Z Transform (H.1)
Definition

20170131

Based on
Complex Analysis for Mathematics and Engineering
J. Mathews

z - Transform

$$X(z) = \sum_{k=-\infty}^{+\infty} x[k]\, z^{-k}$$

$$z = |r|\, e^{j2\pi F} = |r|\, e^{j\Omega}$$

$$x[n] \longleftrightarrow X(z)$$

One Sided z-transform

$$X(z) = \sum_{k=0}^{+\infty} x[k]\, z^{-k}$$

Inverse z-Transform

$$X(z) = \mathcal{Z}\left[\{x_n\}_{n=0}^{\infty}\right] = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$x[n] \rightarrow X(z)$

$$x_n = x[n] = \mathcal{Z}^{-1}\left[X(z)\right] = \frac{1}{2\pi i}\int_C X(z) z^{n-1}\, dz$$

$x[n] \leftarrow X(z)$

Admissible Form of z-transform

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

$X(z)$: admissible z-transform

if $X(z)$ is a rational function

$$\boxed{X(z) = \frac{P(z)}{Q(z)}} = \frac{b_0 + b_1 z^1 + b_2 z^2 + \cdots + b_{p-1} z^{p-1} + b_p z^p}{a_0 + a_1 z^1 + a_2 z^2 + \cdots + a_{q-1} z^{q-1} + a_q z^q}$$

$P(z)$: a polynomial of degree p

$Q(z)$: a polynomial of degree q

Residue Theorem

D: simply connected domain
C: simple closed contour (CCW) in D

if $f(z)$ is analytic inside C and on C
except at the points $\boxed{z_1, z_2, \cdots, z_k}$ in C

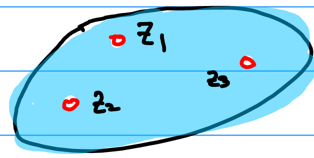
then $$\frac{1}{2\pi i}\int_C f(z)\,dz = \sum_{j=1}^{k} \mathrm{Res}(f(z), z_j)$$

singular points of $f(z)$: $z_1, z_2, \cdots, z_k$

Integration of a function of a complex var.

$$\boxed{\oint_C f(z)\,dz} = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$

finite number k of
singular points z_k
residue theorem

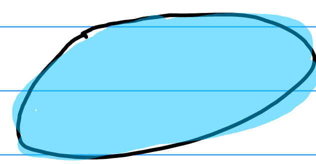


$$\boxed{\oint_C f(z)\,dz} = 0 \quad \text{if } \boxed{f(z) = F'(z)} \text{ on } C$$

: $F(z)$ is an antiderivative of $f(z)$
fundamental theorem of calculus

$$\boxed{\oint_C f(z)\,dz} = 0 \quad \text{if } f(z) \text{ is analytic within and on } C$$

no singularity



Thomas J. Cavicchi
Digital Signal Processing, Wiley, 2000

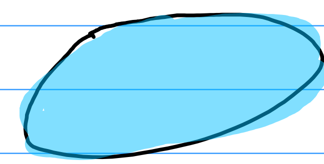
$\boxed{\oint_C f(z)\,dz} = 0$ if $f(z)$ is continuous in D and

$\boxed{f(z) = F'(z)}$: $F(z)$ is an antiderivative of $f(z)$

fundamental theorem of calculus

$\boxed{\oint_C f(z)\,dz} = 0$ if $f(z)$ is analytic within and on C

no singularity



can expand $f(z)$ about any point z_m over powers of $(z - z_m)$

whether or not $f(z)$ is singular at z_m or at other points between z and z_m

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

① Laurent Series Expansion of $f(z)$ at z_m

general n_1 — depend on $f(z)$ and z_m

② z-transform of $a_n^{\{m\}}$

general n_1 — depend on $f(z)$

$\boxed{z_m = 0}$

③ Taylor Series Expansion of $f(z)$ at z_m

positive n_1 — depend on $f(z)$ and z_m $(n_1 \geq 0)$

④ MacLaurin Series Expansion of $f(z)$ at z_m

positive n_1 — depend on $f(z)$ $(n_1 \geq 0)$

$\boxed{z_m = 0}$

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$\ast$ Expansion of $f(z)$ about any point z_m
over powers of $(z - z_m)$

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_m)^{n+1}} \, dz$$ for general $f(z)$

$$a_n^{\{m\}} = \sum_k \mathrm{Res}\left[\frac{f(z)}{(z - z_m)^{n+1}}, z_k \right]$$ for general $f(z)$

$$a_n^{\{m\}} = \frac{1}{n!} f^{(n)}(z_m) \qquad n_1 \geq 0$$ for analytic $f(z)$ within C

analytic $f(z)$ $\longrightarrow$ $\dfrac{f(z)}{(z - z_m)^{n+1}}$ has a pole at z_m
order of $n+1$

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

$\boxed{z_m}$: possible poles of $f(z)$

not necessarily poles

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_m)^{n+1}} \, dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k \right)$$

$\boxed{z_k}$: poles of $\dfrac{f(z)}{(z - z_m)^{n+1}}$

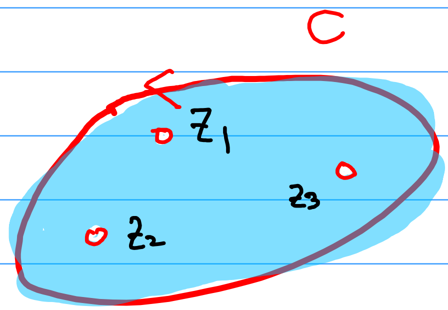
within C

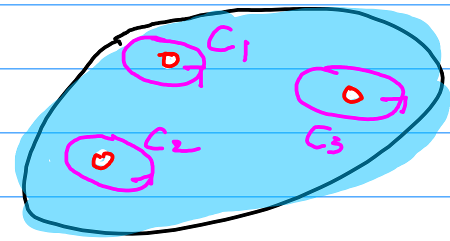
$$= \frac{1}{n!} f^{(n)}(z_m) \qquad n_1 \geq 0$$

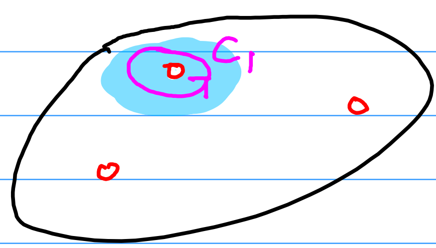
Residue Theorem

assumed there are (m) singularities (poles) of $f(z)$ in a region

C_m is taken to enclose only one pole z_m



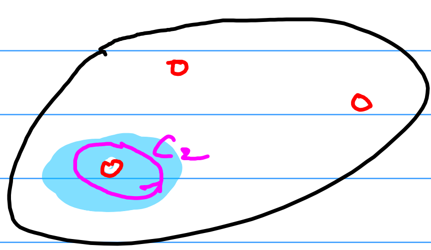




$a_n^{\{1\}}$ expanded at z_1

C_1 encloses z_1 only

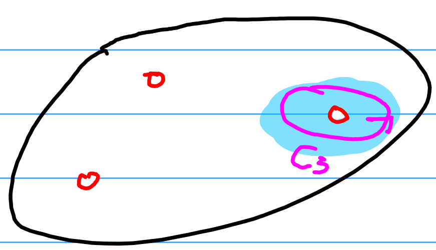
$\tilde{a}_{-1}^{\{1\}} = Res(f(z), z_1)$



$a_n^{\{2\}}$ expanded at z_2

C_2 encloses z_2 only

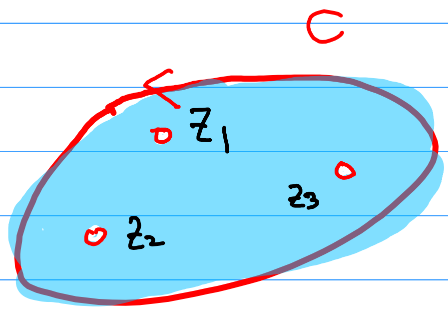
$\tilde{a}_{-1}^{\{2\}} = Res(f(z), z_2)$



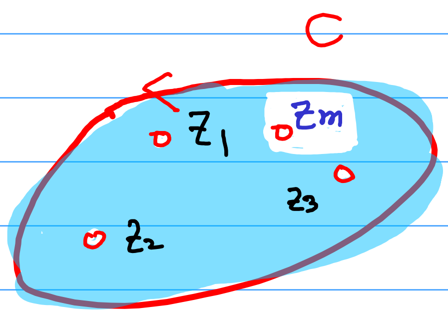
$a_n^{\{3\}}$ expanded at z_3

C_3 encloses z_3 only

$\tilde{a}_{-1}^{\{3\}} = Res(f(z), z_3)$



$$\boxed{\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_m)^{n+1}}\,dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}},\, z_k\right) \end{aligned}}$$



$n \geq 0$ $\quad z_1, \; z_2, \; z_3, \; z_m$

$n < 0$ $\quad z_1, \; z_2, \; z_3$

$n = -1$

$$\boxed{\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_m)^{n+1}}\,dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right) \end{aligned}}
\qquad
\boxed{\begin{aligned} a_{-1}^{\{m\}} &= \frac{1}{2\pi i}\oint_C f(z)\,dz \\ &= \sum_k \mathrm{Res}\left(f(z), z_k\right) \end{aligned}}$$

