

# Series Solution (H1) Legendre Functions

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## Bessel's Equation

Zill & Wright 5.3.1

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

order  $\nu$

## Legendre's Equation

Zill & Wright 5.3.2

$$(1-x^2) y'' - 2x y' + n(n+1) y = 0$$

order  $n$

Zill & Wright 3.6

## Cauchy-Euler Equation

$$x^2 y'' + x y' - \alpha^2 y = 0 \quad \alpha \geq 0$$

## Bessel's Equation

Zill & Wright 5.3.1

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

order  $\nu$

Suppose a solution  $y = \sum_{n=0}^{\infty} c_n x^{n+r}$

$$y = c_0 x^r + c_1 x^{r+1} + c_2 x^{r+2} + c_3 x^{r+3} + \dots$$

## Legendre's Equation

Zill & Wright 5.3.2

$$(1-x^2) y'' - 2x y' + n(n+1) y = 0$$

order  $n$

$$y = \sum_{k=0}^{\infty} c_k x^k$$

$$y = c_0 x^0 + c_1 x^1 + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots$$

$$y = \sum_{k=0}^{\infty} C_k x^k$$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$+ n(n+1)$	$y = \sum_{k=0}^{\infty} C_k x^k$
$- 2x$	$y' = \sum_{k=1}^{\infty} C_k k x^{k-1}$
$(1-x^2)$	$y'' = \sum_{k=2}^{\infty} C_k k(k-1) x^{k-2}$



$$y = \sum_{k=0}^{\infty} C_k x^k$$

$$y' = \sum_{k=1}^{\infty} C_k k x^{k-1}$$

$$y'' = \sum_{k=2}^{\infty} C_k k(k-1) x^{k-2}$$

$$y = \sum_{k=0}^{\infty} C_k x^k = C_0 x^0 + C_1 x^1 + C_2 x^2 + C_3 x^3 + C_4 x^4 + \dots$$

$$y' = \sum_{k=1}^{\infty} C_k k x^{k-1} = \boxed{0} + 1 \cdot C_1 x^0 + 2 \cdot C_2 x^1 + 3 \cdot C_3 x^2 + 4 \cdot C_4 x^3 + \dots$$

$$y'' = \sum_{k=2}^{\infty} C_k k(k-1) x^{k-2} = \boxed{0} \boxed{0} + 2 \cdot 1 \cdot C_2 x^0 + 3 \cdot 2 \cdot C_3 x^1 + 4 \cdot 3 \cdot C_4 x^2 + \dots$$

$$y = \sum_{k=0}^{\infty} C_k x^k$$

$$y' = \sum_{k=1}^{\infty} C_k k x^{k-1}$$

$$y'' = \sum_{k=2}^{\infty} C_k k(k-1) x^{k-2}$$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$(1-x^2) \sum_{k=2}^{\infty} C_k k(k-1) x^{k-2} - 2x \sum_{k=1}^{\infty} C_k k x^{k-1} + n(n+1) \sum_{k=0}^{\infty} C_k x^k = 0$$

$$\sum_{k=2}^{\infty} C_k k(k-1) x^{k-2} - \sum_{k=2}^{\infty} C_k k(k-1) x^k - \sum_{k=1}^{\infty} 2C_k k x^k + n(n+1) \sum_{k=0}^{\infty} C_k x^k = 0$$

$$\sum_{k=0}^{\infty} C_{k+2} (k+2)(k+1) x^k - \sum_{k=2}^{\infty} C_k k(k-1) x^k - \sum_{k=1}^{\infty} 2C_k k x^k + n(n+1) \sum_{k=0}^{\infty} C_k x^k = 0$$

$n(n+1) = \lambda$

$$\sum_{k=0}^{\infty} C_{k+2} (k+2)(k+1) x^k - \sum_{k=2}^{\infty} C_k k(k-1) x^k - \sum_{k=1}^{\infty} 2C_k k x^k + \lambda \sum_{k=0}^{\infty} C_k x^k = 0$$

$$\sum_{k=0}^{\infty} C_{k+2} (k+2)(k+1) x^k - \sum_{k=2}^{\infty} C_k k(k-1) x^k - \sum_{k=1}^{\infty} 2C_k k x^k + \lambda \sum_{k=0}^{\infty} C_k x^k$$

$$= \left\{ \text{for } k=0 \text{ of } C_n \right\} + \left\{ \text{for } k=1 \text{ of } C_n \right\} + \sum_{k=2}^{\infty} \left\{ * \right\}$$

