Series Solution (H1)
Legendre Functions

20160102

Bessel's Equation

Zill & Wright 5.3.1

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

order ν

Legendre's Equation

Zill & Wright 5.3.2

$$(1 - x^2) y'' - 2x y' + n(n+1) y = 0$$

order n

Zill & Wright 3.6

Cauchy-Euler Equation

$$x^2 y'' + x y' - \alpha^2 y = 0 \qquad \alpha \geq 0$$

Bessel's Equation

Zill & Wright 5.3.1

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

order ν

Suppose a solution $\boxed{y = \sum_{n=0}^{\infty} c_n x^{n+r}}$

$$y = c_0 x^r + c_1 x^{r+1} + c_2 x^{r+2} + c_3 x^{r+3} + \cdots\cdots$$

Legendre's Equation

Zill & Wright 5.3.2

$$(1 - x^2) y'' - 2x y' + n(n+1) y = 0$$

order n

$$\boxed{y = \sum_{k=0}^{\infty} c_k x^k}$$

$$y = c_0 x^0 + c_1 x^1 + c_2 x^2 + c_3 x^3 + c_4 x^4 \ldots\ldots$$

$$y = \sum_{k=0}^{\infty} c_k x^k$$

$$(1-x^2)y'' - 2x y' + n(n+1) y = 0$$

$$+n(n+1) \quad y = \sum_{k=0}^{\infty} c_k x^k$$

$$-2x \quad y' = \sum_{k=1}^{\infty} c_k k x^{k-1}$$

$$(1-x^2) \quad y'' = \sum_{k=2}^{\infty} c_k k(k-1) x^{k-2}$$

$$\boxed{\begin{aligned} y &= \sum_{k=0}^{\infty} c_k x^k \\ y' &= \sum_{k=1}^{\infty} c_k k x^{k-1} \\ y'' &= \sum_{k=2}^{\infty} c_k k(k-1) x^{k-2} \end{aligned}}$$

$$y = \sum_{k=0}^{\infty} c_k x^k = c_0 x^0 + c_1 x^1 + c_2 x^2 + c_3 x^3 + c_4 x^4 \ldots\ldots$$

$$y' = \sum_{k=1}^{\infty} c_k k x^{k-1} = \boxed{0} \quad 1 \cdot c_1 x^0 + 2 c_2 x^1 + 3 c_3 x^2 + 4 c_4 x^3 \ldots\ldots$$

$$y' = \sum_{k=2}^{\infty} c_k k(k-1) x^{k-2} = \boxed{0}\ \boxed{0} \quad 2 \cdot 1 c_2 x^0 + 3 \cdot 2 c_3 x^1 + 4 \cdot 3 c_4 x^2 \ldots\ldots$$

$$\boxed{\begin{aligned} y &= \sum_{k=0}^{\infty} C_k x^k \\ y' &= \sum_{k=1}^{\infty} C_k k x^{k-1} \\ y'' &= \sum_{k=2}^{\infty} C_k k(k-1) x^{k-2} \end{aligned}}$$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$\boxed{(1 - x^2)\sum_{k=2}^{\infty} C_k k(k-1)x^{k-2}} \boxed{-2x\sum_{k=1}^{\infty} C_k k x^{k-1}} + n(n+1)\sum_{k=0}^{\infty} C_k x^k$$

$$\boxed{\sum_{k=2}^{\infty} C_k k(k-1)x^{k-2}} - \sum_{k=2}^{\infty} C_k k(k-1)x^k$$

$$\boxed{\sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1)x^k} \qquad \boxed{-\sum_{k=1}^{\infty} 2C_k k x^k}$$

$$\boxed{n(n+1) \stackrel{\text{def}}{=} \lambda}$$

$$\sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1)x^k - \sum_{k=2}^{\infty} C_k k(k-1)x^k - \sum_{k=1}^{\infty} 2C_k k x^k + \lambda\sum_{k=0}^{\infty} C_k x^k$$

$$\sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1)x^k - \sum_{k=2}^{\infty} C_k\, k(k-1)x^k - \sum_{k=1}^{\infty} 2C_k\, k\, x^k + \lambda \sum_{k=0}^{\infty} C_k x^k$$

