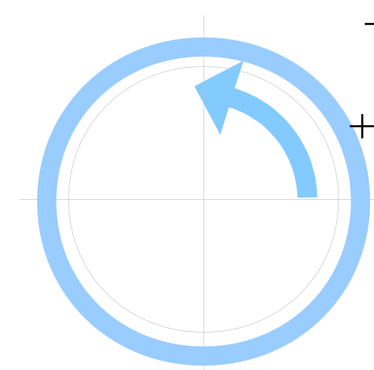
$$+3\pi\,(rad)$$



$$+\frac{5\omega_s}{4} = +\frac{\omega_s}{4} + \omega_s$$

$$+\frac{\pi}{2}\,(rad)$$

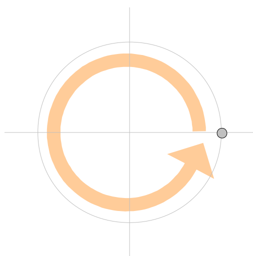
$$+\frac{5\pi}{2}\,(rad)$$



Co-terminal Angles (1)

For 1 sample

$2\pi\ (rad)\ /\ T_s\ (sec)$

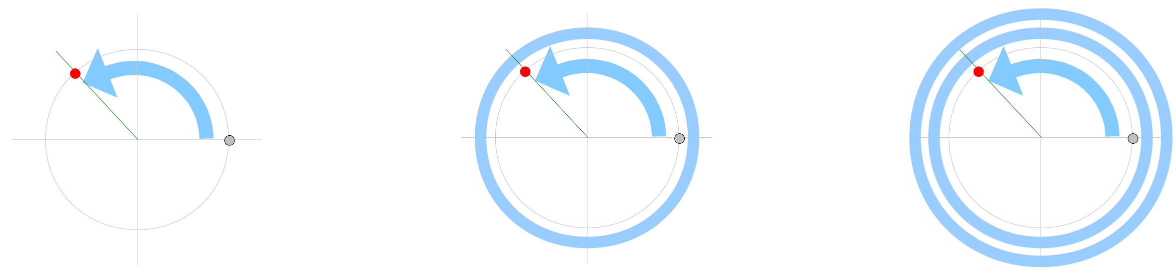


$\hat{\omega} = \omega \cdot T_s \quad (rad/sample)$

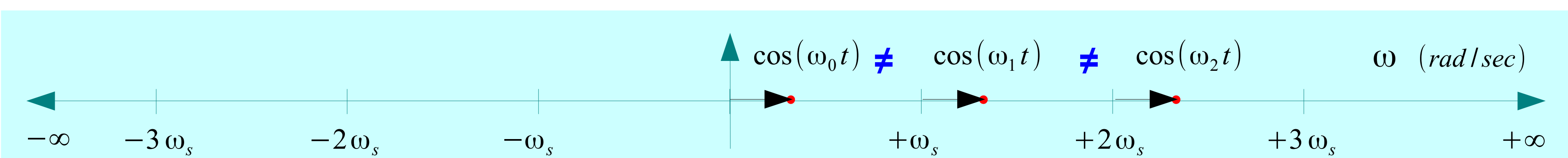
$= \omega / f_s \quad (rad/sample)$

