Trigonometry Functions (5B)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Derivatives

$$\lim_{x \to 0} \frac{\sin x}{x} = 1,$$

/erified using the unit circ

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0,$$

$$\frac{d}{dx} \sin x = \cos x, \qquad \frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx} \cos x = -\sin x, \qquad \frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx} \tan x = \sec^2 x, \qquad \frac{d}{dx} \arctan x = \frac{1}{1 + x^2}$$

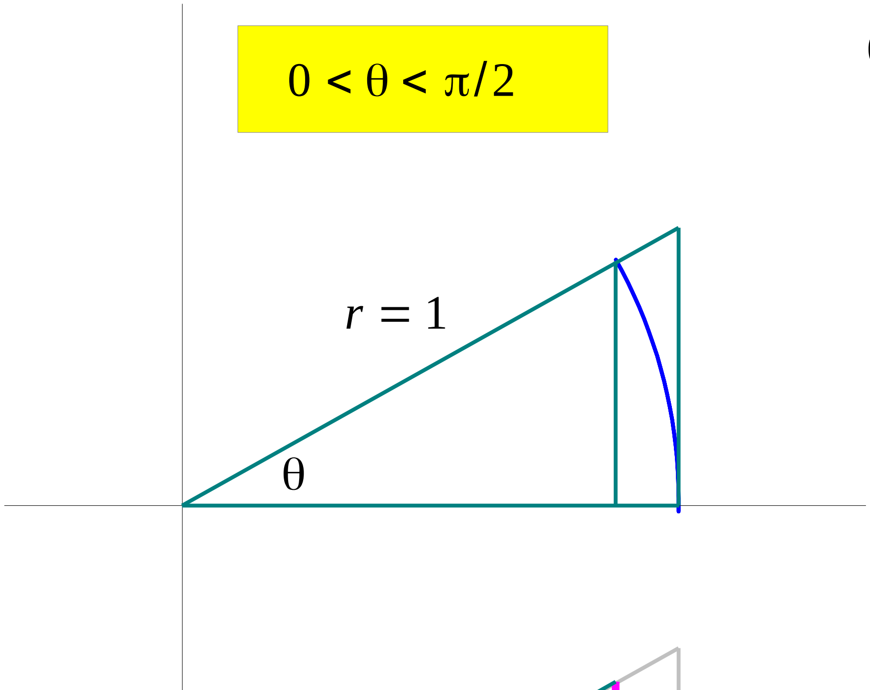
$$\frac{d}{dx} \cot x = -\csc^2 x, \qquad \frac{d}{dx} \operatorname{arccot} x = \frac{-1}{1 + x^2}$$

$$\frac{d}{dx} \sec x = \tan x \sec x, \qquad \frac{d}{dx} \operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2 - 1}}$$

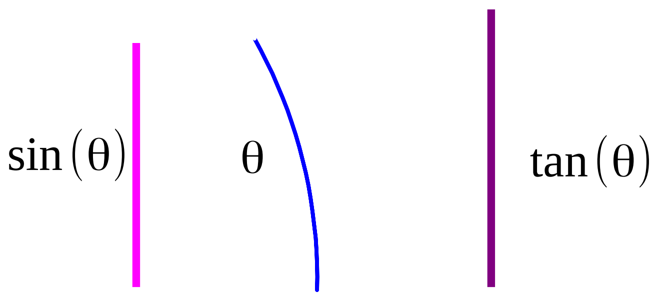
$$\frac{d}{dx} \csc x = -\csc x \cot x, \qquad \frac{d}{dx} \operatorname{arccsc} x = \frac{-1}{|x|\sqrt{x^2 - 1}}$$

http://en.wikipedia.org/wiki/Derivative

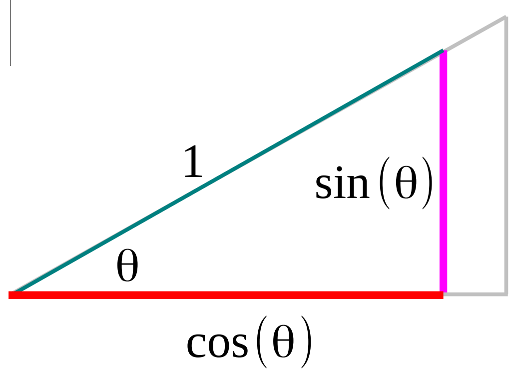
$0 < \theta < \pi/2$



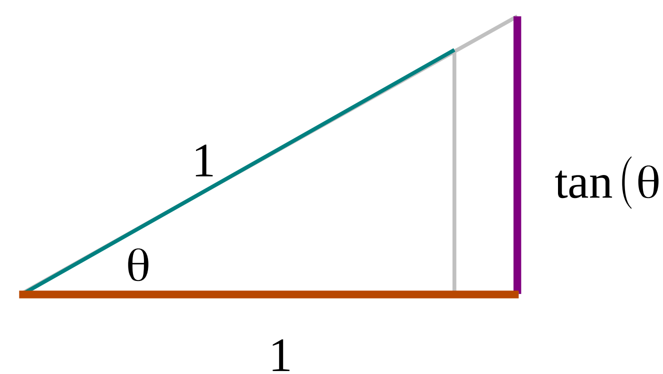
$$\theta = \frac{l}{2\pi r} 2\pi = \frac{l}{r} = l \ (rad)$$



