Laurent Series and z-Transform - Geometric Series Causality A

20180606

2 formulas of z

① $$\frac{3}{2}\,\frac{-1}{(z-0.5)(z-2)} = \left(\frac{1}{z-0.5} - \frac{1}{z-2}\right)$$

$$\Big\updownarrow z^{-1}$$

② $$\frac{3}{2}\,\frac{-z^2}{(z-2)(z-0.5)} = \left(\frac{0.5z}{(z-0.5)} - \frac{2z}{(z-2)}\right)$$

$$\boxed{\frac{3}{2}\,\frac{-1}{(z-0.5)(z-2)} \quad \xleftrightarrow{z^{-1}} \quad \frac{3}{2}\,\frac{-z^2}{(z-2)(z-0.5)}}$$

$$\frac{3}{2}\,\frac{-1}{(z-0.5)(z-2)} = \frac{3}{2}\,\frac{2}{3}\left(\frac{1}{z-0.5} - \frac{1}{z-2}\right)$$

$$= \left(\frac{1}{z-0.5} - \frac{1}{z-2}\right)$$

$$\Big\downarrow z^{-1}$$

$$\frac{3}{2}\,\frac{-1}{(z^{-1}-0.5)(z^{-1}-2)} = \frac{3}{2}\,\frac{2}{3}\left(\frac{1}{z^{-1}-0.5} - \frac{1}{z^{-1}-2}\right)$$

$$= \left(\frac{2}{2z^{-1}-1} - \frac{0.5}{0.5z^{-1}-1}\right)$$

$$= \left(\frac{2z}{2-z} - \frac{0.5z}{0.5-z}\right)$$

$$= \left(\frac{-2z}{z-2} + \frac{0.5z}{z-0.5}\right)$$

$$= z\left(\frac{-2}{z-2} + \frac{0.5}{z-0.5}\right)$$

$$= z\left(\frac{-\frac{3}{2}z}{(z-2)(z-0.5)}\right)$$

$$= \frac{3}{2}\,\frac{-z^2}{(z-2)(z-0.5)}$$

$$\frac{3}{2}\,\frac{-z^2}{(z-2)(z-0.5)} = \frac{3}{2}\,\frac{2}{3}\left(\frac{0.5z}{(z-0.5)} - \frac{2z}{(z-2)}\right)$$

$f(z), g(z)$: causal form of $\boxed{\text{Laurent}}$ series

$$\frac{\text{nominator polynomial of } z}{\text{denominator polynomial of } z}$$

$f(z^{-1}), g(z^{-1})$: anti-causal form of $\boxed{\text{Laurent}}$ series

$$\frac{\text{nominator polynomial of } z^{-1}}{\text{denominator polynomial of } z^{-1}}$$

$X(z), Y(z)$: causal form of $\boxed{z\text{- Trans}}$

$$\frac{\text{nominator polynomial of } z^{-1}}{\text{denominator polynomial of } z^{-1}}$$

$X(z^{-1}), Y(z^{-1})$: anti-causal form of $\boxed{z\text{- Trans}}$

$$\frac{\text{nominator polynomial of } z}{\text{denominator polynomial of } z}$$

2 formulas

$\boxed{\frac{1}{z - p}}$ $\boxed{\frac{1}{z^{-1} - p}}$

2 representations each

$\boxed{\frac{1}{z - p}}$

$\boxed{-\frac{p^{-1}}{1 - p^{-1}z}} \triangleq f(z) = X(z^{-1})$

$\qquad\qquad\qquad\quad \| \qquad\quad \|$

$\boxed{\frac{z^{-1}}{1 - pz^{-1}}} \triangleq Y(z) = g(z^{-1})$

$\boxed{\frac{1}{z^{-1} - p}}$

$\boxed{\frac{z}{1 - pz}} \triangleq g(z) = Y(z^{-1})$

$\qquad\qquad\qquad\quad \| \qquad\quad \|$

$\boxed{-\frac{p^{-1}}{1 - p^{-1}z^{-1}}} \triangleq X(z) = f(z^{-1})$

$$\begin{array}{c} f(z) \\ \| \\ Y(z) \end{array} \qquad \begin{array}{c} g(z) \\ \| \\ X(z) \end{array}$$

$$\begin{array}{ccc} X(z^{-1}) & = & f(z) \\ \| & & \| \\ g(z^{-1}) & = & Y(z) \end{array} \qquad \begin{array}{ccc} g(z) & = & Y(z^{-1}) \\ \| & & \| \\ X(z) & = & f(z^{-1}) \end{array}$$