$= \{ k=0$ 일 때 $\} + \{ k=1$ 일 때 $\} + \sum_{k=2}^{\infty} \{ \ * \ \}$

$k=0$ $\quad C_2 \cdot 2 \cdot 1 \cdot x^0 + \lambda C_0 = 2C_2 + \lambda C_0$

$k=1$ $\quad C_3 \cdot 3 \cdot 2 \cdot x^1 - 2C_1 \cdot 1 \cdot x^1 + \lambda C_1 x^1 = 6C_3 x - 2C_1 x + \lambda C_1 x$

$k \geq 2$ $\quad \sum_{k=2}^{\infty} \Big[C_{k+2}(k+2)(k+1) \big[-C_k\, k(k-1) - 2C_k\, k + \lambda C_k \big] \Big] x^k$

$\big[C_k(-k^2 + k - 2k + \lambda) \big] \qquad -k^2 - k$

$\big[C_k[\lambda - k(k+1)] \big]$

$$(2C_2 + \lambda C_0) + (6C_3 - 2C_1 + \lambda C_1)x + \sum_{k=2}^{\infty} \Big[C_{k+2}(k+2)(k+1) + [\lambda - k(k+1)]\, C_k \Big] x^k = \overset{\text{always}}{0}$$

$$\underbrace{(2c_2+\lambda c_0)}_{=0} + \underbrace{(6c_3 - 2c_1 + \lambda c_1)}_{=0}x + \sum_{k=2}^{\infty}\Big[\underbrace{c_{k+2}(k+2)(k+1) + [\lambda - k(k+1)]\,c_k}_{=0}\Big]x^k = \overset{\text{always}}{0}$$

$$\Rightarrow \begin{cases} (2c_2+\lambda c_0) = 0 \\ (6c_3 - 2c_1 + \lambda c_1) = 0 \\ c_{k+2}(k+2)(k+1) + [\lambda - k(k+1)]\,c_k = 0 \end{cases}$$

$$\lambda = n(n+1)$$

$$
\begin{cases}
(2c_2 + \lambda c_0) = 0 \\
(6c_3 - 2c_1 + \lambda c_1) = 0 \\
c_{k+2}(k+2)(k+1) + [\lambda - k(k+1)]\, c_k = 0
\end{cases}
\qquad \lambda = n(n+1)
$$

$k=0$

$$(2c_2 + \lambda c_0) = 0$$

$$\Rightarrow n(n+1)c_0 + 2c_2 = 0$$

$k=1$

$$(6c_3 - 2c_1 + n(n+1)c_1) = 0 \qquad -2 + n^2 + n = n^2 + n - 2$$

$$\Rightarrow \underline{(n-1)(n+2)}\, c_1 + 6c_3 = 0$$

$k \geq 2$

$$(j+2)(j+1)c_{j+2} + [\lambda - j(j+1)]\, c_j = 0$$

$$(j+2)(j+1)c_{j+2} + [n(n+1) - j(j+1)]\, c_j = 0$$

$$[(n-j)(n+j+1)]$$

$$
\boxed{
\begin{aligned}
& n^2 + n - j^2 - j \\
&= n^2 - j^2 + n - j \\
&= (n-j)(n+j) + (n-j) \\
&= (n-j)(n+j+1)
\end{aligned}
}
$$

$$
\begin{cases}
n(n+1)c_0 + 2c_2 = 0 & \leftarrow (k=0) \\
(n-1)(n+2)c_1 + 6c_3 = 0 & \leftarrow (k=1) \\
(j+2)(j+1)c_{j+2} + (n-j)(n+j+1)c_j = 0 & \leftarrow (k \geq 2)
\end{cases}
$$