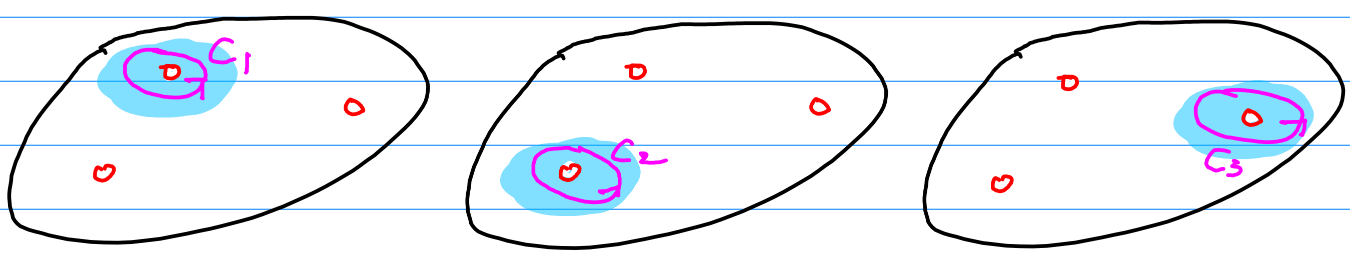
$$a_{-1}^{\{m\}} = \frac{1}{2\pi i}\oint_C f(z)\,dz = \sum_k \mathrm{Res}(f(z), z_k)$$

if C encloses only one pole

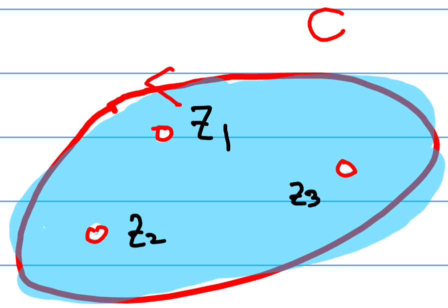
$$a_{-1}^{\{0\}} = \frac{1}{2\pi i}\oint_{C_0} f(z)\,dz = \mathrm{Res}(f(z), z_0)$$

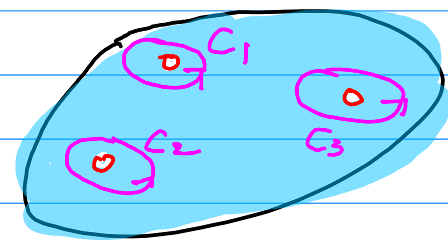
$$\boxed{\tilde{a}_{-1}^{\{m\}} = \mathrm{Res}(f(z), z_m)}$$

the residue of $f(z)$ at z_m using C_m



$\tilde{a}_{-1}^{\{1\}} = \mathrm{Res}(f(z), z_1)$ $\tilde{a}_{-1}^{\{2\}} = \mathrm{Res}(f(z), z_2)$ $\tilde{a}_{-1}^{\{3\}} = \mathrm{Res}(f(z), z_3)$





$$\boxed{\oint_C f(z)\,dz = 2\pi j \sum_{k=1}^{M} \tilde{a}_{-1}^{\{k\}} = 2\pi j \sum_{k=1}^{M} \mathrm{Res}\big(f(z), z_k\big)}$$

residue theorem

$$\boxed{a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k\right)}$$

Laurent coefficient

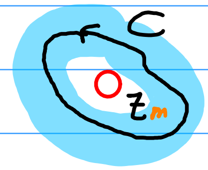
C encloses k poles

C_k encloses only the k-th pole

$\tilde{a}_{-1}^{\{k\}}$ the residue of the k-th pole enclosed by C, z_k

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} dz'$$



$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

C is in the same region of analyticity of $f(z)$

typically a circle centered on z_m

z_k within C : singularities of $\dfrac{f(z)}{(z-z_m)^{n+1}}$

$n_1 = n_{f,m}$ depends on $f(z)$, z_m

$a_n^{\{m\}}$ depends on $f(z)$, z_m, region of analyticity

Whether $f(z)$ is singular at $z = z_m$ or not

or at other points between z and z_m

We can expand $f(z)$ about any point z_m

over powers of $(z - z_m)$.

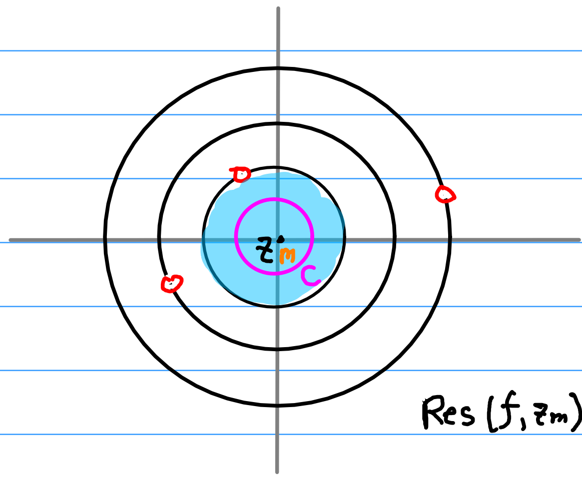
$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} \, dz'$$

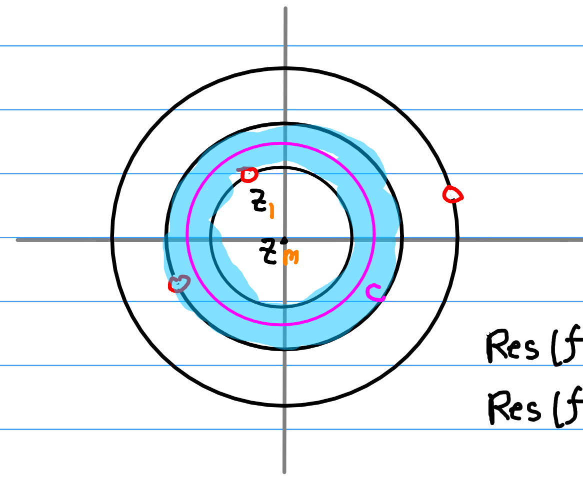
$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k \right)$$

z_k within C : singularities of $\dfrac{f(z)}{(z-z_m)^{n+1}}$

$$\begin{cases} \text{poles of } f(z) \cup \; z = z_m & n \geq 0 \\ \text{poles of } f(z) & n < 0 \end{cases}$$

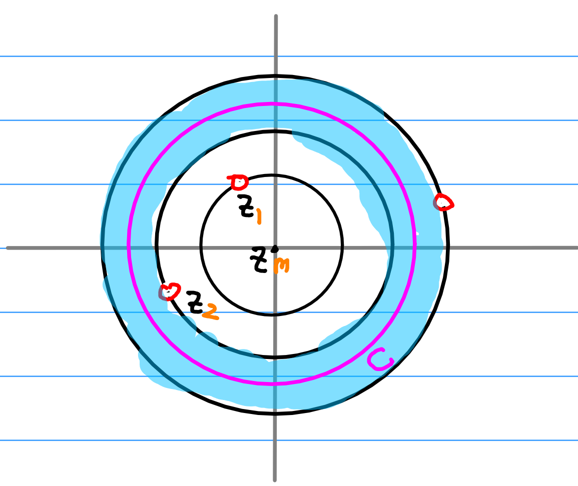


$\mathrm{Res}(f, z_m)$

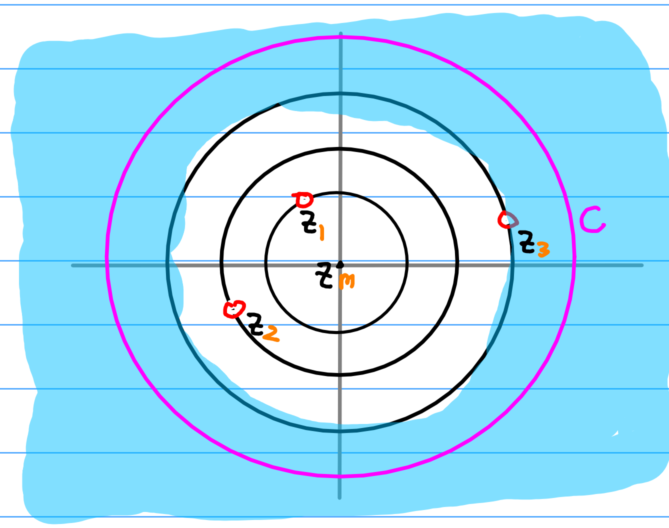


$\mathrm{Res}(f, z_1)$

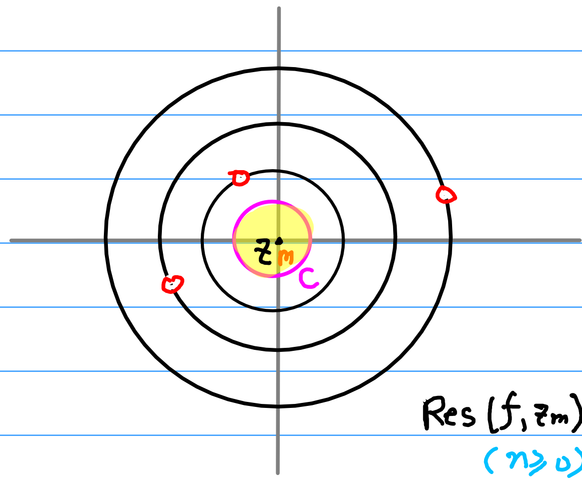
$\mathrm{Res}(f, z_m)$



$\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_m)$

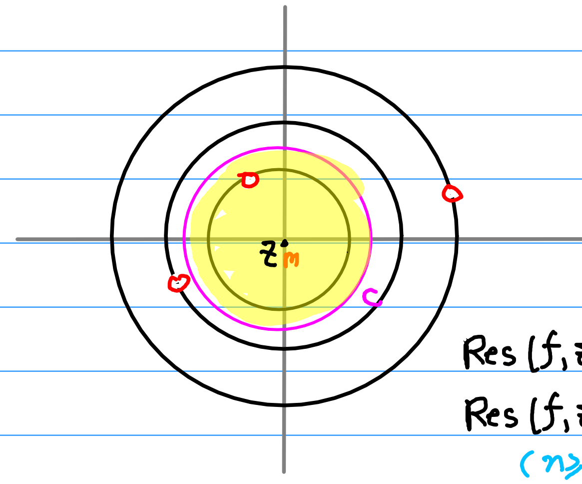


$\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_3)$
$+ \mathrm{Res}(f, z_m)$