$\sin(\theta) < \theta < \tan(\theta)$

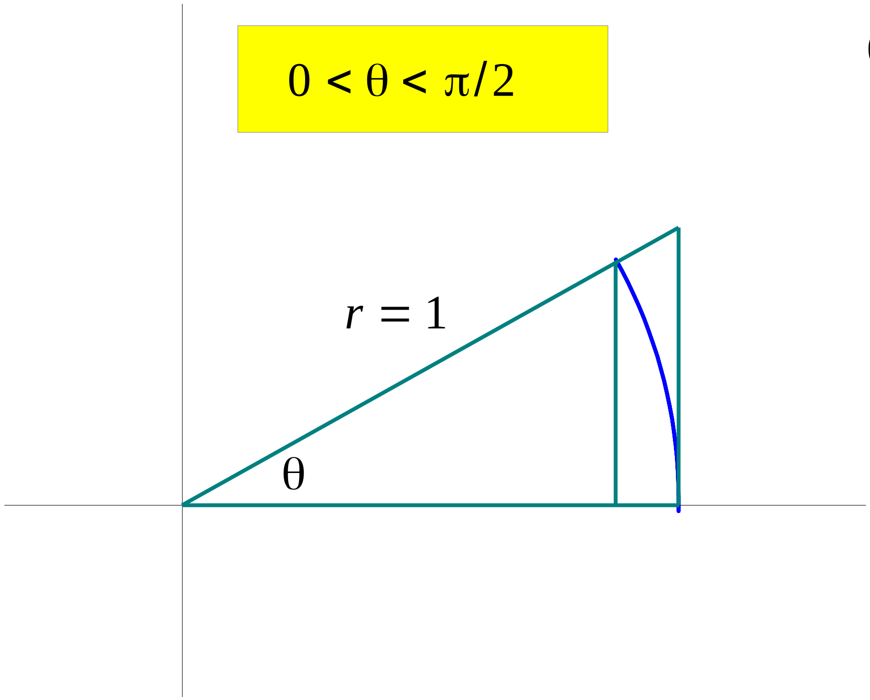


$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

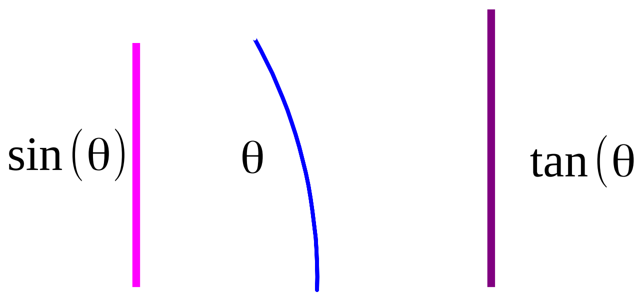


Inequalities

$0 < \theta < \pi/2$



$$\theta = \frac{l}{2\pi r} 2\pi = \frac{l}{r} = l \ (rad)$$



$$\sin(\theta) < \theta < \tan(\theta)$$

$$\frac{\sin(\theta)}{\theta} < 1 < \frac{\tan(\theta)}{\theta}$$

Sinc(x)

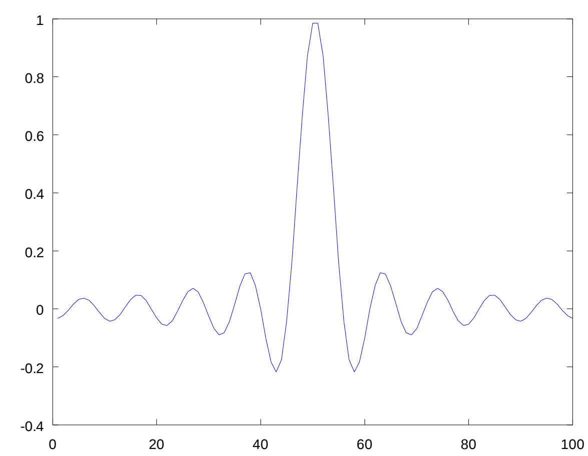
$$0 < \sin(\theta) < 1 \qquad 0 < \sin(\theta) < 1$$