$k=0$

$$C_2 \cdot 2 \cdot 1 \cdot x^0 + \lambda C_0 = \underline{2C_2 + \lambda C_0}$$

$k=1$

$$C_3 \cdot 3 \cdot 2 \cdot x^1 - 2C_1 \cdot 1 \cdot x^1 + \lambda C_1 x^1 = \underline{6C_3 x - 2C_1 x + \lambda C_1 x}$$

$k \geq 2$

$$\sum_{k=2}^{\infty} \left[ C_{k+2} (k+2)(k+1) \left[ -C_k k(k-1) - 2C_k k + \lambda C_k \right] x^k \right]$$

$$\left[ C_k \left( -k^2 + k - 2k + \lambda \right) \right] x^k \quad -k^2 - k$$

$$\left[ C_k \left[ \lambda - k(k+1) \right] \right]$$

$$\underline{(2C_2 + \lambda C_0)} + \underline{(6C_3 - 2C_1 + \lambda C_1) x} +$$

$$\sum_{k=2}^{\infty} \left[ C_{k+2} (k+2)(k+1) + \left[ \lambda - k(k+1) \right] C_k \right] x^k = \text{always } 0$$

$$\begin{aligned}
 & \overset{0}{=} (2c_2 + \lambda c_0) + \overset{0}{=} (6c_3 - 2c_1 + \lambda c_1)x + \\
 & \sum_{k=2}^{\infty} \left[ \underline{c_{k+2} (k+2)(k+1) + [\lambda - k(k+1)] c_k} \right] x^k = \overset{\text{always}}{0}
 \end{aligned}$$

$$\Rightarrow \begin{cases} (2c_2 + \lambda c_0) = 0 \\ (6c_3 - 2c_1 + \lambda c_1) = 0 \\ c_{k+2} (k+2)(k+1) + [\lambda - k(k+1)] c_k = 0 \end{cases} \quad \lambda = n(n+1)$$



$$\begin{cases} (2c_2 + \lambda c_0) = 0 \\ (6c_3 - 2c_1 + \lambda c_1) = 0 \\ c_{k+2} (k+2)(k+1) + [\lambda - k(k+1)] c_k = 0 \end{cases} \quad \lambda = n(n+1)$$

$k=0$

$$(2c_2 + \lambda c_0) = 0$$

$$\Rightarrow n(n+1)c_0 + 2c_2 = 0$$

$k=1$

$$(6c_3 - 2c_1 + n(n+1)c_1) = 0 \quad -2 + n^2 + n = n^2 + n - 2$$

$$\Rightarrow (n-1)(n+2)c_1 + 6c_3 = 0$$

$k \geq 2$

$$(j+2)(j+1)c_{j+2} + [\lambda - j(j+1)]c_j = 0$$

$$(j+2)(j+1)c_{j+2} + [n(n+1) - j(j+1)]c_j = 0$$

$$[ (n-j)(n+j+1) ]$$

$\begin{aligned} n^2 + n - j^2 - j \\ = n^2 - j^2 + n - j \\ = (n-j)(n+j) + (n-j) \\ = (n-j)(n+j+1) \end{aligned}$
--

$$\begin{cases} n(n+1)c_0 + 2c_2 = 0 & \leftarrow (k=0) \\ (n-1)(n+2)c_1 + 6c_3 = 0 & \leftarrow (k=1) \\ (j+2)(j+1)c_{j+2} + (n-j)(n+j+1)c_j = 0 & \leftarrow (k \geq 2) \end{cases}$$

$$\begin{cases} n(n+1)c_0 + 2c_2 = 0 & \leftarrow (k=0) \\ (n+1)(n+2)c_1 + 6c_3 = 0 & \leftarrow (k=1) \\ (j+2)(j+1)c_{j+2} + (n-j)(n+j+1)c_j = 0 & \leftarrow (k \geq 2) \end{cases}$$

$$c_2 = -\frac{n(n+1)}{2!} c_0$$

$$c_3 = -\frac{(n+1)(n+2)}{3!} c_1$$

$$c_{j+2} = -\frac{(n-j)(n+1+j)}{(j+2)(j+1)} c_j \quad j = 2, 3, 4, \dots$$

$$c_2 = - \frac{n(n+1)}{2!} c_0$$

$$c_3 = - \frac{(n+1)(n+2)}{3!} c_1$$

$$c_{j+2} = - \frac{(n-j)(n+1+j)}{(j+2)(j+1)} c_j \quad j = 2, 3, 4, \dots$$

$j=2$

$$\begin{aligned} c_4 &= - \frac{(n-2)(n+3)}{4 \cdot 3} c_2 = - \frac{(n-2)(n+3)}{4 \cdot 3} \left( - \frac{n(n+1)}{2!} c_0 \right) \\ &= \frac{(n-2)n(n+1)(n+3)}{4!} c_0 \end{aligned}$$