$\mathrm{Res}(f, z_m)$

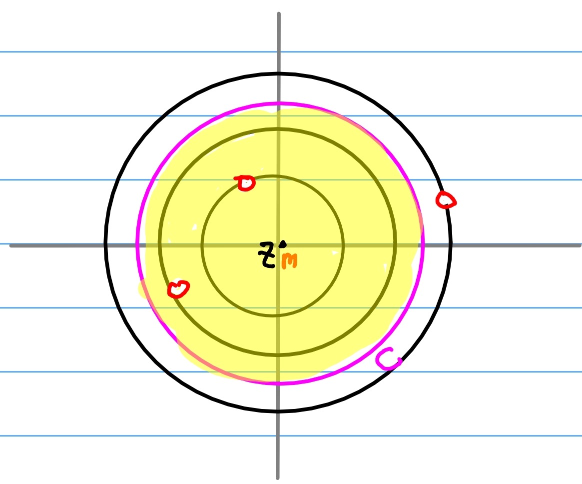
$(n \geq 0)$



$\mathrm{Res}(f, z_1)$

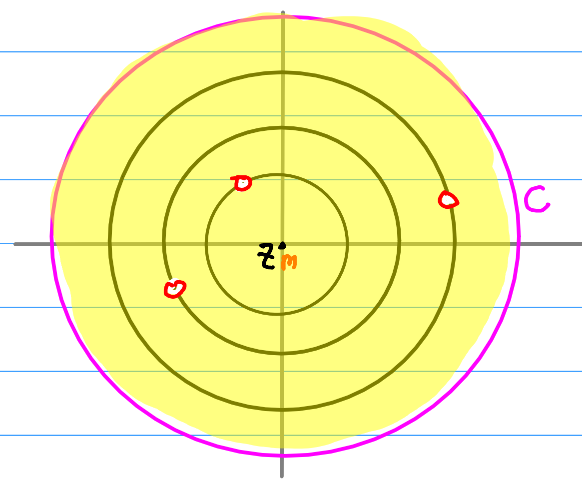
$\mathrm{Res}(f, z_m)$

$(n \geq 0)$



$\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_m)$

$(n \geq 0)$



$\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_3)$

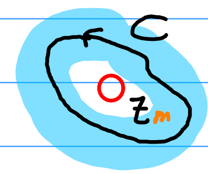
$+ \mathrm{Res}(f, z_m)$

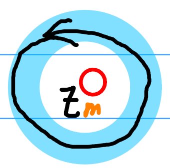
$(n \geq 0)$

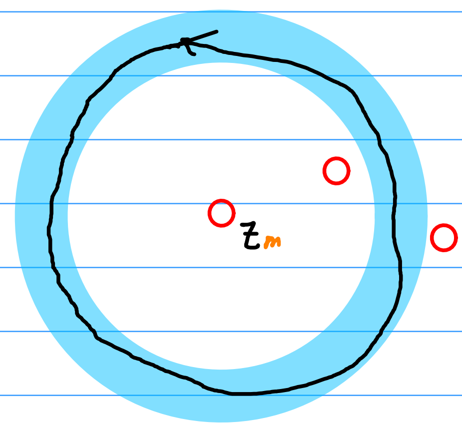
$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

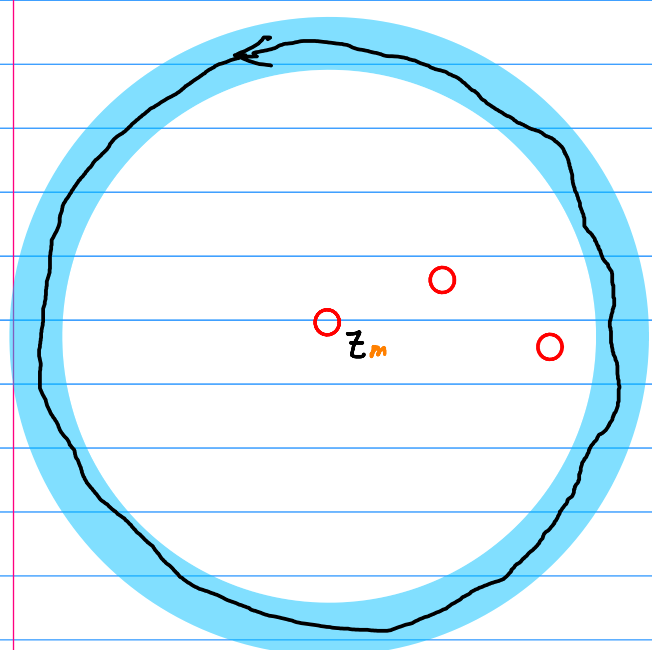
$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} \, dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}} , z_k \right)$$









Laurent's Theorem

f : analytic within the annular domain D

$$r < |z - z_0| < R$$

then

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k \quad , \text{ valid for } \quad r < |z - z_0| < R$$

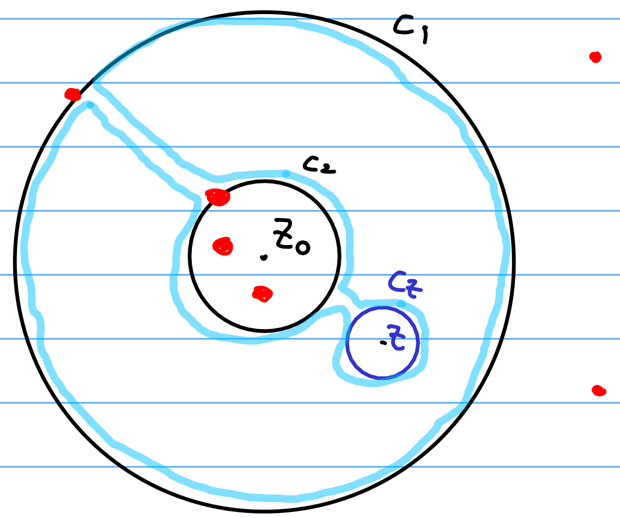
$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} \, ds , \qquad k = 0, \pm 1, \pm 2, \cdots$$

C : a simple closed curve
that lies entirely within D
that encloses z_0

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s) \, ds \Rightarrow \oint_C f(s) \, ds = 2\pi i \cdot a_{-1}$$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s) \, ds = Res(f(z), z_0)$$

$$= \begin{cases} \lim_{z \to z_0} (z - z_0) f(z) & \text{(simple)} \\ \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) & \text{(order } n) \end{cases}$$



z_0 : expansion point

z : evaluation point

which poles of $f(z)$ lie between the point of evaluation z and the point z_0 about which the expansion is formed

$\dfrac{f(z')}{(z'-z_0)}$ is analytic between C_1 & C_2

deformation theorem $\quad C_1 - C_2$ coincide

common contour C

Cauchy's Residue Theorem

$f(z)$: analytic on and within C

except a finite number of singular points

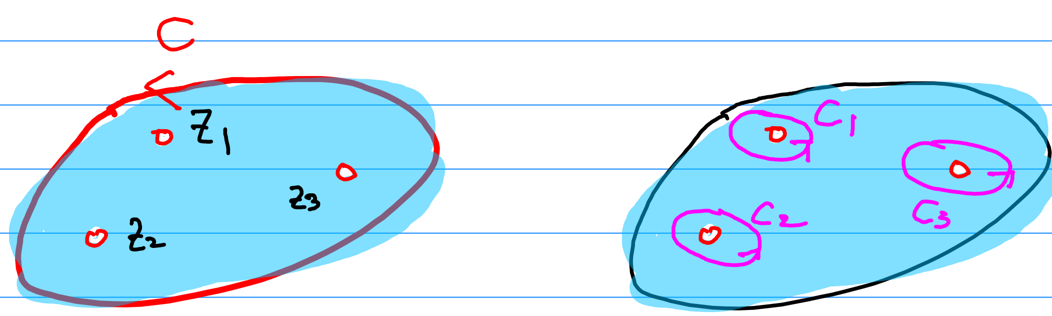
$z_1, z_2, \cdots, z_n$ within C

then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$

D : a simply connected domain

C : a simple closed contour in D



$\boxed{z_1}$ $\quad f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_1)^k \qquad a_{-1}^{\{1\}} = \frac{1}{2\pi i} \oint_{C_1} f(s)\,ds = \mathrm{Res}(f(z), z_1)$

$\boxed{z_2}$ $\quad f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_2)^k \qquad a_{-1}^{\{2\}} = \frac{1}{2\pi i} \oint_{C_2} f(s)\,ds = \mathrm{Res}(f(z), z_2)$

$\boxed{z_3}$ $\quad f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_3)^k \qquad a_{-1}^{\{3\}} = \frac{1}{2\pi i} \oint_{C_3} f(s)\,ds = \mathrm{Res}(f(z), z_3)$

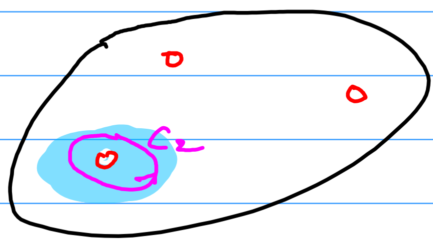
$\boxed{z_1}$

Laurent series expansion at z_1

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_1)^k$$

$$a_{-1}^{\{1\}} = \frac{1}{2\pi i} \oint_{C_1} f(s)\, ds = \mathrm{Res}(f(z), z_1)$$