$$\begin{cases} n(n+1)\,c_0 + 2\,c_2 = 0 & \longleftarrow (k=0) \\ (n-1)(n+2)\,c_1 + 6\,c_3 = 0 & \longleftarrow (k=1) \\ (j+2)(j+1)\,c_{j+2} + (n-j)(n+j+1)\,c_j = 0 & \longleftarrow (k \geq 2) \end{cases}$$

$$\boxed{\begin{aligned} c_2 &= -\frac{n(n+1)}{2!}\,c_0 \\ c_3 &= -\frac{(n-1)(n+2)}{3!}\,c_1 \\ c_{j+2} &= -\frac{(n-j)(n+1+j)}{(j+2)(j+1)}\,c_j \qquad j = 2, 3, 4, \dots \end{aligned}}$$

$$\boxed{\begin{aligned} c_2 &= -\frac{n(n+1)}{2!}\,c_0 \\ c_3 &= -\frac{(n-1)(n+2)}{3!}\,c_1 \\ c_{j+2} &= -\frac{(n-j)(n+1+j)}{(j+2)(j+1)}\,c_j \qquad j = 2, 3, 4, \dots \end{aligned}}$$

$j=2$

$$c_4 = -\frac{(n-2)(n+3)}{4\cdot 3}\,c_2 = -\frac{(n-2)(n+3)}{4\cdot 3}\left(-\frac{n(n+1)}{2!}\,c_0\right)$$

$$= \frac{(n-2)\,n\,(n+1)(n+3)}{4!}\,c_0$$

$j=3$

$$c_5 = -\frac{(n-3)(n+4)}{5\cdot 4}\,c_3 = -\frac{(n-3)(n+4)}{5\cdot 4}\left(-\frac{(n-1)(n+2)}{3!}\,c_1\right)$$

$$= \frac{(n-3)(n-1)(n+2)(n+4)}{5!}\,c_1$$

$j=4$

$$c_6 = -\frac{(n-4)(n+5)}{6\cdot 5}\,c_4 \Rightarrow$$

$$= -\frac{(n-4)(n+5)}{6\cdot 5}\left(\frac{(n-2)\,n\,(n+1)(n+3)}{4!}\,c_0\right)$$

$$= -\frac{(n-4)(n-2)\,n\,(n+1)(n+3)(n+5)}{6!}\,c_0$$

$j=5$

$$c_7 = -\frac{(n-5)(n+6)}{7\cdot 6}\,c_5$$

$$= -\frac{(n-5)(n+6)}{7\cdot 6}\left(\frac{(n-3)(n-1)(n+2)(n+4)}{5!}\,c_1\right)$$

$$= -\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}\,c_1$$

$$\boxed{\begin{aligned} C_2 &= -\frac{n(n+1)}{2!}C_0 \\ C_3 &= -\frac{(n-1)(n+2)}{3!}C_1 \\ C_{j+2} &= -\frac{(n-j)(n+1+j)}{(j+2)(j+1)}C_j \qquad j = 2, 3, 4, \dots \end{aligned}}$$