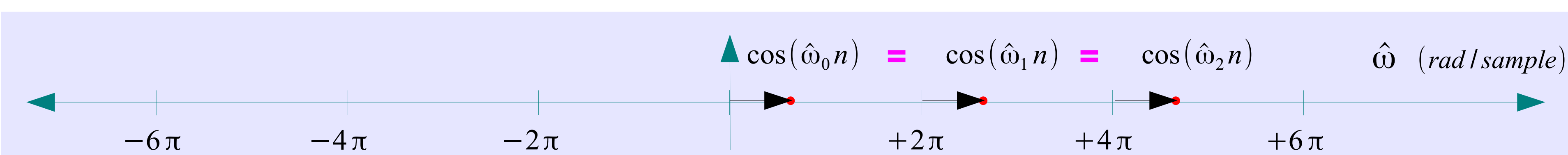
For T_s second

$\hat{\omega} = \omega \cdot T_s \quad (rad/sample)$

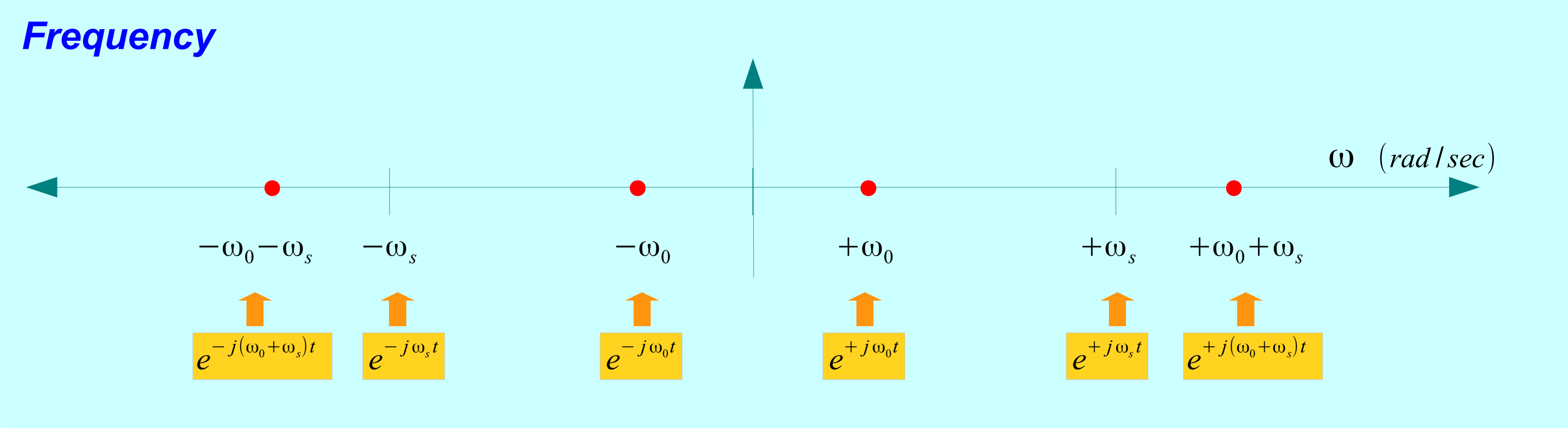


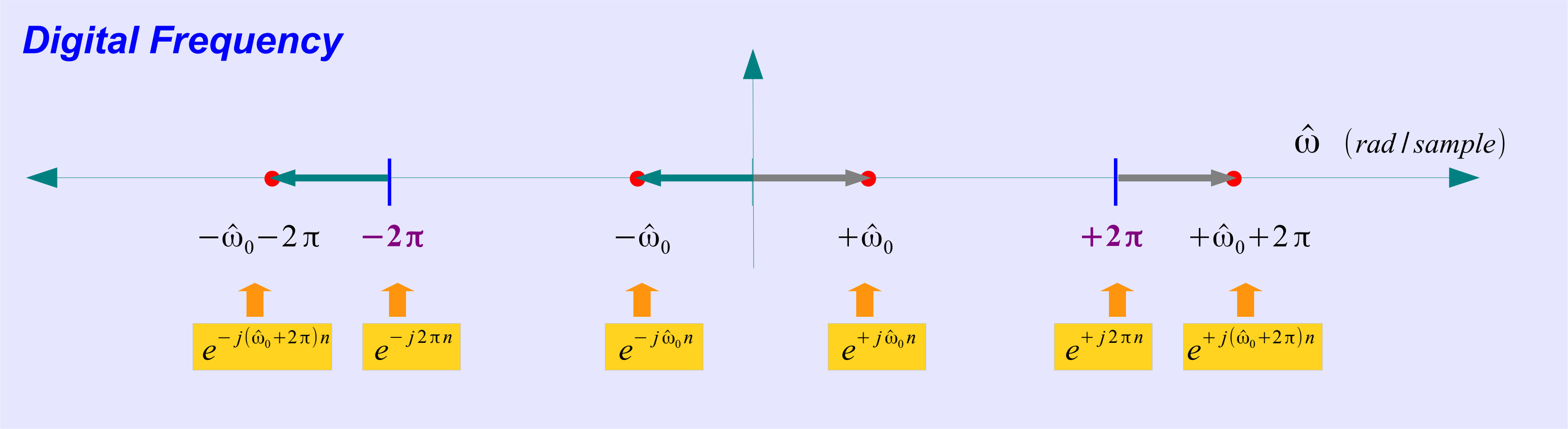
f_0	$f_0 + f_s$	$f_0 + 2f_s$
$\omega_0 = 2\pi f_0$	$\omega_1 = 2\pi(f_0 + f_s)$	$\omega_2 = 2\pi(f_0 + 2f_s)$
$\hat{\omega}_0 \quad (rad/sample)$	$\hat{\omega}_0 + 2\pi \quad (rad/sample)$	$\hat{\omega}_0 + 4\pi \quad (rad/sample)$