$\boxed{z_2}$

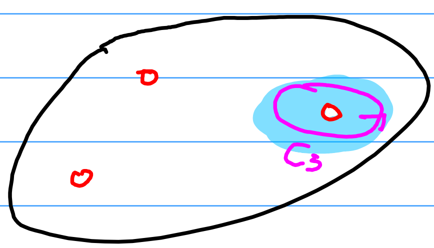


Laurent series expansion at z_2

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_2)^k$$

$$a_{-1}^{\{2\}} = \frac{1}{2\pi i} \oint_{C_2} f(s)\, ds = \mathrm{Res}(f(z), z_2)$$

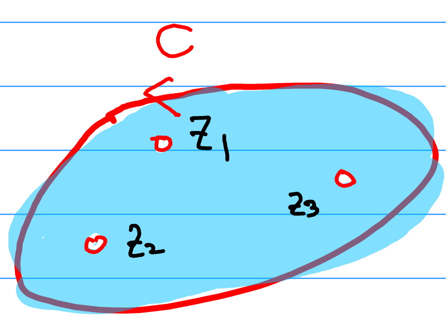
$\boxed{z_3}$



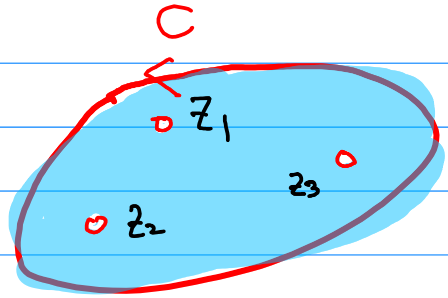
Laurent series expansion at z_3

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_3)^k$$

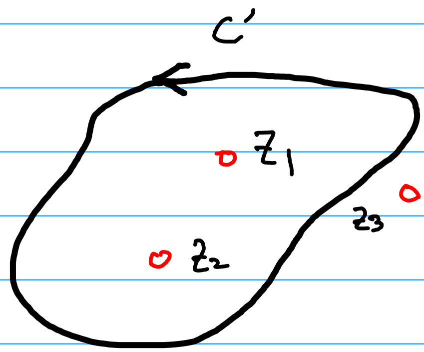
$$a_{-1}^{\{3\}} = \frac{1}{2\pi i} \oint_{C_3} f(s)\, ds = \mathrm{Res}(f(z), z_3)$$



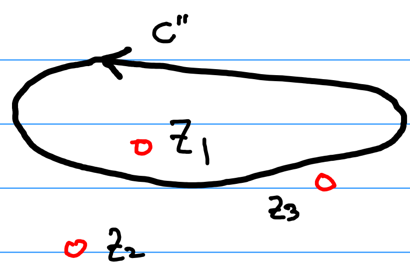
$$\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$



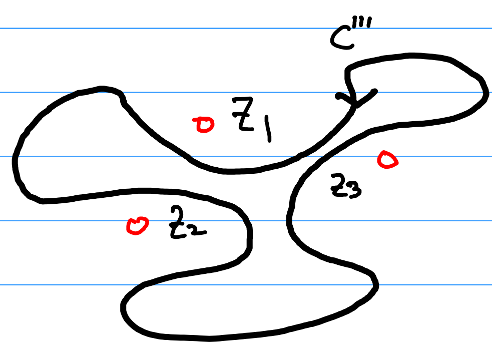
$$\int_C f(z)\,dz = 2\pi i \operatorname{Res}(f(z), z_1) + 2\pi i \operatorname{Res}(f(z), z_2) + 2\pi i \operatorname{Res}(f(z), z_3)$$



$$\int_{C'} f(z)\,dz = 2\pi i \operatorname{Res}(f(z), z_1) + 2\pi i \operatorname{Res}(f(z), z_2)$$

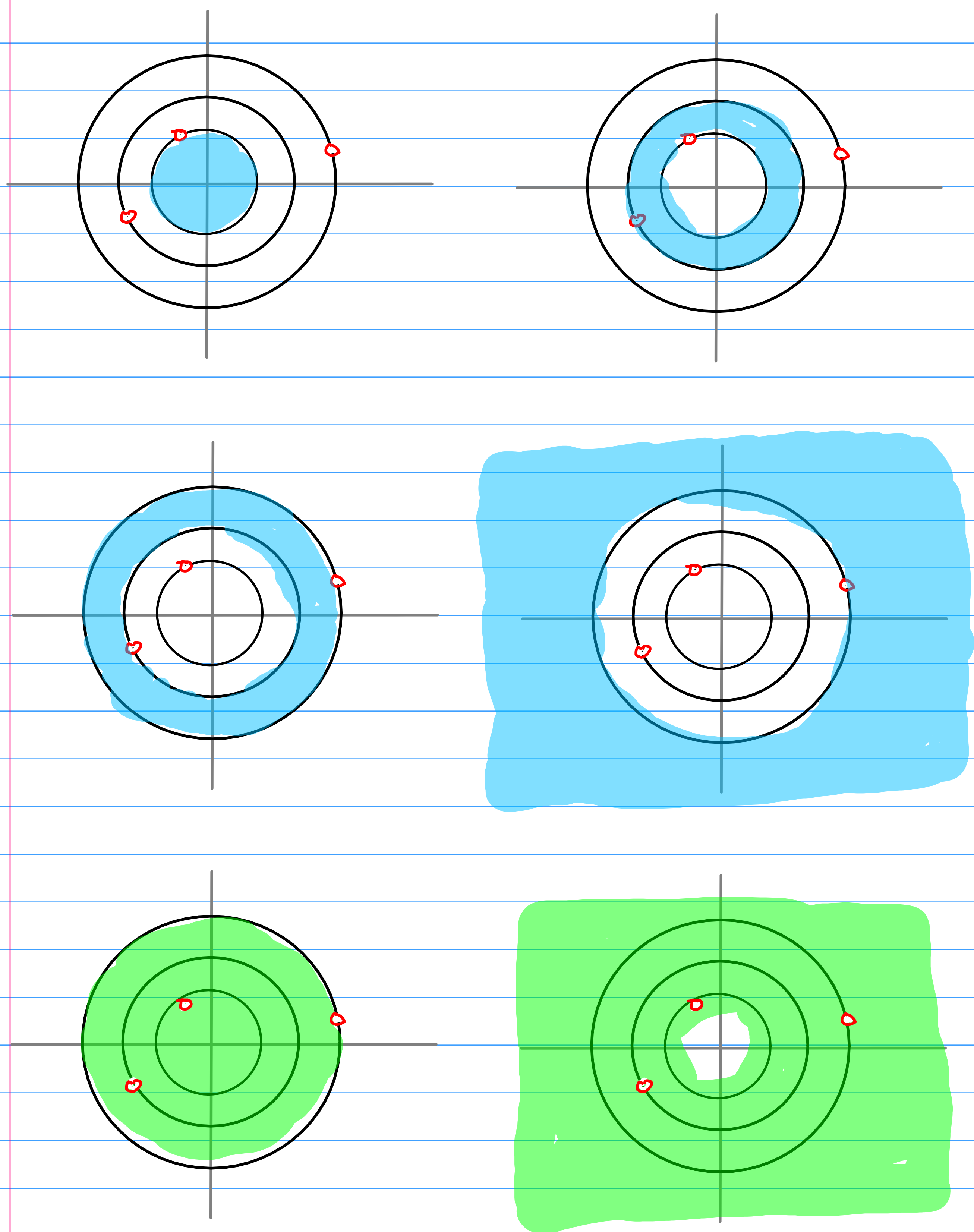


$$\int_{C''} f(z)\,dz = 2\pi i \operatorname{Res}(f(z), z_1)$$



$$\int_{C''} f(z)\,dz = 0$$

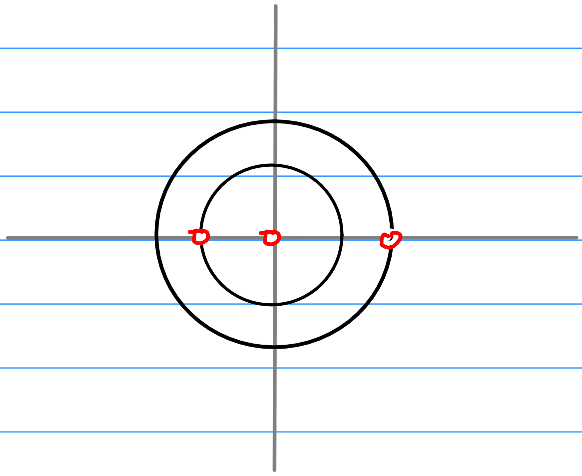
Different D, Different Laurent Series



z-transform

$\cdot\ f(z) = \dfrac{12}{z(2-z)(1+z)} = \dfrac{4}{z}\left(\dfrac{1}{1+z} + \dfrac{1}{2-z}\right)$