$\boxed{\text{Laurent}}$ $f(z), g(z)$: causal, $f(z^{-1}), g(z^{-1})$: anti-causal

$\boxed{\text{z- Trans}}$ $X(z), Y(z)$: causal, $X(z^{-1}), Y(z^{-1})$: anti-causal

$$\begin{tikzcd}
\boxed{\dfrac{1}{z-p}} \arrow[r, shift left, "\frac{p^{-1}}{p^{-1}}"] \arrow[d, shift right, "\frac{z^{-1}}{z^{-1}}"'] & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z}} \arrow[l, shift left] \\
\boxed{\dfrac{z^{-1}}{1-pz^{-1}}} \arrow[u, shift right] &
\end{tikzcd}$$

$= f(z) \longleftrightarrow -(p^{-1})^{n+1}$

$X(z^{-1})$

$\boxed{\dfrac{z^{-1}}{1-pz^{-1}}} = g(z^{-1})$

$Y(z) \longleftrightarrow (p)^{n-1}$

$$\begin{tikzcd}
\boxed{\dfrac{1}{z^{-1}-p}} \arrow[r, shift left, "\frac{p^{-1}}{p^{-1}}"] \arrow[d, shift right, "\frac{z^{-1}}{z^{-1}}"'] & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}} \arrow[l, shift left] \\
\boxed{\dfrac{z}{1-pz}} \arrow[u, shift right] &
\end{tikzcd}$$

$= f(z^{-1})$

$X(z) \longleftrightarrow -(p^{-1})^{n+1}$

$\boxed{\dfrac{z}{1-pz}} = g(z) \longleftrightarrow (p)^{n-1}$

$Y(z^{-1})$

2 formulas of z : $f(z)$, $g(z)$

2 representations : $f(z^{-1})$, $g(z^{-1})$

* Simple Pole Forms

$$
\begin{tikzcd}
p & \boxed{\dfrac{1}{z-p}} \arrow[rr, leftrightarrow, "z^{-1}"] \arrow[d, leftrightarrow, "\frac{z^{-1}}{z^{-1}}\cdot\,\cdot\frac{z}{z}"] & & \boxed{\dfrac{1}{z^{-1}-p}} \arrow[d, leftrightarrow, "\frac{z}{z}\cdot\,\cdot\frac{z^{-1}}{z^{-1}}"] & p^{-1} \\
p & \boxed{\dfrac{z^{-1}}{1-pz^{-1}}} \arrow[rr, leftrightarrow, "z^{-1}"] & & \boxed{\dfrac{z}{1-pz}} & p^{-1}
\end{tikzcd}
$$

* Geometric Series Forms

$$
\begin{tikzcd}
\overset{n \geq 0}{\boxed{\begin{matrix} -(p^{-1})^{n+1} \\ \text{causal } f(z) \end{matrix}}} & \overset{|z|<p}{\boxed{-\dfrac{p^{-1}}{1-p^{-1}z}}} \arrow[rr, leftrightarrow, "z^{-1}"] \arrow[d, equal, "\cdot\frac{p}{p}\frac{z^{-1}}{z^{-1}}"] & & \overset{|z|>p^{-1}}{\boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}} \arrow[d, equal, "\frac{p}{p}\frac{z}{z}\cdot"'] & \overset{n \geq 0}{\boxed{\begin{matrix} \text{causal } X(z) \\ -(p^{-1})^{n+1} \end{matrix}}} \\
\underset{n \geq 1}{\boxed{\begin{matrix} \text{causal } X(z) \\ (p)^{n-1} \end{matrix}}} & \underset{|z|>p}{\boxed{\dfrac{z^{-1}}{1-pz^{-1}}}} \arrow[rr, leftrightarrow, "z^{-1}"] & & \underset{|z|<p^{-1}}{\boxed{\dfrac{z}{1-pz}}} & \underset{n \geq 1}{\boxed{\begin{matrix} (p)^{n-1} \\ \text{causal } f(z) \end{matrix}}}
\end{tikzcd}
$$