$j=2$ $$C_4 = \frac{(n-2)\,n\,(n+1)(n+3)}{4!}C_0$$

$j=3$ $$C_5 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}C_1$$

$j=4$ $$C_6 = -\frac{(n-4)(n-2)\,n\,(n+1)(n+3)(n+5)}{6!}C_0$$

$j=5$ $$C_7 = -\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}C_1$$

$-2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -1$

$$C_6 = -\frac{(n-4)(n-2)\,n\,\big|\,(n+1)(n+3)(n+5)}{6!}C_0$$

$-2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -1$

$$C_7 = -\frac{(n-5)(n-3)(n-1)\,\big|\,(n+2)(n+4)(n+6)}{7!}C_1$$

C_0 $+$

C_1 $+$

C_2 $-$

C_3 $-$

C_4 $+$

C_5 $+$

C_6 $-$

C_7 $-$

$n=6$ $\quad c_1, c_3, c_5, c_7, c_9, c_{11}, \cdots$

$c_2, c_4, c_6, \boxed{0, 0, 0, \cdots}$ finite

$$\boxed{(1-x^2)y'' - 2xy' + n(n+1)y = 0}$$

$$c_6 = -\frac{(n-4)(n-2)\,n\,(n+1)(n+3)(n+5)}{6!}c_0$$

-2 -2 -2 -2 -2 -1

When $n=6$

$$0 = c_8 = +\frac{\cancelto{0}{(n-6)}(n-4)(n-2)\,n\,(n+1)(n+3)(n+5)(n+7)}{8!}c_1$$

$$0 = c_{10} = -\frac{(n-8)\cancelto{0}{(n-6)}(n-4)\cdots(n+5)(n+7)(n+9)}{10!}c_1$$

$$0 = c_{12} = +\frac{(n-10)(n-8)\cancelto{0}{(n-6)}\cdots(n+7)(n+9)(n+11)}{12!}c_1$$

$$0 = c_{14} = -\cancelto{0}{(n-6)}$$

$$0 = c_{16} = +\cancelto{0}{(n-6)}$$

$\vdots$

$n=7$

$c_1,\ c_3,\ c_5,\ c_7,\ \boxed{0,\ 0,\ \cdots}$ finite

$c_2,\ c_4,\ c_6,\ c_8,\ c_{10},\ c_{12},\ \cdots$

$$\boxed{(1-x^2)y'' - 2xy' + n(n+1)y = 0}$$

$$c_7 = -\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}c_1$$

-2, -2, -2, -2, -2, -1

when $n=7$

$$0 = c_9 = +\frac{\cancel{(n-7)}^{0}(n-5)(n-3)\cdots(n+4)(n+6)(n+8)}{9!}c_1$$

$$0 = c_{11} = -\frac{(n-9)\cancel{(n-7)}^{0}(n-5)\cdots(n+6)(n+8)(n+10)}{11!}c_1$$

$$0 = c_{13} = +\frac{(n-11)(n-9)\cancel{(n-7)}^{0}\cdots(n+8)(n+10)(n+12)}{13!}c_1$$

$$0 = c_{15} = -\quad \cancel{(n-7)}^{0}$$

$$0 = c_{17} = +\quad \cancel{(n-7)}^{0}$$

⋮

$$\boxed{(1-x^2)y'' - 2xy' + n(n+1)y = 0}$$

$$\boxed{y = \sum_{j=0}^{\infty} c_j x^j} = \boxed{\sum_{\text{even } j}^{\infty} c_j x^j} + \boxed{\sum_{\text{odd } j}^{\infty} c_j x^j}$$

y축 대칭

원점 대칭

$$= \left(c_0 x^0 + c_2 x^2 + c_4 x^4 + \cdots \right) \Rightarrow y_1(x)$$

$$+ \left(c_1 x^1 + c_3 x^3 + c_5 x^5 + \cdots \right) \Rightarrow y_2(x)$$

even func

odd func

$\downarrow$

linearly independent

$$c_4 = \frac{(n-2)n(n+1)(n+3)}{4!} c_0$$

$$c_5 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} c_1$$

$$c_6 = -\frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!} c_0$$

$$c_7 = -\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!} c_1$$

$$\boxed{(1-x^2)y'' - 2xy' + n(n+1)y = 0}$$ order: n

$$C_4 = \frac{(n-2)n(n+1)(n+3)}{4!} C_0$$

$$C_5 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} C_1$$

$$C_6 = -\frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!} C_0$$

$$C_7 = -\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!} C_1$$

$$y = \underbrace{\boxed{y_1(x)}}_{\text{odd fn}} + \underbrace{\boxed{y_2(x)}}_{\text{even fn}}$$