pole : $z=0$, $z=2$, $z=-1$



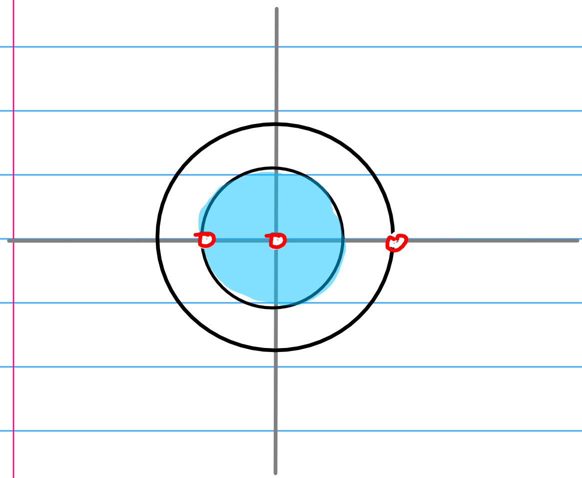
$0 < |z| < 1$

$$f(z) = -3 + 9z/2 - 15z^2/4 + 33z^3/8 + \cdots + 6/z$$

$|z| > 2$

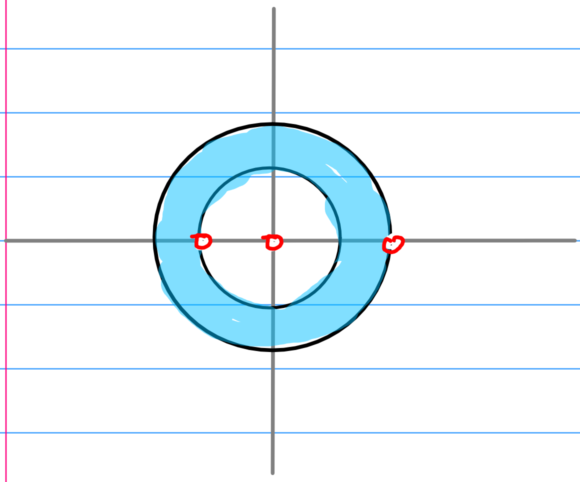
$$\frac{1}{1+z} = \frac{1}{z}\frac{1}{(1+z^{-1})} \qquad \frac{1}{2+z} = -\frac{1}{z}\frac{1}{1-2z^{-1}}$$

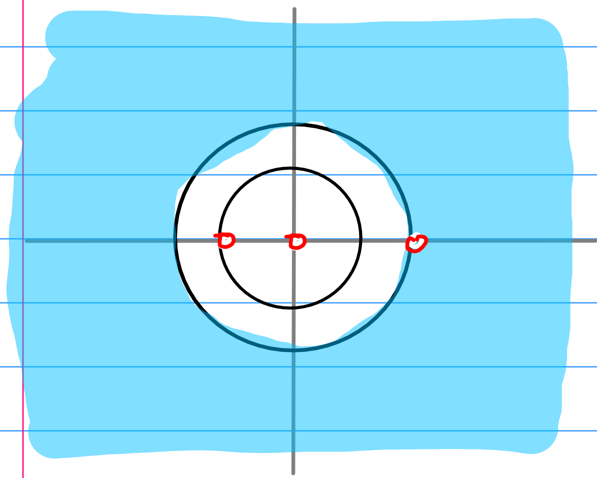
$$f(z) = -(12/z^3)(1 + 1/z + 3/z^2 + 5/z^3 + 11/z^4 + \cdots)$$



$0 < |z| < 1$

$$f(z) = -3 + 9z/2 - 15z^2/4 + 33z^3/8 + \cdots + 6/z$$





$|z| > 2$

$$\frac{1}{1+z} = \frac{1}{z}\frac{1}{(1+z^{-1})} \qquad \frac{1}{2+z} = -\frac{1}{z}\frac{1}{1-2z^{-1}}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

Complex Variables and Ap
Brown & Churchill

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

$D_1 : |z| < 1$

$D_2 : 1 < |z| < 2$

$D_3 : 2 < |z|$

① D_1 $\quad |z| < 1, \quad |\frac{z}{2}| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{2}\frac{1}{1-(\frac{z}{2})}$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n \quad |z| < 1$$

② D_2 $\quad 1 < |z| < 2 \Rightarrow |\frac{1}{z}| < 1, \quad |\frac{z}{2}| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-(\frac{1}{z})} + \frac{1}{2}\frac{1}{1-(\frac{z}{2})}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

③ D_3 $\quad 2 < |z| \quad |\frac{2}{z}| < 1 \quad |\frac{1}{z}| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-(\frac{1}{z})} - \frac{1}{z}\frac{1}{1-(\frac{2}{z})}$$

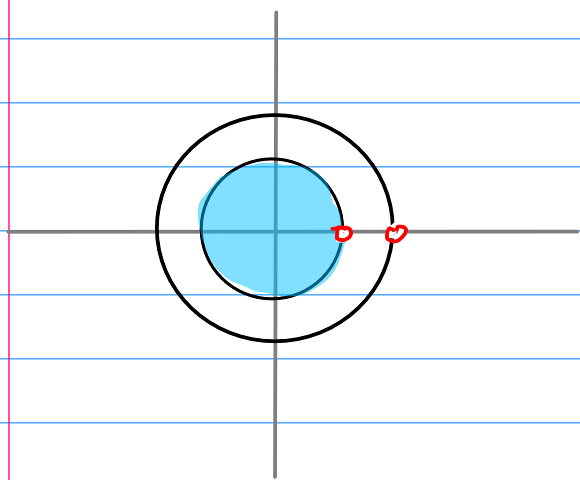
$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

① D_1 $\quad |z|<1$, $\quad \left|\frac{z}{2}\right|<1$

$$\frac{f(z)}{z^{n+1}} = \frac{-1}{(z-1)(z-2)z^{n+1}}$$



$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \left(2^{-n+1} - 1\right) z^n \quad |z|<1$$

$$a_n = \frac{f(z)}{z^{n+1}} = \frac{1}{(z-1)(z-2)z^{n+1}} \qquad \frac{1}{z-1} - \frac{1}{z-2}$$

$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \quad (\text{order } n)$$

$$\frac{d}{dz}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)\left((z-1)^{-2} - (z-2)^{-2}\right)$$

$$\frac{d^2}{dz^2}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)(-2)\left((z-1)^{-3} - (z-2)^{-3}\right)$$

$$\frac{d^3}{dz^3}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)(-2)(-3)\left((z-1)^{-4} - (z-2)^{-4}\right)$$

$$\frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n n! \left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

$$\frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n \lim_{z \to 0} \left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

$$= (-1)^n \left((-1)^{-n-1} - (-2)^{-n-1}\right)$$

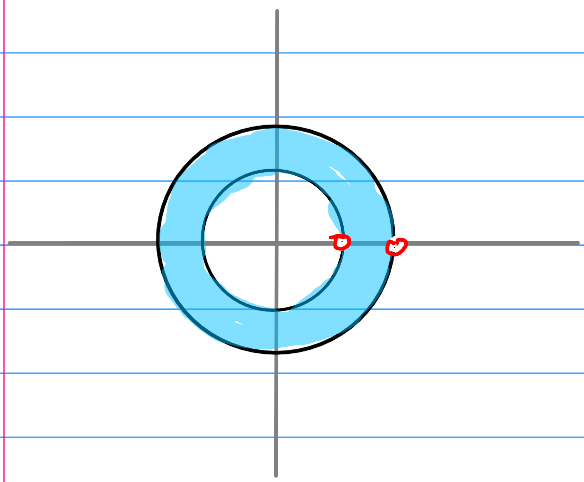
$$= -1 + 2^{-n-1}$$

$$a_n = -1 + 2^{-n-1} \qquad (n \geq 0)$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \left(2^{-n+1} - 1\right) z^n \qquad |z|<1$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

② D_2 $\quad 1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$



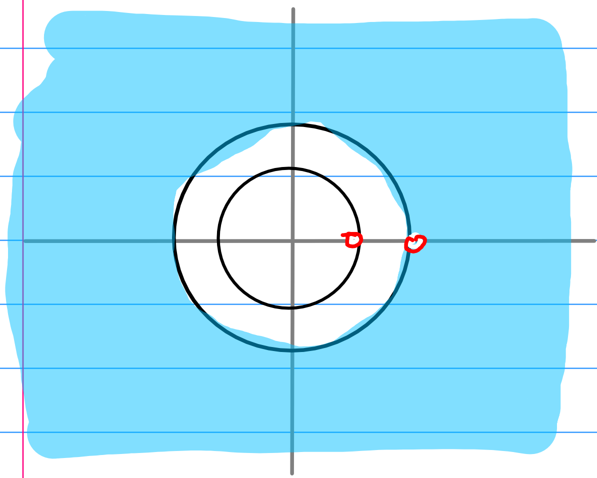
$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\left(\frac{1}{z}\right)} + \frac{1}{2} \frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

③ D_3 $\quad 2 < |z| \quad \left|\frac{2}{z}\right| < 1 \quad \left|\frac{1}{z}\right| < 1$