Frequency and Digital Frequency

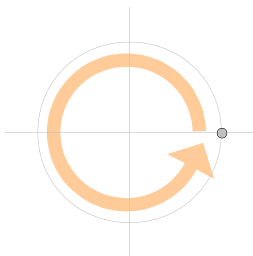




Co-terminal Angles

For 1 sample

$2\pi\ (rad)\ /\ T_s\ (sec)$

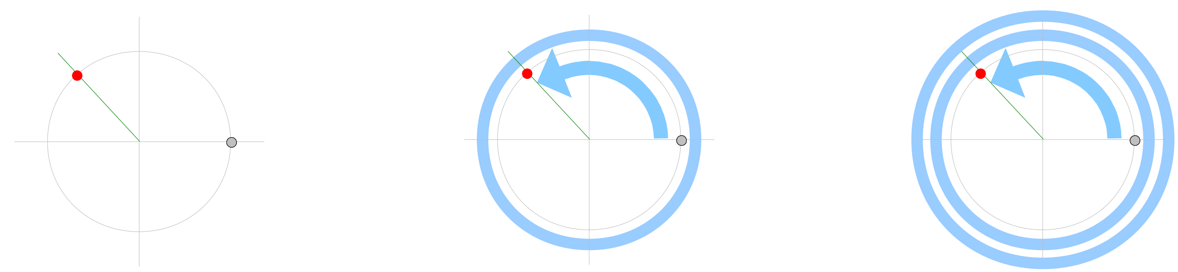


$\hat{\omega} = \omega \cdot T_s \quad (rad/sample)$

$= \omega / f_s \quad (rad/sample)$

For T_s second

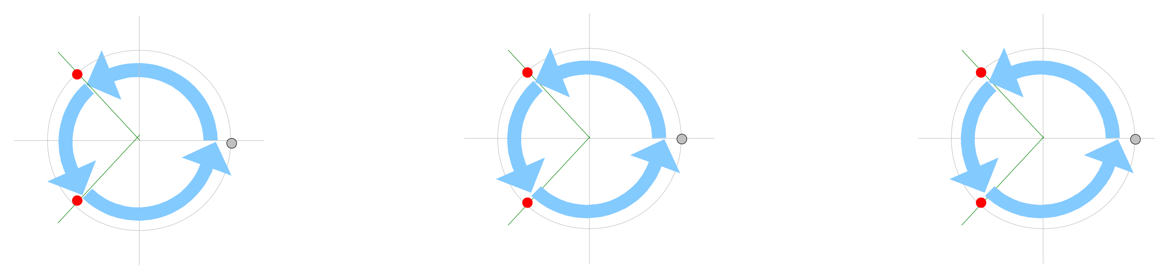
$\hat{\omega} = \omega \cdot T_s \quad (rad/sample)$



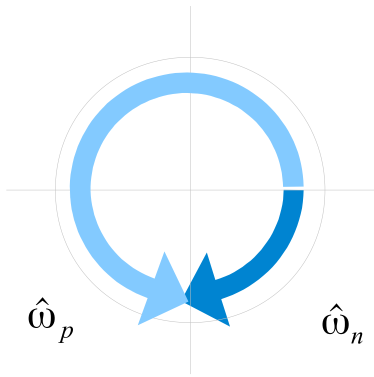
f_0	$f_0 + f_s$	$f_0 + 2f_s$
$\omega_0 = 2\pi f_0$	$\omega_1 = 2\pi(f_0 + f_s)$	$\omega_2 = 2\pi(f_0 + 2f_s)$
$\hat{\omega}_0 \quad (rad/sample)$	$\hat{\omega}_0 + 2\pi \quad (rad/sample)$	$\hat{\omega}_0 + 4\pi \quad (rad/sample)$

Co-terminal Angles

The same angular positions after each sample time.