Laurent Series

$f(z)\ (|z|<p) \leftrightarrow a_n\ (n \geq 0)$

$f(z^{-1})\ (|z|>p^{-1}) \leftrightarrow a_{-n}\ (n < 1)$

$f(z)\ (|z|>p) \leftrightarrow -a_n\ (n < 0)$

$f(z^{-1})\ (|z|<p^{-1}) \leftrightarrow -a_{-n}\ (n \geq 1)$

(A) $f(z)$ for $\boxed{|z|<p}$, $g(z)$ for $\boxed{|z|<p^{-1}}$ Laurent S

Geometric Series Forms

$$p \qquad \boxed{f(z)} = \underset{|z|<p}{\boxed{-\frac{p^{-1}}{1-p^{-1}z}}} \qquad\qquad \boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}}$$

$$\boxed{\frac{z^{-1}}{1-pz^{-1}}} \qquad\qquad \underset{|z|<p^{-1}}{\boxed{\frac{z}{1-pz}}} = \boxed{g(z)} \qquad p^{-1}$$

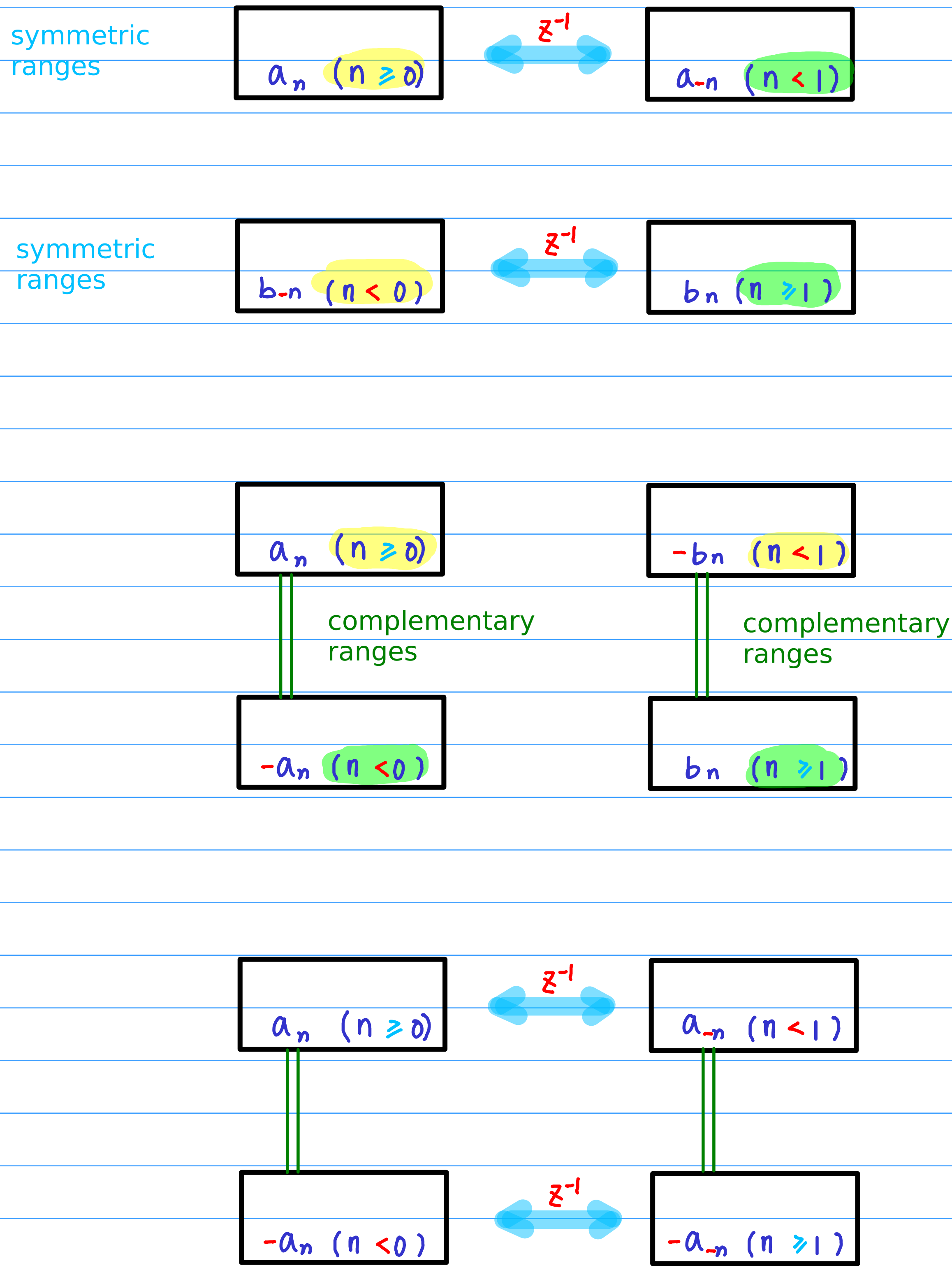
(B) $f(z^{-1})$ for $\boxed{|z|>p^{-1}}$, $g(z^{-1})$ for $\boxed{|z|>p}$

