Example Random Processes

Young W Lim

September 11, 2019

Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

1. Gaussian Random Processes

2. Poisson Random Process

Gaussian Random Process

N Gaussian random variables

Definition

$$f_X(x_1, \cdots, x_N; t_1, \cdots, t_N) =$$

$$\frac{exp\left\{-(1/2)\left[x - \overline{X}\right]^t [C_X]^{-1} \left[x - \overline{X}\right]\right\}}{\sqrt{(2\pi)^N |[C_X]|}}$$

The Covariance Matrix
N Gaussian random variables

Definition

$$\overline{X}_i = E\left[X_i\right] = E\left[X(t_i)\right]$$

$$C_{ik} = C_{X_i X_k} = E\left[\left(X_i - \overline{X}_i\right)\left(X_k - \overline{X}_k\right)\right]$$

$$= E\left[\left(X_i - E\left[X(t_i)\right]\right)\left(X_k - E\left[X(t_k)\right]\right)\right]$$

$$C_{ik} = C_{X_i X_k} = C_{XX}\left(t_i t_k\right)$$

$$= R_{XX}(t_i, t_k) - E\left[X(t_i)\right] E\left[X(t_k)\right]$$

Stationary Gaussian Process

N Gaussian random variables

Definition

$$\overline{X}_i = E[X_i] = E[X(t_i)] = \overline{X} = const$$

$$C_{XX}(t_i, t_k) = C_{XX}(t_k - t_i)$$

$$R_{XX}(t_i, t_k) = R_{XX}(t_k - t_i)$$

Jointly Gaussian Process

N Gaussian random variables

Definition

the two random processes $X(t)$ and $Y(t)$ are jointly Gaussian
if the random variables $X(t_1), \dots, X(t_N), Y(t'_1), \dots, Y(t'_M)$
at times$t_1, \dots, t_N$ for $X(t)$ and $t'_1, \dots, t'_M$ for $Y(t)$
are jointly gaussian for any $N, t_1, \dots, t_N, M, t'_1, \dots, t'_M$

Stationary Gaussian Markov Process

N Gaussian random variables

Definition

$$C_{XX}(\tau) = \sigma^2 e^{-\beta|\tau|}$$

$$C_{XX}[k] = \sigma^2 a^{-|k|}$$

$$a = e^{\beta T_S}$$

Poisson Random Process

N Gaussian random variables

Definition

$$p\left[X(t=k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}\right], \qquad k = 0, 1, 2, \cdots$$

$$f_X(x) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k)$$

Poisson Random Process - mean and 2nd moment

N Gaussian random variables

Definition

$$E[X(t)] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_{-\infty}^{\infty} x \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k) dx$$

$$= \sum_{k=0}^{\infty} \frac{k(\lambda t)^k e^{-\lambda t}}{k!} = \lambda t$$

$$E\left[X^2 t)\right] = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_{-\infty}^{\infty} x^2 \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k) dx$$

$$= \sum_{k=0}^{\infty} \frac{k^2(\lambda t)^k e^{-\lambda t}}{k!} = \lambda t(1+\lambda t)$$

Poisson Random Process - joint probability density
N Gaussian random variables

Definition

$$P[X(t_1) = k_1] = \frac{(\lambda t_1)^{k_1} e^{-\lambda t_1}}{k_1!} \qquad k = 0, 1, 2, \cdots$$

$$P[X(t_2) = k_2 | X(t_1) = k_1] = \frac{(\lambda (t_2 - t_1))^{k_2 - k_1} e^{-\lambda (t_2 - t_1)}}{(k_2 - k_1)!}$$

$$P(k_1, k_2) = P[X(t_2) = k_2 | X(t_1) = k_1] \cdot P[X(t_1) = k_1]$$

$$= \frac{(\lambda t_1)^{k_1} (\lambda (t_2 - t_1))^{k_2 - k_1} e^{-\lambda t_2}}{k_1! (k_2 - k_1)!}$$

$$f_X(x_1, x_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=k_1}^{\infty} P(k_1, k_2) \delta(x_1 - k_1) \delta(x_2 - k_2)$$

Bernoulli Random Process

N Gaussian random variables

Definition

$$X[n] = \sum_{m=1}^{n} I[m]$$

$$f_X(x) = \sum_{k=0}^{n} P(k)\delta(x-k)$$

$$P(k) = \left(\begin{array}{c} n \\ k \end{array} \right) p^k (1-p)^{n-k}$$

$$E\left[X\left[n\right]\right] = np$$

$$Var\left[X\left[n\right]\right] = np(1-p)$$