$j=3$

$$\begin{aligned} c_5 &= - \frac{(n-3)(n+4)}{5 \cdot 4} c_3 = - \frac{(n-3)(n+4)}{5 \cdot 4} \left( - \frac{(n+1)(n+2)}{3!} c_1 \right) \\ &= \frac{(n-3)(n+1)(n+2)(n+4)}{5!} c_1 \end{aligned}$$

$j=4$

$$\begin{aligned} c_6 &= - \frac{(n-4)(n+5)}{6 \cdot 5} c_4 \Rightarrow \\ &= - \frac{(n-4)(n+5)}{6 \cdot 5} \left( \frac{(n-2)n(n+1)(n+3)}{4!} c_0 \right) \\ &= - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!} c_0 \end{aligned}$$

$j=5$

$$\begin{aligned} c_7 &= - \frac{(n-5)(n+6)}{7 \cdot 6} c_5 \\ &= - \frac{(n-5)(n+6)}{7 \cdot 6} \left( \frac{(n-3)(n+1)(n+2)(n+4)}{5!} c_1 \right) \\ &= - \frac{(n-5)(n-3)(n+1)(n+2)(n+4)(n+6)}{7!} c_1 \end{aligned}$$

$$C_2 = - \frac{n(n+1)}{2!} C_0$$

$$C_3 = - \frac{(n+1)(n+2)}{3!} C_1$$

$$C_{j+2} = - \frac{(n-j)(n+1+j)}{(j+2)(j+1)} C_j \quad j = 2, 3, 4, \dots$$

$$j=2 \quad C_4 = \frac{(n-2)n(n+1)(n+3)}{4!} C_0$$

$$j=3 \quad C_5 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} C_1$$

$$j=4 \quad C_6 = - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!} C_0$$

$$j=5 \quad C_7 = - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!} C_1$$

$$C_6 = - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!} C_0$$

Diagram showing the derivation of  $C_6$  from  $C_0$ . A vertical line separates the terms  $(n-4)(n-2)n$  on the left and  $(n+1)(n+3)(n+5)$  on the right. Green arrows with a label  $-2$  point from  $n$  to  $(n-4)$ ,  $(n-2)$ , and  $(n+1)$ . Blue arrows with labels  $-2$ ,  $-1$ , and  $-1$  point from  $(n+1)$ ,  $(n+3)$ , and  $(n+5)$  to  $n$ . The entire expression is circled in blue.

$$C_7 = - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!} C_1$$

Diagram showing the derivation of  $C_7$  from  $C_1$ . A vertical line separates the terms  $(n-5)(n-3)(n-1)$  on the left and  $(n+2)(n+4)(n+6)$  on the right. Blue arrows with a label  $-2$  point from  $(n-1)$  to  $(n-5)$ ,  $(n-3)$ , and  $(n+2)$ . Green arrows with labels  $-2$ ,  $-2$ , and  $-1$  point from  $(n+2)$ ,  $(n+4)$ , and  $(n+6)$  to  $(n-1)$ . The entire expression is circled in green.

$C_0$  +

$C_1$  +

$C_2$  -

$C_3$  -

$C_4$  +

$C_5$  +

$C_6$  -

$C_n$  -

$$n=6$$

$$c_1, c_3, c_5, c_7, c_9, c_{11}, \dots$$

$$c_2, c_4, c_6, \boxed{0, 0, 0, \dots} \text{ finite}$$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$c_6 = - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!} c_0$$

When  $n=6$

$$0 = c_8 = + \frac{(n-6)(n-4)(n-2)n(n+1)(n+3)(n+5)(n+7)}{8!} c_1$$

$$0 = c_{10} = - \frac{(n-8)(n-6)(n-4) \dots (n+5)(n+7)(n+9)}{10!} c_1$$

$$0 = c_{12} = + \frac{(n-10)(n-8)(n-6) \dots (n+7)(n+9)(n+11)}{12!} c_1$$

$$0 = c_{14} = - (n-6)$$

$$0 = c_{16} = + (n-6)$$

•  
•  
•  
•

$$n=7$$

$C_1, C_3, C_5, C_7, \boxed{0, 0, \dots}$  finite  
 $C_2, C_4, C_6, C_8, C_{10}, C_{12}, \dots$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$C_7 = - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!} C_1$$