Geometric Series Forms

$$f(z) = \underset{|z|<p}{\boxed{-\frac{p^{-1}}{1-p^{-1}z}}} \quad \xleftrightarrow{z^{-1}} \quad \underset{|z|>p^{-1}}{\boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}}} = \boxed{f(z^{-1})}$$

$$\boxed{g(z^{-1})} = \underset{|z|>p}{\boxed{\frac{z^{-1}}{1-pz^{-1}}}} \quad \xleftrightarrow{z^{-1}} \quad \underset{|z|<p^{-1}}{\boxed{\frac{z}{1-pz}}} = g(z)$$

Laurent Series $\boxed{a_n} \leftrightarrow f(z)$



Laurent Series $a_n \leftrightarrow \boxed{f(z)}$

ROC's with reciprocal poles

$$f(z)\ (|z|<p) \xleftrightarrow{z^{-1}} f(z^{-1})\ (|z^{-1}|<p)$$

ROC's with reciprocal poles

$$g(z^{-1})\ (|z^{-1}|<p^{-1}) \xleftrightarrow{z^{-1}} g(z)\ (|z|<p^{-1})$$

$f(z)\ (|z|<p)$

complementary ROC

$g(z^{-1})\ (|z|>p)$

$f(z^{-1})\ (|z|>p^{-1})$

complementary ROC

$g(z)\ (|z|<p^{-1})$

$$f(z)\ (|z|<p) \xleftrightarrow{z^{-1}} f(z^{-1})\ (|z|>p^{-1})$$

$$f(z)\ (|z|>p) \xleftrightarrow{z^{-1}} f(z^{-1})\ (|z|<p^{-1})$$

Laurent Series $a_n \leftrightarrow f(z)$ $b_n \leftrightarrow g(z)$

Geometric Series Forms

$|z| < p$ $\qquad\qquad$ $|z| > p^{-1}$

$$f(z) = \boxed{-\frac{p^{-1}}{1-p^{-1}z}} \quad \overset{z^{-1}}{\longleftrightarrow} \quad \boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} = f(z^{-1})$$

$\cdot \frac{p}{p}\frac{z^{-1}}{z^{-1}}$ $\qquad\qquad$ $\cdot \frac{p}{p}\frac{z}{z}$

$$g(z^{-1}) = \boxed{\frac{z^{-1}}{1-pz^{-1}}} \quad \overset{z^{-1}}{\longleftrightarrow} \quad \boxed{\frac{z}{1-pz}} = g(z)$$

$|z| > p$ $\qquad\qquad$ $|z| < p^{-1}$

$f(z)$ $(\lvert z\rvert < p)$	$\overset{z^{-1}}{\longleftrightarrow}$	$f(z^{-1})$ $(\lvert z^{-1}\rvert < p)$	$g(z)$ $(\lvert z\rvert > p^{-1})$
a_n $(n \geq 0)$		a_{-n} $(n < 1)$	$-b_n$ $(n < 1)$

$f(z)$ $(\lvert z\rvert > p)$	$g(z^{-1})$ $(\lvert z^{-1}\rvert < p^{-1})$	$\overset{z^{-1}}{\longleftrightarrow}$	$g(z)$ $(\lvert z\rvert < p^{-1})$
$-a_n$ $(n < 0)$	b_{-n} $(n < 0)$		b_n $(n \geq 1)$

Laurent Series using only $\boxed{a_n} \leftrightarrow \boxed{f(z)}$

$$
\begin{tikzcd}
 & \boxed{\begin{array}{c} f(z)\ (|z|<p) \\ a_n\ (n \geq 0) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, equal] & \boxed{\begin{array}{c} f(z^{-1})\ (|z^{-1}|<p) \\ a_{-n}\ (n<1) \end{array}} \arrow[d, equal] & \boxed{\begin{array}{c} g(z)\ (|z|>p^{-1}) \\ -b_n\ (n<1) \end{array}} \\
\boxed{\begin{array}{c} f(z)\ (|z|>p) \\ -a_n\ (n<0) \end{array}} & \boxed{\begin{array}{c} g(z^{-1})\ (|z^{-1}|<p^{-1}) \\ b_{-n}\ (n<0) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{c} g(z)\ (|z|<p^{-1}) \\ b_n\ (n \geq 1) \end{array}} &
\end{tikzcd}
$$