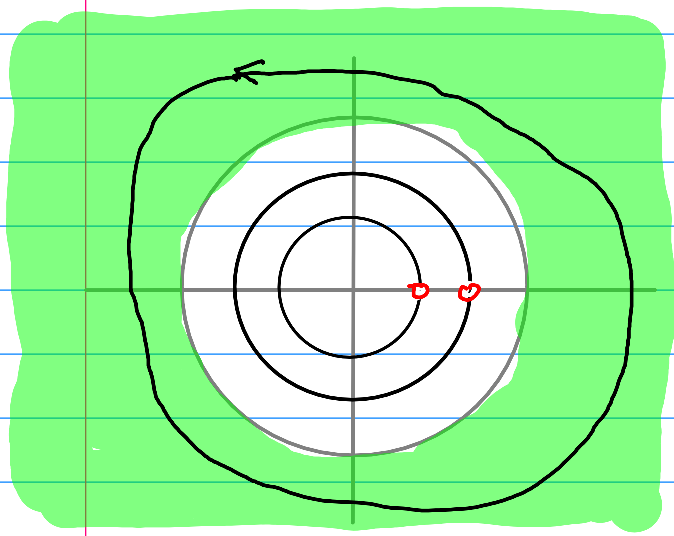


$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$



$x[n]$

$$= \frac{1}{2\pi i} \int_C \boxed{X(z)\, z^{n-1}}\, dz$$

$$= \sum_{j=1}^{k} \text{Res}\left(\boxed{X(z)\, z^{n-1}}, z_j\right)$$

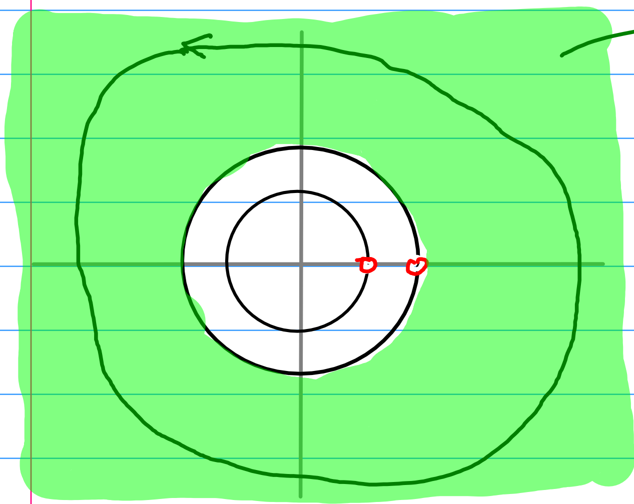
$$X(z) = \frac{-1}{(z-1)(z-2)}$$

$$X(z)\, z^{n-1} = \frac{-1}{(z-1)(z-2)} z^{n-1}$$

$$\text{Res}\left(\boxed{X(z)\, z^{n-1}}, 1\right) = (z-1)\frac{-1}{(z-1)(z-2)} z^{n-1} \Big|_{z=1} = 1$$

$$\text{Res}\left(\boxed{X(z)\, z^{n-1}}, 2\right) = (z-2)\frac{-1}{(z-1)(z-2)} z^{n-1} \Big|_{z=2} = -2^{n-1}$$

$$x[n] = 1 - 2^{n-1}$$



ROC (Region of Convergence)

$$|z| > 2 \Rightarrow \frac{2}{|z|} < 1$$

$$\left(\frac{2}{z}\right)^0 + \left(\frac{2}{z}\right)^1 + \left(\frac{2}{z}\right)^2 + \cdots \xrightarrow[\text{converge}]{} \frac{1}{1 - \frac{2}{z}}$$

$$|z| > 2 \Rightarrow \frac{1}{|z|} < 1$$

$$\left(\frac{1}{z}\right)^0 + \left(\frac{1}{z}\right)^1 + \left(\frac{1}{z}\right)^2 + \cdots \xrightarrow[\text{converge}]{} \frac{1}{1 - \frac{1}{z}}$$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

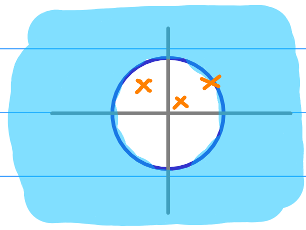
$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$\left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots$$
$$+ \frac{-1}{2}\left\{ \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots \right\} \xrightarrow[\text{converge}]{} \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{(z-1)(z-2)}$$

$$(1-2^0)z^{-1} + (1-2^1)z^{-2} + (1-2^2)z^{-3} + \cdots \xrightarrow[\text{converge}]{} \frac{-1}{(z-1)(z-2)} \quad (|z| > 2)$$

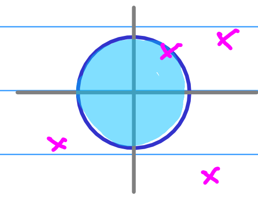
$$x[n] = 1 - 2^{n-1} \longleftrightarrow X(z) = \frac{-1}{(z-1)(z-2)} \quad (|z| > 2)$$

causal $x[n]=0$ $(n<0)$



ROC: outside a circle

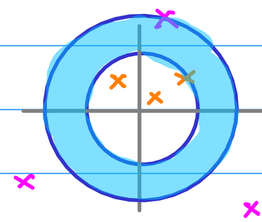
anti-causal $x[n]=0$ $(n>0)$



ROC: inside a circle

bi-causal $x[n]$

Overlapped ROC



$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$f(z) = \sum_{n=n_1}^{\infty} a_n z^n \qquad z_m = 0 \qquad a_n^{\{0\}} \to a_n$$

Laurent Series at $z=0$

$$f(z) = \cdots + a_{-2} z^{-2} + a_{-1} z^{-1} + a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + \cdots$$

Z-transform

$\boxed{\text{Bi-causal}}$ $\quad X(z) = \cdots + x[-2] z^2 + x[-1] z^1 + x[0] z^0 + x[1] z^{-1} + x[2] z^{-2} + x[3] z^{-3} + \cdots$

$\boxed{\text{Causal}}$ $\quad X(z) = x[0] z^0 + x[1] z^{-1} + x[2] z^{-2} + x[3] z^{-3} + \cdots$

$\boxed{\text{Anti-causal}}$ $\quad X(z) = \cdots + x[-2] z^2 + x[-1] z^1 + x[0] z^0$

$$a_n \longleftrightarrow x[-n]$$

$$a_{-n} \longleftrightarrow x[n]$$

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z'-z_m)^{n+1}} \, dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}} , z_k \right)$$

analytic at z_m

$n_1 \geq 0$ Taylor Series

general n_1 $z_m = 0$ MacLaurin Series

singular at z_m

general n_1 Laurent Series

general n_1 $z_m = 0$ z - Transform

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z'-z_m)^{n+1}} dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

$$\boxed{z_m = 0} \quad \boxed{a_{-n}^{\{0\}} = h(n)} \quad \boxed{n \to -n}$$

$$H(z) = \sum_{n=n_1}^{\infty} h(-n)\, z^n$$

$$h(n) = \frac{1}{2\pi i} \oint_c \frac{H(z')}{z'^{\,n+1}} dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{H(z)}{z^{n+1}}, z_k\right)$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n}$$

$$h(n) = \frac{1}{2\pi i} \oint_c H(z')\, z'^{\,n-1} dz'$$

$$= \sum_k \mathrm{Res}\left(H(z)\, z^{n-1}, z_k\right)$$

C is in the same region of analyticity of $f(z)$

typically a circle centered on z_m

z_k within C : singularities of $\frac{f(z)}{(z-z_m)^{n+1}}$

C is in the same region of analyticity of $H(z)$

typically a circle centered on z_m

generally a circle centered on the origin

may enclose any or all singularities of $H(z)$

often the unit circle

z_k within C : singularities of $H(z)\, z^{n-1}$

$$\boxed{\begin{array}{ll} H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n} & z \in \text{R.O.C} \\ h(n) = \frac{1}{2\pi i} \oint_C H(z')\, z'^{\,n-1}\, dz' & C \text{ in R.O.C.} \\ \quad = \sum_k \operatorname{Res}\left(H(z)\, z^{n-1}, z_k\right) & \end{array}}$$

① a power series representation
of a function $f(z)$ of a complex variable z

② a transform $H(z)$ of a sequence of 1

$$X(z) = \frac{z}{z-\frac{1}{2}} \qquad \text{pole } z_0 = \frac{1}{2}$$

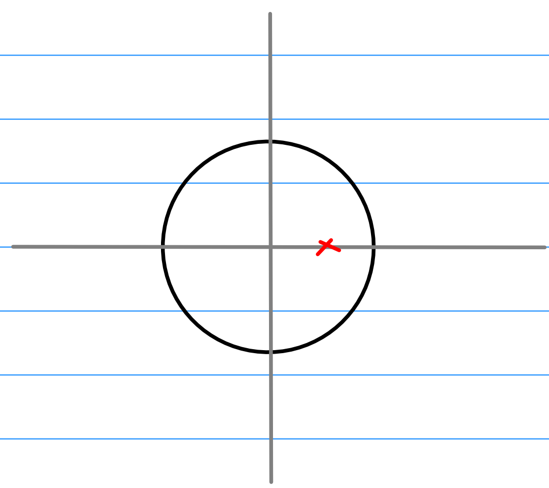
$$x[n] = \mathrm{Res}\left(X(z)\, z^{n-1},\ z_0\right) = \mathrm{Res}\left(\frac{z}{z-\frac{1}{2}}\, z^{n-1},\ \frac{1}{2}\right)$$

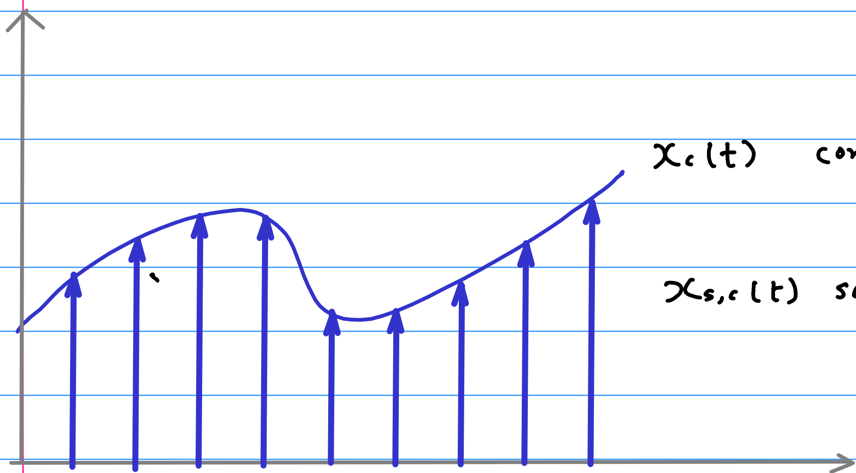
$$= \mathrm{Res}\left(\frac{z^n}{z-\frac{1}{2}},\ \frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{z^n}{z-\frac{1}{2}} = \left(\frac{1}{2}\right)^n$$