*(Note: In the original image, blue arrows point from the terms (n-5), (n-3), (n-1) to the denominator 7!, and green arrows point from the terms (n+2), (n+4), (n+6) to the denominator 7!. The coefficient C7 is circled in blue and C1 is circled in green.)*

When  $n=7$

$$0 = C_9 = + \frac{\cancel{(n-7)}^0 (n-5)(n-3) \dots (n+4)(n+6)(n+8)}{9!} C_1$$

$$0 = C_{11} = - \frac{(n-9) \cancel{(n-7)}^0 (n-5) \dots (n+6)(n+8)(n+10)}{11!} C_1$$

$$0 = C_{13} = + \frac{(n-11)(n-9) \cancel{(n-7)}^0 \dots (n+8)(n+10)(n+12)}{13!} C_1$$

$$0 = C_{15} = - \cancel{(n-7)}^0$$

$$0 = C_{17} = + \cancel{(n-7)}^0$$

⋮  
⋮  
⋮  
⋮

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$y = \sum_{j=0}^{\infty} c_j x^j = \sum_{\text{even } j} c_j x^j + \sum_{\text{odd } j} c_j x^j$$

$y$  중 대칭  
 항만 따기

$$= (c_0 x^0 + c_2 x^2 + c_4 x^4 + \dots) \Rightarrow y_1(x)$$

$$+ (c_1 x^1 + c_3 x^3 + c_5 x^5 + \dots) \Rightarrow y_2(x)$$

Even func

odd func



linearly  
independent

$$c_4 = \frac{(n-2)(n+1)(n+3)}{4!} c_0$$


$$c_5 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} c_1$$

$$c_6 = - \frac{(n-4)(n-2)(n+3)(n+5)}{6!} c_0$$

$$c_7 = - \frac{(n-5)(n-3)(n+2)(n+4)(n+6)}{7!} c_1$$



$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Order:  $n$  

$$C_4 = \frac{(n-2)n(n+1)(n+3)}{4!} C_0$$

$$C_5 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} C_1$$

$$C_6 = - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!} C_0$$

$$C_7 = - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!} C_1$$

$$y = \overset{\text{odd fn}}{\boxed{y_1(x)}} + \overset{\text{even fn}}{\boxed{y_2(x)}}$$

even  $(n)$       finite      infinite ...      terms

odd  $(n)$       infinite ...      finite      terms

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Order:  $n$  ☆

$$y = \overset{\text{even fn}}{y_1(x)} + \overset{\text{odd fn}}{y_2(x)}$$

even  $(n)$

finite  
 $[c_0, c_2, \dots, c_n]$

infinite ...

terms

odd  $(n)$

infinite ...


finite

$[c_1, c_3, \dots, c_n]$

terms

\*  $n$ -th degree polynomial solution

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Order:  $n$  

$$y = \overset{\text{even fn}}{y_1(x)} + \overset{\text{odd fn}}{y_2(x)}$$

even  $n$

finite  $y_1(x)$

$$n=4 \quad y_1(x) = C_0 x^0 + C_2 x^2 + C_4 x^4 + \textcircled{0}$$

$$n=6 \quad y_1(x) = C_0 x^0 + C_2 x^2 + C_4 x^4 + C_6 x^6 + \textcircled{0}$$

$$n=8 \quad y_1(x) = C_0 x^0 + C_2 x^2 + C_4 x^4 + C_6 x^6 + C_8 x^8 + \textcircled{0}$$

odd  $n$

finite  $y_2(x)$

$$n=5 \quad y_2(x) = C_1 x^1 + C_3 x^3 + C_5 x^5$$

$$n=7 \quad y_2(x) = C_1 x^1 + C_3 x^3 + C_5 x^5 + C_7 x^7$$

$$n=9 \quad y_2(x) = C_1 x^1 + C_3 x^3 + C_5 x^5 + C_7 x^7 + C_9 x^9$$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{Order: } n$$