$$1 < \theta$$

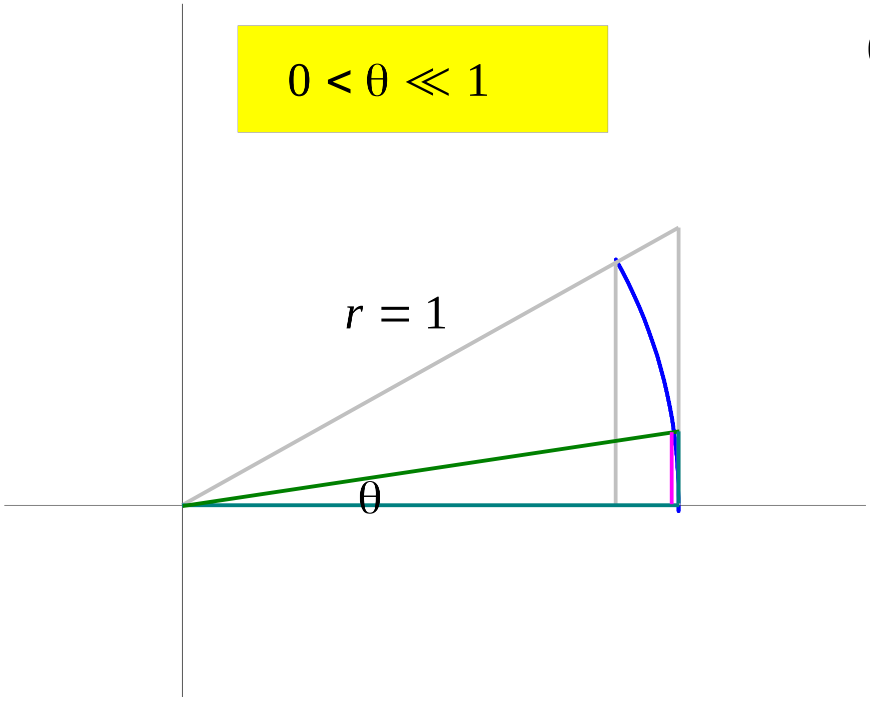
$$-1 < \sin(\theta) < 0 \qquad -1 < \sin(\theta) < 0$$