even n — finite — infinite ... terms

odd n — infinite ... — finite — terms

$$\boxed{(1-x^2)y'' - 2xy' + n(n+1)y = 0}$$ Order : $\bar{n}$ ☆

$$y = \underset{\text{even fn}}{y_1(x)} + \underset{\text{odd fn}}{y_2(x)}$$

even n | finite $[c_0, c_2, \dots, c_n]$ | infinite ... | terms

odd n | infinite ... | finite $[c_1, c_3, \dots, c_n]$ | terms

☆ n-th de ree polynomial solution

$$\boxed{(1-x^2)y'' - 2xy' + n(n+1)y = 0}$$ order : n

$$y = \overset{\text{even fn}}{y_1(x)} + \overset{\text{odd fn}}{y_2(x)}$$

$\boxed{\text{even } n}$ finite $y_1(x)$

$n=4$ $y_1(x) = C_0x^0 + C_2x^2 + C_4x^4 + 0$

$n=6$ $y_1(x) = C_0x^0 + C_2x^2 + C_4x^4 + C_6x^6 + 0$

$n=8$ $y_1(x) = C_0x^0 + C_2x^2 + C_4x^4 + C_6x^6 + C_8x^8 + 0$

$\boxed{\text{odd } n}$ finite $y_2(x)$

$n=5$ $y_2(x) = C_1x^1 + C_3x^3 + C_5x^5$

$n=7$ $y_2(x) = C_1x^1 + C_3x^3 + C_5x^5 + C_7x^7$

$n=9$ $y_2(x) = C_1x^1 + C_3x^3 + C_5x^5 + C_7x^7 + C_9x^9$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad \text{Order: } n$$

$$\boxed{y_1(x)} = c_0x^0 + c_2x^2 + c_4x^4 + \cdots\cdots$$

$$c_2 = -\frac{n(n+1)}{2!}c_0$$

$$c_4 = +\frac{(n-2)\,n(n+1)(n+3)}{4!}c_0$$

$$c_6 = -\frac{(n-4)(n-2)\,n(n+1)(n+3)(n+5)}{6!}c_0$$

$$c_8 = +\frac{(n-6)(n-4)(n-2)\,n(n+1)(n+3)(n+5)(n+7)}{8!}c_0$$

$\boxed{n=2}$

$$c_2 = -\frac{n(n+1)}{2!}c_0 \qquad \frac{1}{2!}$$

$$c_4 = +\frac{\cancel{(n-2)}^{0}\,n(n+1)(n+3)}{4!}c_0 = 0$$

$$c_6 = -\frac{(n-4)\cancel{(n-2)}^{0}\,n(n+1)(n+3)(n+5)}{6!}c_0 = 0$$

$$c_8 = +\frac{(n-6)(n-4)\cancel{(n-2)}^{0}\,n(n+1)(n+3)(n+5)(n+7)}{8!}c_0 = 0$$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$y = \sum_{j=0}^{\infty} c_j x^j$$

$$= \sum_{\text{even } j}^{\infty} c_j x^j + \sum_{\text{odd } j}^{\infty} c_j x^j$$

$$y = \underset{\text{even fn}}{y_1(x)} + \underset{\text{odd fn}}{y_2(x)}$$

even n $\boxed{\text{finite}}$ infinite ...   terms

$[c_0, c_2, \cdots, c_n]$

odd n infinite ... $\boxed{\text{finite}}$ terms

$[c_1, c_3, \cdots, c_n]$

n-th degree polynomial solution

$$\begin{cases} c_0 = 1 & n=0 \\ c_0 = (-1)^{\frac{n}{2}} \dfrac{1\cdot 3\cdots(n-1)}{2\cdot 4\cdots n} & n=2,4,6\cdots \end{cases}$$

$$\begin{cases} c_1 = 1 & n=1 \\ c_1 = (-1)^{\frac{(n-1)}{2}} \dfrac{1\cdot 3\cdots n}{2\cdot 4\cdots (n-1)} & n=1,3,5,\ldots \end{cases}$$