$$y_1(x) = c_0 x^0 + c_2 x^2 + c_4 x^4 + \dots$$

$$c_2 = \frac{n(n+1)}{2!} c_0$$

$$c_4 = \frac{(n-2)n(n+1)(n+3)}{4!} c_0$$

$$c_6 = \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!} c_0$$

$$c_8 = \frac{(n-6)(n-4)(n-2)n(n+1)(n+3)(n+5)(n+7)}{8!} c_0$$

$$n=2$$

$$c_2 = \frac{n(n+1)}{2!} c_0 = \frac{1}{2!} c_0$$

$$c_4 = \frac{\cancel{(n-2)}n(n+1)(n+3)}{4!} c_0 = 0$$

$$c_6 = \frac{(n-4)\cancel{(n-2)}n(n+1)(n+3)(n+5)}{6!} c_0 = 0$$

$$c_8 = \frac{(n-6)(n-4)\cancel{(n-2)}n(n+1)(n+3)(n+5)(n+7)}{8!} c_0 = 0$$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$y = \sum_{j=0}^{\infty} c_j x^j$$

$$= \sum_{\text{even } j} c_j x^j + \sum_{\text{odd } j} c_j x^j$$

$$y = \overset{\text{even fn}}{y_1(x)} + \overset{\text{odd fn}}{y_2(x)}$$

even  $(n)$

**finite**  
 $[c_0, c_2, \dots, c_n]$

infinite ...

terms

odd  $(n)$

infinite ...

**finite**

$[c_1, c_3, \dots, c_n]$

terms

$n$ -th degree polynomial solution

$$\begin{cases} c_0 = 1 & n=0 \\ c_0 = (-1)^{\frac{n}{2}} \frac{1 \cdot 3 \cdots (n-1)}{2 \cdot 4 \cdots n} & n=2, 4, 6, \dots \end{cases}$$

$$\begin{cases} c_1 = 1 & n=1 \\ c_1 = (-1)^{\frac{(n-1)}{2}} \frac{1 \cdot 3 \cdots n}{2 \cdot 4 \cdots (n-1)} & n=3, 5, \dots \end{cases}$$

# \* Convergence. Condition for infinite series

$$y = \sum_{j=0}^{\infty} c_j x^j = \sum_{\text{even } j} c_j x^j + \sum_{\text{odd } j} c_j x^j$$

$$y = \overset{\text{even fn}}{y_1(x)} + \overset{\text{odd fn}}{y_2(x)}$$

even  $(n)$

infinite ... terms  
converge?

odd  $(n)$

infinite ... terms  
converge?

$$y = \sum_{j=0}^{\infty} c_j x^j = \sum_{\text{even } j} c_j x^j + \sum_{\text{odd } j} c_j x^j$$

$$c_2 = -\frac{n(n+1)}{2!} c_0$$

$$c_3 = -\frac{(n+1)(n+2)}{3!} c_1$$

$$c_{j+2} = -\frac{(n-j)(n+1+j)}{(j+2)(j+1)} c_j \quad j=2, 3, 4, \dots$$

$$\left| \frac{c_{j+2} x^{j+2}}{c_j x^j} \right| < 1 \quad \text{as } j \rightarrow \infty$$

$$\left| \frac{(n-j)(n+1+j)}{(j+2)(j+1)} x^2 \right| < 1 \quad |x| < 1 \quad \boxed{-1 < x < +1}$$

Converges to zero  $\rightarrow$  a solution

## general solution

$$y = \overset{\text{even fn}}{y_1(x)} + \overset{\text{odd fn}}{y_2(x)}$$

even  $n$

A Legendre  
Polynomial  
Solution

An infinite  
series  
solution

odd  $n$

An infinite  
series  
solution

A Legendre  
Polynomial  
Solution

A Legendre polynomial is a particular solution

particular solution

all coefficients  
are determined



## Legendre Polynomials

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

Order

Legendre Polynomial

$$\downarrow (1-x^2)y'' - 2xy' + n(n+1)y = 0$$



$$n=0 \quad (1-x^2)y'' - 2xy' + 0y = 0$$

$$y = P_0(x) = 1$$

$$n=1 \quad (1-x^2)y'' - 2xy' + 2y = 0$$

$$y = P_1(x) = x$$

$$n=2 \quad (1-x^2)y'' - 2xy' + 6y = 0$$

$$y = P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$n=3 \quad (1-x^2)y'' - 2xy' + 12y = 0$$

$$y = P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\begin{cases} c_0 = 1 & n=0 \\ c_0 = (-1)^{\frac{n}{2}} \frac{1 \cdot 3 \cdots (n-1)}{2 \cdot 4 \cdots n} & n=2, 4, 6, \dots \end{cases}$$