$$a_{-n} = -b_n \qquad\qquad -a_{-n} = b_n$$

$$
\begin{tikzcd}
\boxed{\begin{array}{c} f(z)\ (|z|<p) \\ a_n\ (n \geq 0) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, equal] & \boxed{\begin{array}{c} f(z^{-1})\ (|z|>p^{-1}) \\ a_{-n}\ (n<1) \end{array}} \arrow[d, equal] \\
\boxed{\begin{array}{c} f(z)\ (|z|>p) \\ -a_n\ (n<0) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{c} f(z^{-1})\ (|z|<p^{-1}) \\ -a_{-n}\ (n \geq 1) \end{array}}
\end{tikzcd}
$$

Z - Transform

$X(z^{-1})$ $(|z| < p)$ $\leftrightarrow$ x_{-n} $(n < 1)$

$X(z)$ $(|z| > p^{-1})$ $\leftrightarrow$ x_n $(n \geq 0)$

$X(z^{-1})$ $(|z| > p)$ $\leftrightarrow$ $-x_{-n}$ $(n \geq 1)$

$X(z)$ $(|z| < p^{-1})$ $\leftrightarrow$ $-x_n$ $(n < 0)$

Ⓐ $X(z)$ for $\boxed{|z| > p^{-1}}$, $Y(z)$ for $\boxed{|z| > p}$ z-Transform

Geometric Series Forms

$$\boxed{-\frac{p^{-1}}{1 - p^{-1}z}}$$

$$|z| > p^{-1} \qquad \boxed{-\frac{p^{-1}}{1 - p^{-1}z^{-1}}} = \boxed{X(z)} \quad p^{-1}$$