* Convergence Condition for infinite series

$$y = \sum_{j=0}^{\infty} c_j x^j = \sum_{\text{even } j}^{\infty} c_j x^j + \sum_{\text{odd } j}^{\infty} c_j x^j$$

$$y = \underset{\text{even fn}}{y_1(x)} + \underset{\text{odd fn}}{y_2(x)}$$

even n infinite ... terms

converge?

odd n infinite ... terms

converge?

$$y = \sum_{j=0}^{\infty} c_j x^j = \sum_{\text{even } j}^{\infty} c_j x^j + \sum_{\text{odd } j}^{\infty} c_j x^j$$

$$\boxed{\begin{aligned} c_2 &= -\frac{n(n+1)}{2!} c_0 \\ c_3 &= -\frac{(n-1)(n+2)}{3!} c_1 \\ c_{j+2} &= -\frac{(n-j)(n+1+j)}{(j+2)(j+1)} c_j \qquad j = 2, 3, 4, \dots \end{aligned}}$$

$$\left| \frac{c_{j+2}\, x^{j+2}}{c_j\, x^j} \right| < 1 \quad \text{as } j \to \infty$$

$$\left| \frac{(n-j)(n+1+j)}{(j+2)(j+1)} x^2 \right| < 1 \qquad |x| < 1 \qquad \boxed{-1 < x < +1}$$

$\underbrace{\text{Converges}}$ to zero $\rightarrow$ a solution general solution

even fn   odd fn

$$y = y_1(x) + y_2(x)$$

even n	A Legendre Polynomial Solution	An infinite series solution
odd n	An infinite series solution	A Legendre Polynomial Solution

A Legendre polynomial is a particular solution

all coefficients are determined

Legendre Polynomials

$P_0(x) = 1$ $\qquad$ $P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$ $\qquad$ $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$ $\qquad$ $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

Order $\downarrow$ $\qquad$ Legendre Polynomial $\downarrow$

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

$n=0$ $\quad (1 - x^2)y'' - 2xy' + 0\,y = 0 \qquad y = P_0(x) = 1$

$n=1$ $\quad (1 - x^2)y'' - 2xy' + 2\,y = 0 \qquad y = P_1(x) = x$

$n=2$ $\quad (1 - x^2)y'' - 2xy' + 6\,y = 0 \qquad y = P_2(x) = \frac{1}{2}(3x^2 - 1)$

$n=3$ $\quad (1 - x^2)y'' - 2xy' + 12\,y = 0 \qquad y = P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$$\begin{cases} C_0 = 1 & n=0 \\ C_0 = (-1)^{\frac{n}{2}} \dfrac{1\cdot 3\cdots(n-1)}{2\cdot 4\cdots n} & n=2,4,6\cdots \end{cases}$$

$n=2$ $$C_0 = (-1)^{\frac{2}{2}}\frac{1}{2} = (-1)\cdot\frac{1}{2} = -\frac{1}{2}$$

$$C_2 = -\frac{n(n+1)}{2!} C_0 = -\frac{\cancel{2}\cdot 3}{\cancel{2}}\left(-\frac{1}{2}\right) = +\frac{3}{2}$$

$$y = -\frac{1}{2}x^0 + \frac{3}{2}x^2 = P_2(x)$$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$\begin{cases} C_1 = 1 & n=1 \\ C_1 = (-1)^{\frac{(n-1)}{2}} \dfrac{1\cdot 3\cdots n}{2\cdot 4\cdots (n-1)} & n=1,3,5,\cdots \end{cases}$$