$$n=2$$

$$c_0 = (-1)^{\frac{2}{2}} \frac{1}{2} = (-1) \cdot \frac{1}{2} = -\frac{1}{2}$$

$$c_2 = -\frac{n(n+1)}{2!} c_0 = -\frac{2 \cdot 3}{2} \left(-\frac{1}{2}\right) = +\frac{3}{2}$$

$$y = -\frac{1}{2}x^0 + \frac{3}{2}x^2 = P_2(x)$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$\begin{cases} c_1 = 1 & n=1 \\ c_1 = (-1)^{\frac{(n-1)}{2}} \frac{1 \cdot 3 \cdots n}{2 \cdot 4 \cdots (n-1)} & n=1, 3, 5, \dots \end{cases}$$

$$n=3$$

$$c_1 = (-1)^{\frac{3-1}{2}} \frac{1 \cdot 3}{2} = (-1) \frac{3}{2} = -\frac{3}{2}$$

$$c_3 = -\frac{(n+1)(n+2)}{3!} c_1 = -\frac{(3+1)(3+2)}{6} \left(-\frac{3}{2}\right)$$

$$= +\frac{4 \cdot 5}{6} \left(+\frac{3}{2}\right)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$= \frac{5}{2}$$

$$y = c_1 x^1 + c_3 x^3 = -\frac{3}{2}x + \frac{5}{2}x^3 = P_3(x)$$

$n=0$ 

$$(1-x^2)y'' - 2xy' + 0y = 0$$

$\underbrace{\quad}_0$ 
 $\underbrace{\quad}_0$ 
 $\underbrace{\quad}_1$

$$y = P_0(x) = 1$$

$$y' = 0 \quad y'' = 0$$

 $n=1$ 

$$(1-x^2)y'' - 2xy' + 2y = 0$$

$\underbrace{\quad}_0$ 
 $\underbrace{\quad}_1$ 
 $\underbrace{\quad}_x$

$$y = P_1(x) = x$$

$$y' = 1 \quad y'' = 0$$

 $n=2$ 

$$(1-x^2)y'' - 2xy' + 6y = 0$$

$\underbrace{\quad}_3$ 
 $\underbrace{\quad}_{3x}$ 
 $\underbrace{\quad}_{\frac{1}{2}(3x^2-1)}$

$$y = P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$y' = 3x \quad y'' = 3$$

 $n=3$ 

$$(1-x^2)y'' - 2xy' + 12y = 0$$

$$y = P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$y' = \frac{15}{2}x^2 - \frac{3}{2} \quad y'' = 15x$$

## \* Orthogonality

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\int_{-1}^1 1 \cdot x \, dx = 0$$

$$\int_{-1}^1 1 \cdot \frac{1}{2}(3x^2 - 1) \, dx = \frac{1}{2} \int_{-1}^1 3x^2 - 1 \, dx = \frac{1}{2} [x^3 - x]_{-1}^1 = 0$$

$$\int_{-1}^1 1 \cdot \frac{1}{2}(5x^3 - 3x) \, dx = 0$$

$$\int_{-1}^1 x \cdot \frac{1}{2}(3x^2 - 1) \, dx = \frac{1}{2} \int_{-1}^1 3x^3 - x \, dx = 0$$

$$\int_{-1}^1 x \cdot \frac{1}{2}(5x^3 - 3x) \, dx = \frac{1}{2} \int_{-1}^1 5x^4 - 3x^2 \, dx = \frac{1}{2} [x^5 - x^3]_{-1}^1 = 0$$

$$\begin{aligned} \int_{-1}^1 \frac{1}{2}(3x^2 - 1) \cdot \frac{1}{2}(5x^3 - 3x) \, dx &= \frac{1}{4} \int_{-1}^1 15x^5 - 9x^3 - 5x^2 + 3x \, dx \\ &= \frac{1}{4} \int_{-1}^1 15x^5 - 14x^3 + 3x \, dx = 0 \end{aligned}$$

## Properties

$$P_n(-x) = (-1)^n P_n(x)$$

$$P_n(1) = 1$$

$$P_n(-1) = (-1)^n$$

$$P_n(0) = 0, \quad \text{odd } n$$

$$P'_n(0) = 0, \quad \text{even } n$$

## Recurrence Relation

$$(k+1) P_{k+1}(x) - (2k+1)x P_k(x) + k P_{k-1}(x) = 0$$

## Rodrigue's Formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \quad n=0, 1, 2, \dots$$