$$p \qquad \boxed{Y(z)} = \boxed{\frac{z^{-1}}{1 - pz^{-1}}} \quad |z| > p$$

$$\boxed{\frac{z}{1 - pz}}$$

Ⓑ $X(z^{-1})$ for $\boxed{|z| < p}$, $Y(z^{-1})$ for $\boxed{|z| < p^{-1}}$

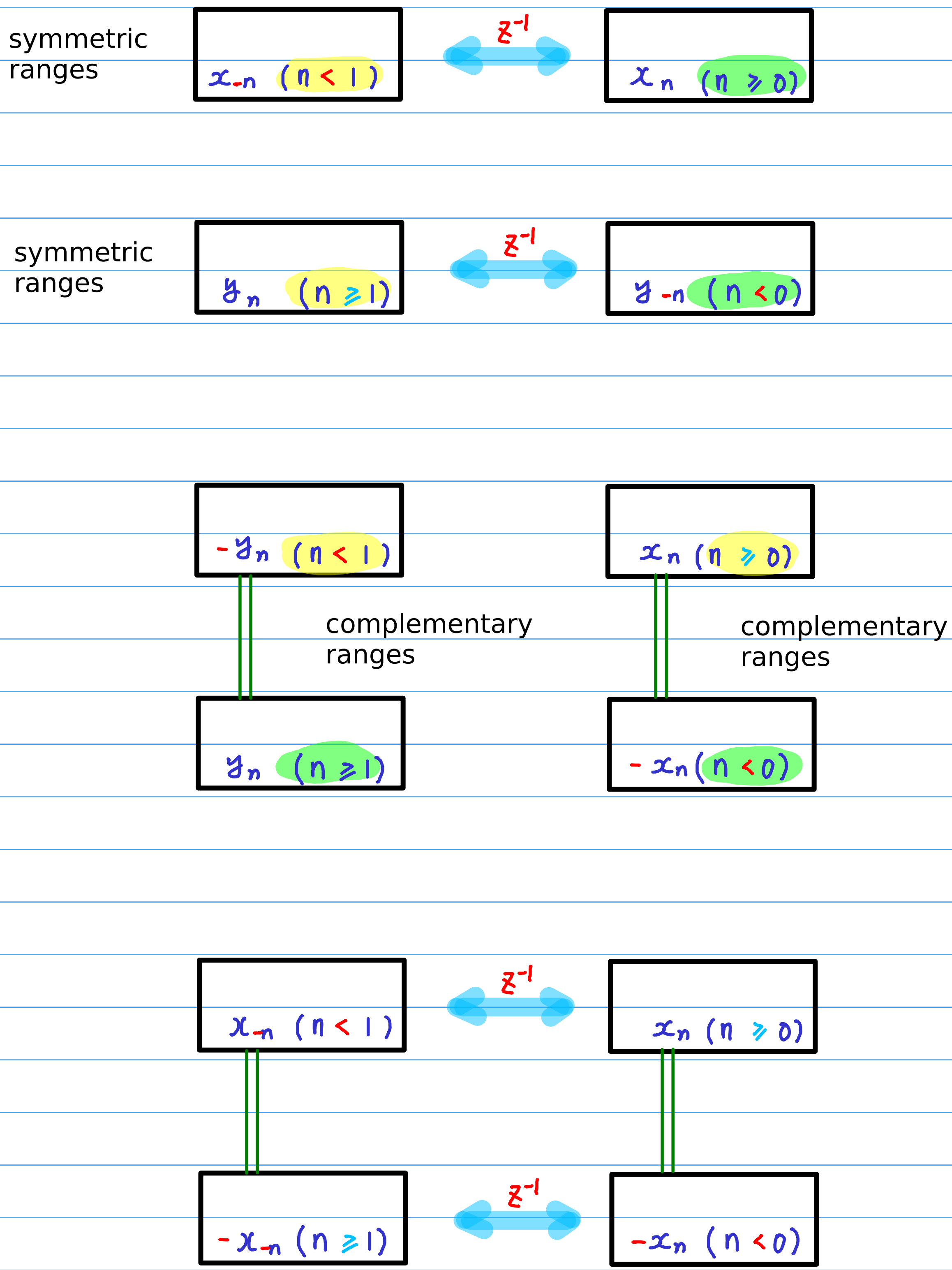
Geometric Series Forms

$$|z| < p \qquad \boxed{X(z^{-1})} = \boxed{-\frac{p^{-1}}{1 - p^{-1}z}} \xleftrightarrow{z^{-1}} \boxed{-\frac{p^{-1}}{1 - p^{-1}z^{-1}}} = X(z) \qquad |z| > p^{-1}$$

$$Y(z) = \boxed{\frac{z^{-1}}{1 - pz^{-1}}} \xleftrightarrow{z^{-1}} \boxed{\frac{z}{1 - pz}} = \boxed{Y(z^{-1})}$$

$$|z| > p \qquad |z| < p^{-1}$$

z - Transform $\boxed{x_n} \leftrightarrow X(z)$



z - Transform $x_n \leftrightarrow \boxed{X(z)}$

ROC's with reciprocal poles

$\boxed{X(z^{-1}) \; (|z^{-1}| > p^{-1})} \xleftrightarrow{z^{-1}} \boxed{X(z) \; (|z| > p^{-1})}$

ROC's with reciprocal poles

$\boxed{Y(z) \; (|z| > p)} \xleftrightarrow{z^{-1}} \boxed{Y(z^{-1}) \; (|z^{-1}| > p)}$

$\boxed{Y(z) \; (|z| < p)}$

complementary ROC

$\boxed{Y(z) \; (|z| > p)}$

$\boxed{X(z) \; (|z| > p^{-1})}$

complementary ROC

$\boxed{X(z) \; (|z| < p)}$

$\boxed{X(z^{-1}) \; (|z| < p)} \xleftrightarrow{z^{-1}} \boxed{X(z) \; (|z| > p^{-1})}$

$\boxed{X(z^{-1}) \; (|z| > p)} \xleftrightarrow{z^{-1}} \boxed{X(z) \; (|z| < p)}$

Z- Transform $X(z) \leftrightarrow x_n$ $Y(z) \leftrightarrow y_n$

Geometric Series Forms

$|z| < p$ $\qquad\qquad\qquad\qquad$ $|z| > p^{-1}$

$$X(z^{-1}) = \boxed{-\frac{p^{-1}}{1-p^{-1}z}} \quad \xleftrightarrow{z^{-1}} \quad \boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} = X(z)$$

$$\cdot\frac{p}{p}\frac{z^{-1}}{z^{-1}} \qquad\qquad \cdot\frac{p}{p}\frac{z}{z}$$

$$Y(z) = \boxed{\frac{z^{-1}}{1-pz^{-1}}} \quad \xleftrightarrow{z^{-1}} \quad \boxed{\frac{z}{1-pz}} = Y(z^{-1})$$

$|z| > p$ $\qquad\qquad\qquad\qquad$ $|z| < p^{-1}$

$\boxed{Y(z)\ (|z|<p) \quad -y_n\ (n<1)}$ $\boxed{X(z^{-1})\ (|z^{-1}|>p^{-1}) \quad x_{-n}\ (n<1)}$ $\xleftrightarrow{z^{-1}}$ $\boxed{X(z)\ (|z|>p^{-1}) \quad x_n\ (n \geq 0)}$