$$x[n] = \frac{1}{2^n} \qquad n \geq 0$$

$$\left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \left(\frac{1}{2}\right)^3 z^{-3} + \cdots = \frac{1}{1-\left(\frac{1}{2} z^{-1}\right)}$$

$$= \frac{z}{z-\frac{1}{2}}$$





$$x_{s,c}(t) = \sum_{n=-\infty}^{+\infty} x(n)\, \delta_c(t - n\Delta t)$$

$$X_{s,c}(s) = \mathcal{L}\{x_{s,c}(t)\} = \int_{-\infty}^{\infty} \boxed{\sum_{n=-\infty}^{+\infty} x(n)\, \delta_c(t - n\Delta t)}\, e^{-st}\, dt$$

$$= \sum_{n=-\infty}^{+\infty} x(n) \int_{-\infty}^{\infty} \delta_c(t - n\Delta t)\, e^{-st}\, dt$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-sn\Delta t} \qquad e^{s\Delta t} \triangleq z$$

$$X_{s,c}(s) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \Big|_{z = e^{s\Delta t}}$$

$$X_{s,c}(s) = X(z) \Big|_{z = e^{s\Delta t}}$$

$$X_{s.c}(s) = \mathcal{L}\{x_{s.c}(t)\} = X(z)\Big|_{z=e^{s\Delta t}}$$

$x_{s.c}(t)$ an impulse train

whose coefficients are given by $x[n] = x_c(n\Delta t)$

z - transform : a special Laurent series

$\boxed{z_m = 0}$ $\boxed{a_{-n}^{\{0\}} = h(n)}$ $\boxed{n \to -n}$

$$\boxed{\begin{aligned} f(z) &= \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n \\ a_n^{\{m\}} &= \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} dz' \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k \right) \end{aligned}}$$

Time Reversal ← Laplace Transform

the transform functions

$X(s) = \int$ over negative powers e^{-st} for $t > 0$

$X(z) = \int$ over negative powers z^{-n} for $n > 0$

the time expansion functions

$x(t) = \int$ over negative powers e^{-st} for $t > 0$

$x[n] = \int$ over negative powers z^{-n} for $n > 0$

Time Reversal ← z^{-1} : unit delay , char eq (modes in z^k)

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Stable System : $h[n]$ must be asbsolutely summable

$$|e^{j\omega n}| = 1$$

$$|z^n| \qquad z = 1$$

$$\infty > M_h > \sum_{n=-\infty}^{\infty} |h[n]| \qquad \text{asolutely summable}$$

$$= \sum_{n=-\infty}^{+\infty} |h[n] e^{-j\omega n}|$$

$$\geq \left| \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n} \right|$$

$$= \left| H(z)\big|_{z=e^{j\omega}} \right|$$

$$\infty > \left| H(z)\big|_{z=e^{j\omega}} \right|$$

a stable system,

$H(z)$ must converge on the unit circle $|z|=1$

ROC (Region of Convergence) must include the unit circle

regardless of causality of $h[n]$

$$H(z)\Big|_{|z|=1} = H(e^{j\hat{\omega}})$$ DTFT of $h[n]$

discrete all stable sequence must have convergent DTFTs

continuous all stable signal must have convergent CTFTs

$C \leftarrow$ unit circle $z = e^{j\hat{\omega}}$

ZT^{-1} $DTFT^{-1}$ identical formulas

$h[n]$ causal

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n]\, z^{-n} = \sum_{n=0}^{+\infty} h[n]\, z^{-n} \qquad n \in [0, \infty)$$

for finite values of n,

each term must be finite as long as $z \neq 0$

For the sum to converge,

$h[n]\, z^{-n}$ must vanish as $n \to \infty$

$|z| > r_h \qquad z_h = r_h e^{j\theta}$

z_h^n is the largest magnitude

geometrically increasing component

$n^m z_e^n$: the most general term

for impulse responses

$n \to \infty \qquad z_e^n$ dominant over n^m for finite m

geometric components — as poles

$$\mathcal{Z}\{z_l^n u[n]\} = \frac{1}{1-\left(\frac{z_l}{z}\right)} = \frac{z}{z - z_l}$$

ROC of a causal sequence $h[n]$

outside the radius of the largest magnitude pole of $H(z)$

ROC of a causal signal $h(t)$

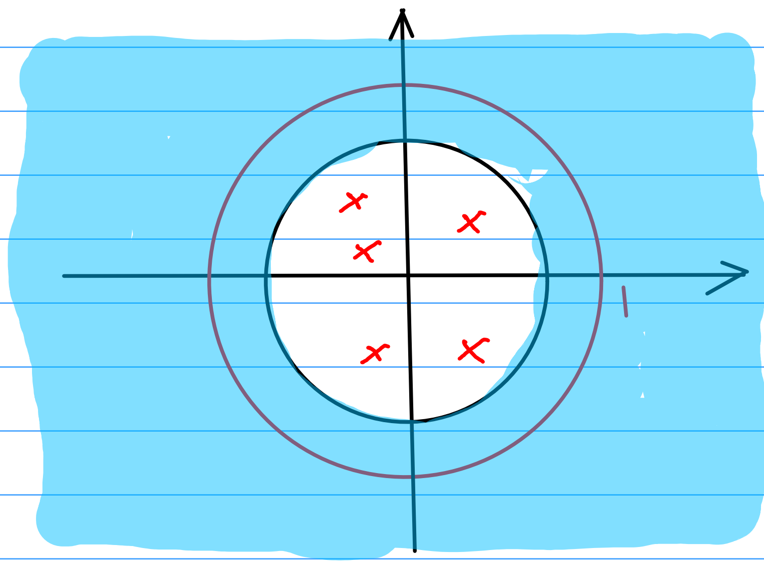
to the right of the rightmost pole of $H_c(s)$

if $h[n]$ is a stable, causal sequence,

the unit circle must be included in the ROC

- Causal $h[n]$

ROC : outside of a circle



- Stable $h[n]$

all poles inside the unit circle

ROC circle must be smaller than the unit circle

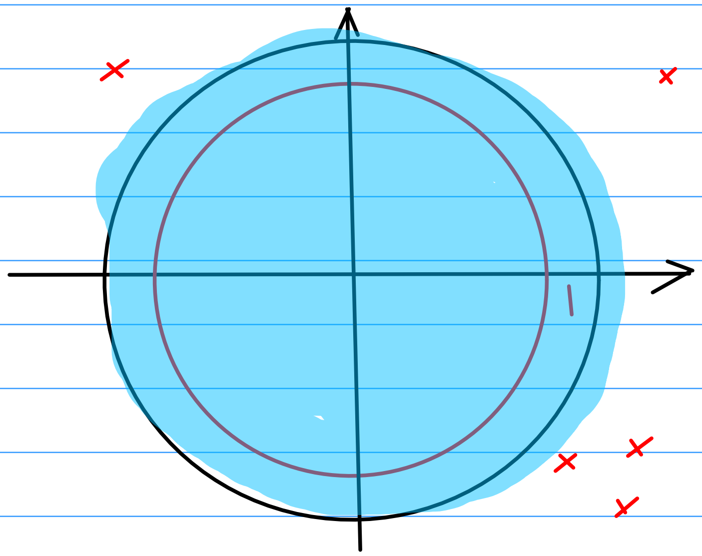
⇒ all the geometric components of $h[n]$: modes must decay with increasing n

all the poles of $H(z)$ must be within the unit circle

all the poles of $H_c(s)$ must be in the left half plane

- anti-Causal $h[n]$

  ROC: in side of a circle



- Stable $h[n]$

  all poles outside the unit circle

  ROC circle must be larger than the unit circle

$\Rightarrow$ all the geometric components of $h[n]$: modes must decay with decreasing n

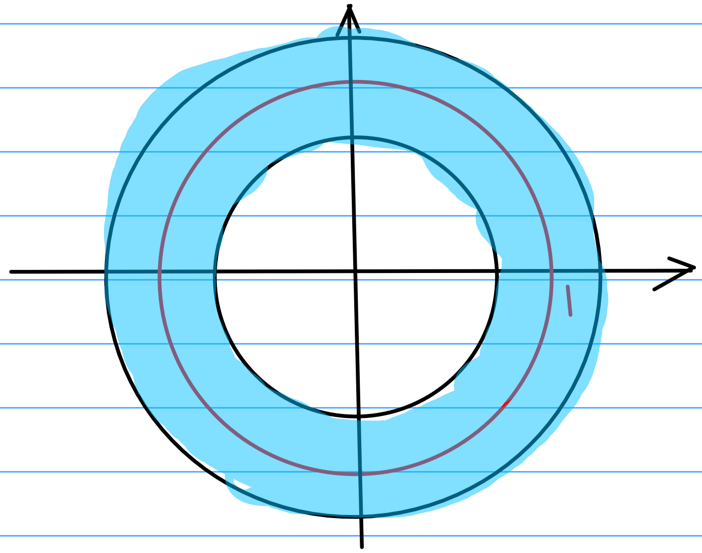
- bi-Causal $h[n]$

  $h_c[n] + h_{ac}[n]$

  outside inside

  max mag < min mag

  Overlapped ROC



- Stable $h[n]$

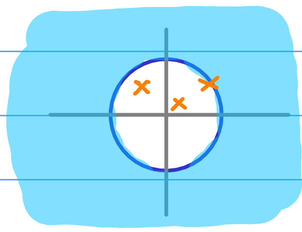
  all poles outside the unit circle

  ROC circle must include the unit circle

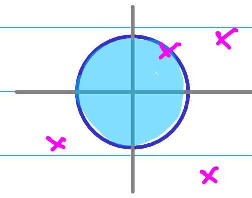
- bi-Causal $h[n]$

$$h[n] = h_c[n] + h_{ac}[n]$$

causal comp. — anti-causal comp



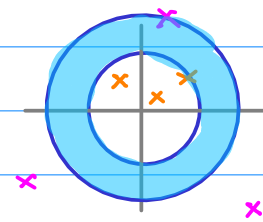
outside a circle



inside a circle

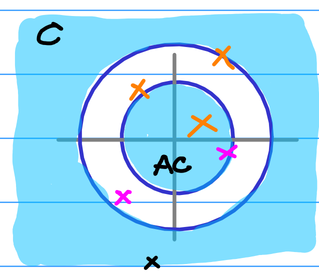
max mag < min mag

Overlapped ROC



max mag > min mag

non-overlapping ROC



- Stable $h[n]$

all poles <u>outside</u> the large circle

<u>inside</u> the small circle

ROC circle must include the <u>unit circle</u>

only one annulus include the <u>unit circle</u>

only one stable sequence

Existence of the z-Transform

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} \frac{x[n]}{z^n}$$

the existence of the z-transform is guaranteed if

$$|X(z)| \leq \sum_{n=0}^{\infty} \frac{|x[n]|}{|z^n|} < \infty \quad \text{for some } |z|$$