$n=3$ $$C_1 = (-1)^{\frac{3-1}{2}}\frac{1\cdot 3}{2} = (-1)\frac{3}{2} = -\frac{3}{2}$$

$$C_3 = -\frac{(n-1)(n+2)}{3!} C_1 = -\frac{(3-1)(3+2)}{6}\left(-\frac{3}{2}\right)$$

$$= +\frac{\cancel{2}\cdot 5}{\cancel{6}}\left(+\frac{\cancel{3}}{2}\right)$$

$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$= \frac{5}{2}$$

$$y = C_1 x^1 + C_3 x^3 = -\frac{3}{2}x + \frac{5}{2}x^3 = P_3(x)$$

$n=0$ $$\boxed{(1-x^2)\underbrace{y''}_{0} - 2x\underbrace{y'}_{0} + 0\underbrace{y}_{1} = 0}$$

$$\boxed{y = P_0(x) = 1}$$

$y' = 0$ $\quad$ $y'' = 0$

$n=1$ $$\boxed{(1-x^2)\underbrace{y''}_{0} - 2x\underbrace{y'}_{1} + 2\underbrace{y}_{x} = 0}$$

$$\boxed{y = P_1(x) = x}$$

$y' = 1$ $\quad$ $y'' = 0$

$n=2$ $$\boxed{(1-x^2)\underbrace{y''}_{3} - 2x\underbrace{y'}_{3x} + 6\underbrace{y}_{\frac{1}{2}(3x^2-1)} = 0}$$

$$\boxed{y = P_2(x) = \frac{1}{2}(3x^2-1)}$$

$y' = 3x$ $\quad$ $y'' = 3$

$n=3$ $$\boxed{(1-x^2)y'' - 2x y' + 12 y = 0}$$

$$\boxed{y = P_3(x) = \frac{1}{2}(5x^3 - 3x)}$$

$y' = \frac{15}{2}x^2 - \frac{3}{2}$ $\quad$ $y'' = 15x$

* Orthogonality

$P_0(x) = 1$ $\qquad$ $P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$ $\qquad$ $P_3(x) = \frac{1}{2}(5x^3-3x)$

$$\int_{-1}^{1} 1 \cdot x \, dx = 0$$

$$\int_{-1}^{1} 1 \cdot \frac{1}{2}(3x^2-1)\,dx = \frac{1}{2}\int_{-1}^{1} 3x^2 - 1 \, dx = \frac{1}{2}\left[x^3 - x\right]_{-1}^{1} = 0$$

$$\int_{-1}^{1} 1 \cdot \frac{1}{2}(5x^3-3x)\,dx = 0$$

$$\int_{-1}^{1} x\,\frac{1}{2}(3x^2-1)\,dx = \frac{1}{2}\int_{-1}^{1} 3x^3 - x = 0$$

$$\int_{-1}^{1} x \cdot \frac{1}{2}(5x^3-3x)\,dx = \frac{1}{2}\int_{-1}^{1} 5x^4 - 3x^2\,dx = \frac{1}{2}\left[x^5 - x^3\right] = 0$$

$$\int_{-1}^{1} \frac{1}{2}(3x^2-1)\,\frac{1}{2}(5x^3-3x)\,dx = \frac{1}{4}\int_{-1}^{1} 15x^5 - 9x^3 - 5x^3 + 3x\,dx$$

$$= \frac{1}{4}\int_{-1}^{1} 15x^5 - 14x^3 + 3x\,dx = 0$$

Properties

$$P_n(-x) = (-1)^n P_n(x)$$

$$P_n(1) = 1$$

$$P_n(-1) = (-1)^n$$

$$P_n(0) = 0, \qquad \text{odd } n$$

$$P_n'(0) = 0, \qquad \text{even } n$$

Recurrence Relation

$$(k+1)\,P_{k+1}(x) - (2k+1)\,x\,P_k(x) + k\,P_{k-1}(x) = 0$$

Rodrigue's Formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \qquad n = 0, 1, 2, \dots$$