Positive & Negative Angles (1)



$+ \qquad -$

$$\hat{\omega}_p - \hat{\omega}_n = 2\pi$$

Positive Normalized Rad Freq

$$\hat{\omega}_p = 2\pi + \hat{\omega}_n$$

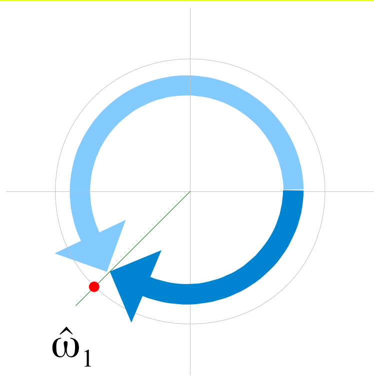
$+ \qquad -$

Negative Normalized Rad Freq

$$\hat{\omega}_n = \hat{\omega}_p - 2\pi$$

$- \qquad +$

$$\frac{f_s}{2} < f_1 < f_s$$



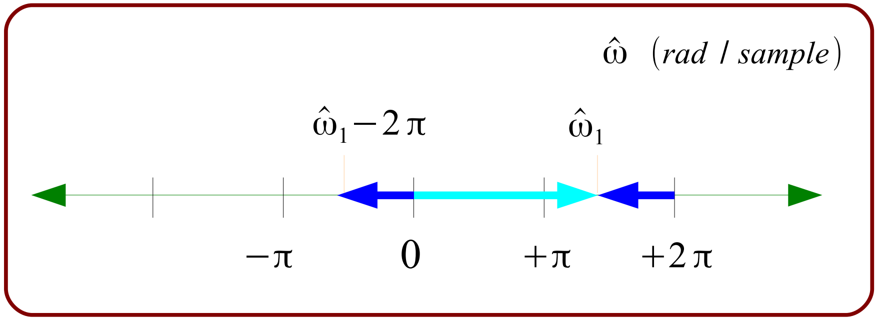
Positive Angle

$$+\pi < \hat{\omega}_1 < 2\pi$$

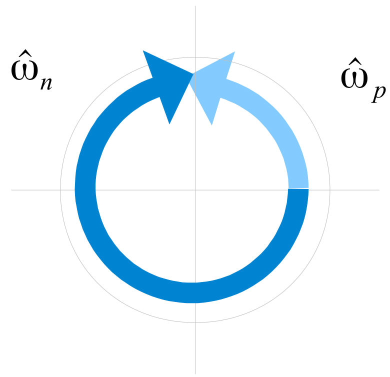
Negative Angle

$$-\pi < \hat{\omega}_1 - 2\pi < 0$$

Normalized Radian Frequency



Positive & Negative Angles (2)