any signal $x[n]$ that grows no faster than
an exponential signal r_0^n, for some r_0
satisfies the above condition

if $|x[n]| \leq r_0^n$ for some r_0

then $|X(z)| \leq \sum_{n=0}^{\infty} \left(\frac{r_0}{|z|}\right)^n = \dfrac{1}{1 - \frac{r_0}{|z|}} \quad |z| > r_0$

therefore $X(z)$ exists for $|z| > r_0$

Almost all practical signal satisfy this condition
$|x[n]| \leq r_0^n$ for some r_0
and z-transformable

Some signal models (e.g. r^{n^2}) grows faster than
the exponential signal r_0^n (for any r_0)
and do not satisfy this condition
and are not z-transformable

Such signals are of little practical or theoretical interest
Even such signals over a finite interval are z-transformable

Region of Convergence

Laplace Transform $\quad Ae^{\alpha t}u(t) \quad \alpha > 0$

z - Transform $\quad A\alpha^n u[n] \quad |\alpha| > 0$

DTFT (X)

$$X(z) = A \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} = A \sum_{n=0}^{\infty} \alpha^n z^{-n} = A \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n$$

converge $\left|\frac{\alpha}{z}\right| < 1 \quad |z| > |\alpha|$

open exterior of
a circle of radius $|\alpha|$

the sum of a geometric series

$$X(z) = A \frac{1}{1 - \frac{\alpha}{z}} = \frac{A}{1 - \alpha z^{-1}} = A \frac{z}{z - \alpha} \qquad |z| > |\alpha|$$

DTFT

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega} n}$$

DTFT

DTFT of the unit sequence $u[n]$

$$X(e^{-j\hat{\omega}n}) = \sum_{n=-\infty}^{+\infty} u[n] e^{-j\hat{\omega}n} = \sum_{n=0}^{\infty} e^{-j\hat{\omega}n}$$

not converge

$\hat{\omega} = 0 \qquad \sum_{n=0}^{\infty} 1^n$ diverge

$\hat{\omega} = \pi \qquad \sum_{n=0}^{\infty} (-1)^n$ oscillates

$\hat{\omega} = \frac{\pi}{2} \qquad \sum_{n=0}^{\infty} (j)^n$

The DTFTs of some commonly used functions do not exist in the strict sense.

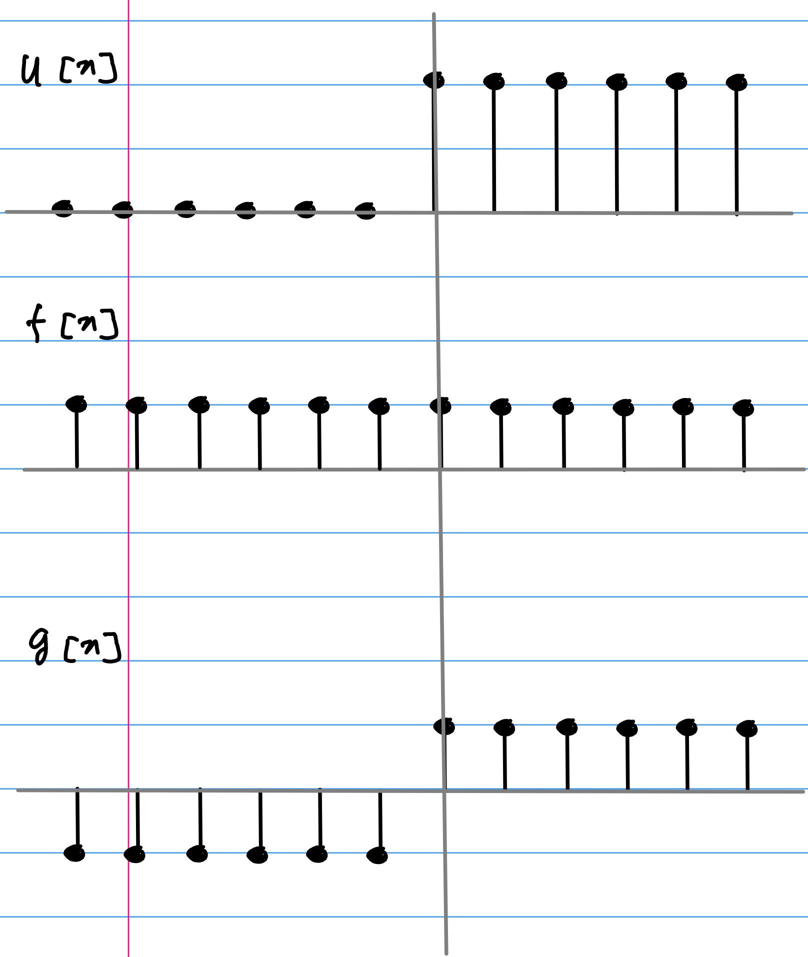
But even though the DTFT does not exist, the z-transform does exist.

$$X(z) = \sum_{n=-\infty}^{+\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$|z| > 1 \qquad X(z) = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$

$X(z) = \frac{z}{z-1}$ pole $z=1$, zero $z=0$

$X(z) = \frac{1}{1-z^{-1}}$ useful when a system is synthesized from a z-domain transfer function http://math.stackexchange.com/questions/1772285/dtft-of-the-unit-step-function



$$f[n] = \frac{1}{2} \qquad -\infty < n < \infty$$

$$g[n] = \begin{cases} \frac{1}{2} & n \geq 0 \\ -\frac{1}{2} & n < 0 \end{cases}$$

$$u[n] = f[n] + g[n]$$

$$\delta[n] = g[n] - g[n-1]$$

$$1 = G(e^{j\hat{\omega}}) - e^{-j\hat{\omega}} G(e^{j\hat{\omega}})$$

$$G(e^{j\hat{\omega}}) = \frac{1}{1 - e^{-j\hat{\omega}}}$$

$$F(e^{j\hat{\omega}}) = \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$$ (impulse train)

$$U(e^{j\hat{\omega}}) = \frac{1}{1 - e^{-j\hat{\omega}}} + \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$$

Discrete Time Exponential γ^n

continuous time exponential $e^{\lambda t}$

$$e^{\lambda t} = \gamma^t \qquad (e^{\lambda})^t = \gamma^t$$

$$e^{\lambda} = \gamma$$

$$\lambda = \ln \gamma$$

$$e^{-0.3t} = (0.7408)^t$$

$$4^t = e^{1.386t}$$

continuous time analysis $e^{\lambda t}$

discrete time analysis γ^n

$$e^{\lambda n} = \gamma^n \qquad (e^{\lambda})^n = \gamma^n$$

$$e^{\lambda} = \gamma$$

$$\lambda = \ln \gamma$$

$e^{\lambda n}$

exponentially grows if $\text{Re}\,\lambda > 0$ (λ in RHP)

exponentially decays if $\text{Re}\,\lambda < 0$ (λ in LHP)

oscillates or constant if $\text{Re}\,\lambda = 0$ (λ in imag axis)

the location of λ in the complex plain indicates whether

1. $e^{\lambda t}$ will grow exponentially
2. $e^{\lambda t}$ will decay exponentially
3. $e^{\lambda t}$ will oscillates with constant amplitude

constant signal : oscillation with zero frequency

$e^{j\Omega n}$ $\quad \lambda = j\Omega$ imaginary axis

constant amplitude oscillating signal

$e^{j\Omega n} = (e^{j\Omega})^n = \gamma^n \quad \gamma = e^{j\Omega} \quad |\gamma| = 1$

$\lambda = j\Omega$ imaginary axis $\rightarrow |\gamma| = 1$ unit circle

if γ lies on the unit circle,

γ^n oscillates with constant amplitude

the imaginary axis in the λ plane

the unit circle in the γ plane

$e^{\lambda n}$ $\lambda = a + jb$ in the LHP $(a<0)$

exponentially decaying

$\gamma = e^{\lambda} = e^{a+jb} = e^{a} e^{jb}$

$|\gamma| = |e^{\lambda}| = |e^{a}| \cdot |e^{jb}| = |e^{a}| = e^{a}$

$|\gamma| = e^{a} < 1$ inside the Unit circle

γ^{n} : exponentially decaying

$|\gamma| = e^{a} > 1$ outside the Unit circle

γ^{n} : exponentially growing

λ-plane		r-plane
the imaginary axis	$\longrightarrow$	the unit circle
the LHP	$\longrightarrow$	inside of the unit circle
the RHP	$\longrightarrow$	outside of the unit circle