$\boxed{Y(z)\ (|z|>p) \quad y_n\ (n \geq 1)}$ $\xleftrightarrow{z^{-1}}$ $\boxed{Y(z^{-1})\ (|z^{-1}|>p) \quad y_{-n}\ (n<0)}$ $\boxed{X(z)\ (|z|<p^{-1}) \quad -x_n\ (n<0)}$

z-Transform using only $\boxed{x_n} \leftrightarrow \boxed{X(z)}$

$$
\begin{tikzcd}
\boxed{\begin{array}{c} Y(z)\ (|z|<p) \\ -y_n\ (n<1) \end{array}} & \boxed{\begin{array}{c} X(z^{-1})\ (|z^{-1}|>p^{-1}) \\ x_{-n}\ (n<1) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, dash, shift left] \arrow[d, dash, shift right] & \boxed{\begin{array}{c} X(z)\ (|z|>p^{-1}) \\ x_n\ (n\geq 0) \end{array}} \arrow[d, dash, shift left] \arrow[d, dash, shift right] & \\
& \boxed{\begin{array}{c} Y(z)\ (|z|>p) \\ y_n\ (n\geq 1) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{c} Y(z^{-1})\ (|z^{-1}|>p) \\ y_{-n}\ (n<0) \end{array}} & \boxed{\begin{array}{c} X(z)\ (|z|<p^{-1}) \\ -x_n\ (n<0) \end{array}}
\end{tikzcd}
$$

$$x_{-n} = -y_n \qquad -x_{-n} = y_n$$

$$
\begin{tikzcd}
\boxed{\begin{array}{c} X(z^{-1})\ (|z|<p) \\ x_{-n}\ (n<1) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, dash, shift left] \arrow[d, dash, shift right] & \boxed{\begin{array}{c} X(z)\ (|z|>p^{-1}) \\ x_n\ (n\geq 0) \end{array}} \arrow[d, dash, shift left] \arrow[d, dash, shift right] \\
\boxed{\begin{array}{c} X(z^{-1})\ (|z|>p) \\ -x_{-n}\ (n\geq 1) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{c} X(z)\ (|z|<p^{-1}) \\ -x_n\ (n<0) \end{array}}
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \boxed{\begin{array}{ll} f(z) & (|z|<p) \\ a_n & (n \geq 0) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, equal] & \boxed{\begin{array}{ll} f(z^{-1}) & (|z^{-1}|<p) \\ a_{-n} & (n<1) \end{array}} \arrow[d, equal] & \boxed{\begin{array}{ll} g(z) & (|z|>p^{-1}) \\ -b_n & (n<1) \end{array}} \\
\boxed{\begin{array}{ll} f(z) & (|z|>p) \\ -a_n & (n<0) \end{array}} & \boxed{\begin{array}{ll} g(z^{-1}) & (|z^{-1}|<p^{-1}) \\ b_{-n} & (n<0) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{ll} g(z) & (|z|<p^{-1}) \\ b_n & (n \geq 1) \end{array}} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\boxed{\begin{array}{ll} Y(z) & (|z|<p) \\ -y_n & (n<1) \end{array}} & \boxed{\begin{array}{ll} X(z^{-1}) & (|z^{-1}|>p^{-1}) \\ x_{-n} & (n<1) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, equal] & \boxed{\begin{array}{ll} X(z) & (|z|>p^{-1}) \\ x_n & (n \geq 0) \end{array}} \arrow[d, equal] & \\
 & \boxed{\begin{array}{ll} Y(z) & (|z|>p) \\ y_n & (n \geq 1) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{ll} Y(z^{-1}) & (|z^{-1}|>p) \\ y_{-n} & (n<0) \end{array}} & \boxed{\begin{array}{ll} X(z) & (|z|<p^{-1}) \\ -x_n & (n<0) \end{array}}
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=large]
\boxed{\begin{array}{l} f(z) \quad (|z|<p) \\ a_n \quad (n \geq 0) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, equal] & \boxed{\begin{array}{l} f(z^{-1}) \quad (|z|>p^{-1}) \\ a_{-n} \quad (n < 1) \end{array}} \arrow[d, equal] \\
\boxed{\begin{array}{l} f(z) \quad (|z|>p) \\ -a_n \quad (n < 0) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{l} f(z^{-1}) \quad (|z|<p^{-1}) \\ -a_{-n} \quad (n \geq 1) \end{array}}
\end{tikzcd}
$$