\+ −

$$\hat{\omega}_p - \hat{\omega}_n = 2\pi$$

Positive Normalized Rad Freq

$$\hat{\omega}_p = 2\pi + \hat{\omega}_n$$

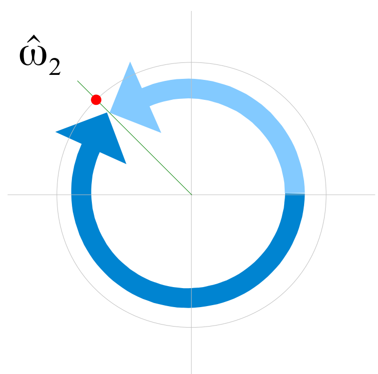
\+ −

Negative Normalized Rad Freq

$$\hat{\omega}_n = \hat{\omega}_p - 2\pi$$

− +

$$-f_s < f_2 < -\frac{f_s}{2}$$



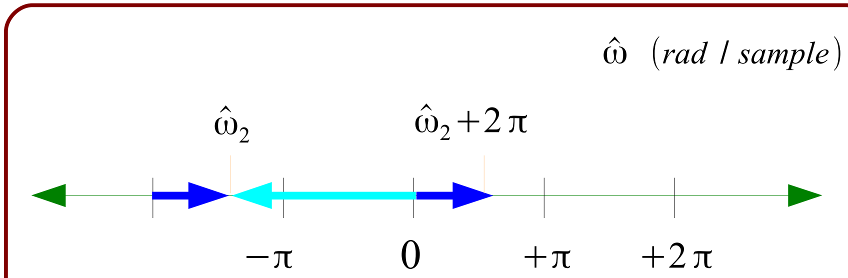
Negative Angle

$$-2\pi < \hat{\omega}_2 < -\pi$$

Positive Angle

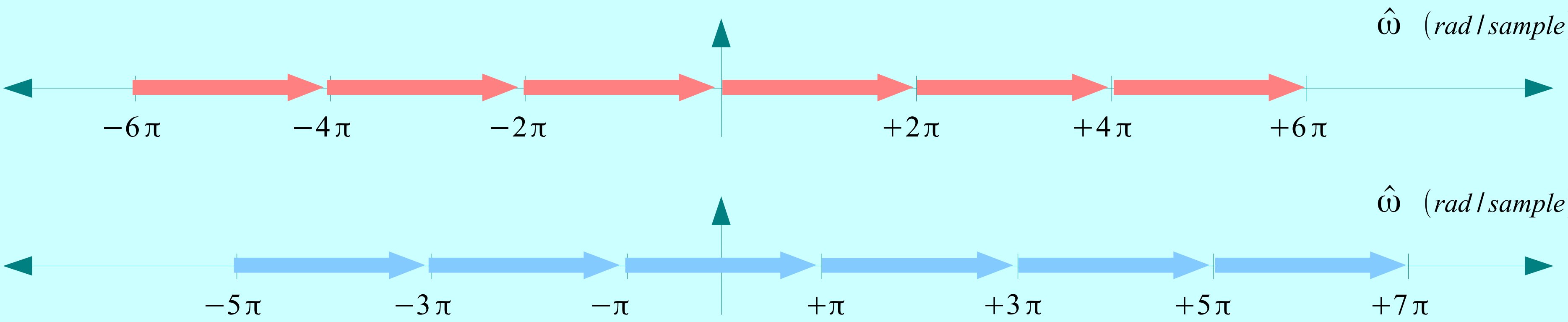
$$0 < 2\pi + \hat{\omega}_2 < \pi$$

Normalized Radian Frequency

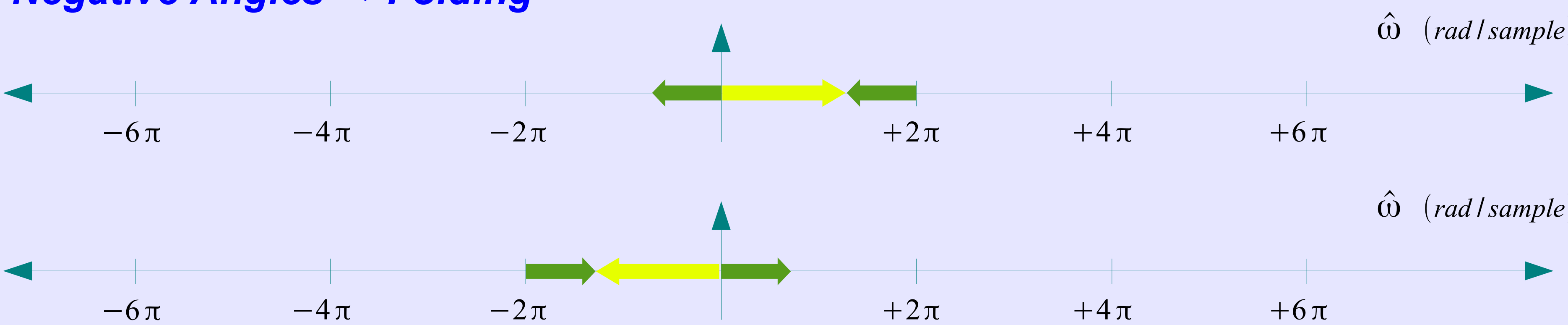


Periodicity and Folding

Co-terminal Angles → Periodic



Negative Angles → Folding



References

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[3] A “graphical interpretation” of the DFT and FFT, by Steve Mann