

# Example Random Processes

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles, Jr. and B. Shi

# Outline

- 1 Gaussian Random Processes
- 2 Poisson Random Process

## Gaussian Random Process

 $N$  Gaussian random variables

## Definition

$$f_X(x_1, \dots, x_N; t_1, \dots, t_N) = \frac{\exp\left\{-\frac{1}{2} [x - \bar{X}]^t [C_X]^{-1} [x - \bar{X}]\right\}}{\sqrt{(2\pi)^N |[C_X]|}}$$

# The Covariance Matrix

$N$  Gaussian random variables

## Definition

$$\bar{X}_i = E[X_i] = E[X(t_i)]$$

$$C_{ik} = C_{X_i X_k} = E[(X_i - \bar{X}_i)(X_k - \bar{X}_k)]$$

$$= E[(X_i - E[X(t_i)])(X_k - E[X(t_k)])]$$

$$C_{ik} = C_{X_i X_k} = C_{XX}(t_i, t_k)$$

$$= R_{XX}(t_i, t_k) - E[X(t_i)]E[X(t_k)]$$

# Stationary Gaussian Process

$N$  Gaussian random variables

## Definition

$$\bar{X}_i = E[X_i] = E[X(t_i)] = \bar{X} = \text{const}$$

$$C_{XX}(t_i, t_k) = C_{XX}(t_k - t_i)$$

$$R_{XX}(t_i, t_k) = R_{XX}(t_k - t_i)$$

# Jointly Gaussian Process

$N$  Gaussian random variables

## Definition

the two random processes  $X(t)$  and  $Y(t)$  are jointly Gaussian if the random variables  $X(t_1), \dots, X(t_N), Y(t'_1), \dots, Y(t'_M)$  at times  $t_1, \dots, t_N$  for  $X(t)$  and  $t'_1, \dots, t'_M$  for  $Y(t)$  are jointly gaussian for any  $N, t_1, \dots, t_N, M, t'_1, \dots, t'_M$

# Stationary Gaussian Markov Process

$N$  Gaussian random variables

## Definition

$$C_{XX}(\tau) = \sigma^2 e^{-\beta|\tau|}$$

$$C_{XX}[k] = \sigma^2 a^{-|k|}$$

$$a = e^{\beta T_s}$$





## Poisson Random Process

 $N$  Gaussian random variables

## Definition

$$p \left[ X(t = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \right], \quad k = 0, 1, 2, \dots$$

$$f_X(x) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x - k)$$

Poisson Random Process - mean and 2nd moment  
 $N$  Gaussian random variables

## Definition

$$\begin{aligned} E[X(t)] &= \int_{-\infty}^{\infty} x f_X(x) dx = \int_{-\infty}^{\infty} x \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x - k) dx \\ &= \sum_{k=0}^{\infty} \frac{k(\lambda t)^k e^{-\lambda t}}{k!} = \lambda t \end{aligned}$$

$$\begin{aligned} E[X^2 t] &= \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_{-\infty}^{\infty} x^2 \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x - k) dx \\ &= \sum_{k=0}^{\infty} \frac{k^2 (\lambda t)^k e^{-\lambda t}}{k!} = \lambda t(1 + \lambda t) \end{aligned}$$

Poisson Random Process - joint probability density  
 $N$  Gaussian random variables

## Definition

$$P[X(t_1) = k_1] = \frac{(\lambda t_1)^{k_1} e^{-\lambda t_1}}{k_1!} \quad k = 0, 1, 2, \dots$$

$$P[X(t_2) = k_2 | X(t_1) = k_1] = \frac{(\lambda(t_2 - t_1))^{k_2 - k_1} e^{-\lambda(t_2 - t_1)}}{(k_2 - k_1)!}$$

$$P(k_1, k_2) = P[X(t_2) = k_2 | X(t_1) = k_1] \cdot P[X(t_1) = k_1]$$

$$= \frac{(\lambda t_1)^{k_1} (\lambda(t_2 - t_1))^{k_2 - k_1} e^{-\lambda t_2}}{k_1! (k_2 - k_1)!}$$

$$f_X(x_1, x_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=k_1}^{\infty} P(k_1, k_2) \delta(x_1 - k_1) \delta(x_2 - k_2)$$

## Bernoulli Random Process

 $N$  Gaussian random variables

## Definition

$$X[n] = \sum_{m=1}^n I[m]$$

$$f_X(x) = \sum_{k=0}^n P(k) \delta(x - k)$$

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$E[X[n]] = np$$

$$\text{Var}[X[n]] = np(1-p)$$