$$a_n = x_{-n} \qquad\qquad a_{-n} = x_n$$

$$
\begin{tikzcd}[column sep=large]
\boxed{\begin{array}{l} X(z^{-1}) \quad (|z|<p) \\ x_{-n} \quad (n < 1) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, equal] & \boxed{\begin{array}{l} X(z) \quad (|z|>p^{-1}) \\ x_n \quad (n \geq 0) \end{array}} \arrow[d, equal] \\
\boxed{\begin{array}{l} X(z^{-1}) \quad (|z|>p) \\ -x_{-n} \quad (n \geq 1) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{l} X(z) \quad (|z|<p^{-1}) \\ -x_n \quad (n < 0) \end{array}}
\end{tikzcd}
$$

Laurent Series

$n \geq 0$ $\quad |z| < p$ $\qquad\qquad$ $|z| > p^{-1}$ $\quad n < 1$

a_n: causal f(z) $\; -(p^{-1})^{n+1}$ $\quad \boxed{-\dfrac{p^{-1}}{1 - p^{-1} z}}$ $\quad \xleftrightarrow{z^{-1}} \quad$ $\boxed{-\dfrac{p^{-1}}{1 - p^{-1} z^{-1}}}$ $\quad$ anti-causal f(z) $\; -(p^{-1})^{-n+1}$: a_{-n}

$\cdot(-1)$ $\qquad \cdot \dfrac{p}{p}\dfrac{z^{-1}}{z^{-1}}$ $\qquad\qquad \cdot \dfrac{p}{p}\dfrac{z}{z}$ $\qquad \cdot(-1)$

b_{-n}: $(p)^{-n-1}$ anti-causal f(z) $\quad \boxed{\dfrac{z^{-1}}{1 - p z^{-1}}}$ $\quad \xleftrightarrow{z^{-1}} \quad$ $\boxed{\dfrac{z}{1 - p z}}$ $\quad (p)^{n-1}$ causal f(z) : b_n

$n < 0$ $\quad |z| > p$ $\qquad\qquad$ $|z| < p^{-1}$ $\quad n \geq 1$

z - Transform

$n < 1$ $\quad |z| < p$ $\qquad\qquad$ $|z| > p^{-1}$ $\quad n \geq 0$

x_{-n}: anti-causal X(z) $\; -(p^{-1})^{-n+1}$ $\quad \boxed{-\dfrac{p^{-1}}{1 - p^{-1} z}}$ $\quad \xleftrightarrow{z^{-1}} \quad$ $\boxed{-\dfrac{p^{-1}}{1 - p^{-1} z^{-1}}}$ $\quad$ causal X(z) $\; -(p^{-1})^{n+1}$: x_n

$\cdot(-1)$ $\qquad \cdot \dfrac{p}{p}\dfrac{z^{-1}}{z^{-1}}$ $\qquad\qquad \cdot \dfrac{p}{p}\dfrac{z}{z}$ $\qquad \cdot(-1)$

y_n: $(p)^{n-1}$ causal X(z) $\quad \boxed{\dfrac{z^{-1}}{1 - p z^{-1}}}$ $\quad \xleftrightarrow{z^{-1}} \quad$ $\boxed{\dfrac{z}{1 - p z}}$ $\quad (p)^{-n-1}$ anti-causal X(z) : y_{-n}

$n \geq 1$ $\quad |z| > p$ $\qquad\qquad$ $|z| < p^{-1}$ $\quad n < 0$

$$f(z) \quad (|z| < p) \quad \leftrightarrow \quad a_n \quad (n \geq 0)$$
$$X(z^{-1}) \quad (|z| < p) \quad \leftrightarrow \quad x_{-n} \quad (n < 1)$$

$$f(z^{-1}) \quad (|z| > p^{-1}) \quad \leftrightarrow \quad a_{-n} \quad (n < 1)$$
$$X(z) \quad (|z| > p^{-1}) \quad \leftrightarrow \quad x_n \quad (n \geq 0)$$

$$f(z) \quad (|z| > p) \quad \leftrightarrow \quad -a_n \quad (n < 0)$$
$$X(z^{-1}) \quad (|z| > p) \quad \leftrightarrow \quad -x_{-n} \quad (n \geq 1)$$

$$f(z^{-1}) \quad (|z| < p^{-1}) \quad \leftrightarrow \quad -a_{-n} \quad (n \geq 1)$$
$$X(z) \quad (|z| < p^{-1}) \quad \leftrightarrow \quad -x_n \quad (n < 0)$$