$$\theta < 1$$

$$\frac{\sin(\theta)}{\theta} < 1 \quad \textit{if} \quad \theta \neq 0$$

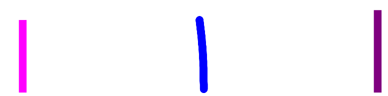


Sin(x) / x



$$0 < \theta \ll 1$$

$$\theta = \frac{l}{2\pi r} 2\pi = \frac{l}{r} = l \ (rad)$$

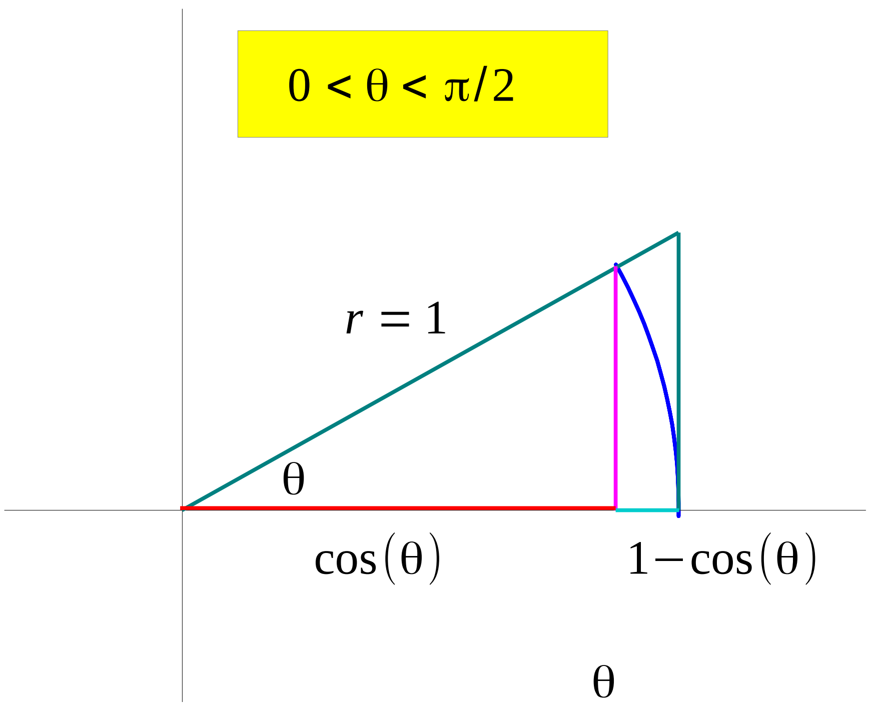


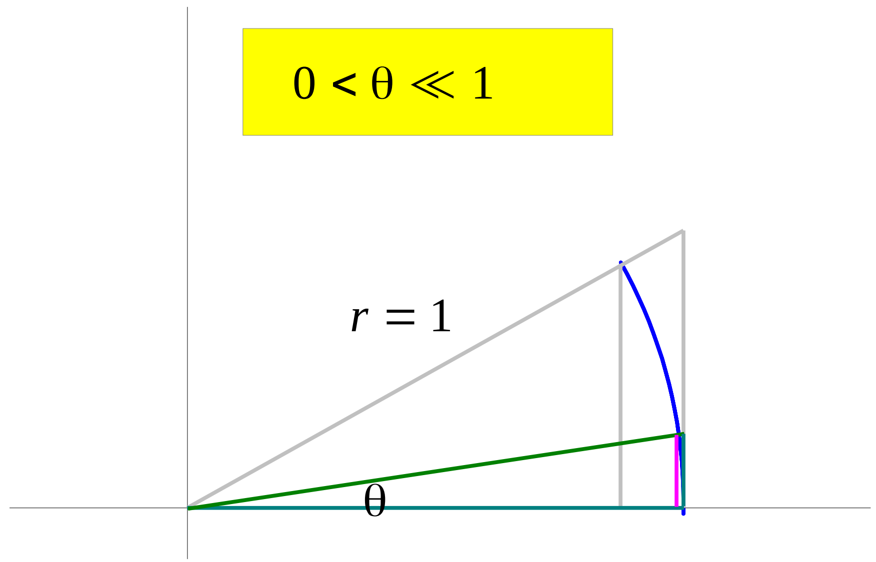
$$\sin(\theta) < \theta < \tan(\theta)$$

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

$$\frac{\sin(\theta)}{\theta} < 1 < \frac{\tan(\theta)}{\theta}$$

$$\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1 \Leftarrow \cos(\theta) < \frac{\sin(\theta)}{\theta} \Leftarrow 1 < \frac{\tan(\theta)}{\theta}$$

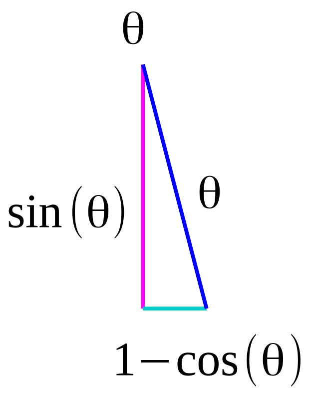




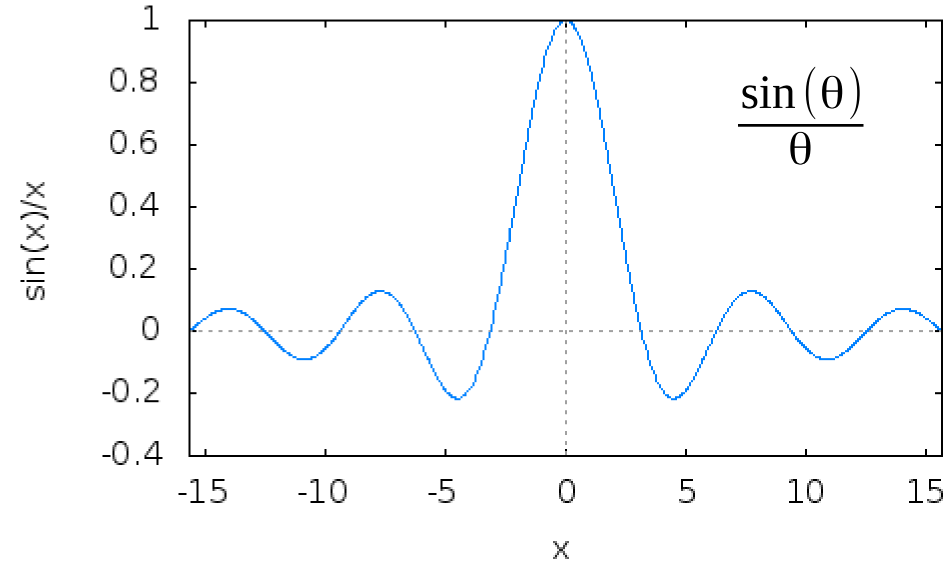
$$\frac{1-\cos(\theta)}{\theta}\frac{1+\cos(\theta)}{1+\cos(\theta)} = \frac{1-\cos^2(\theta)}{\theta(1+\cos(\theta))}$$

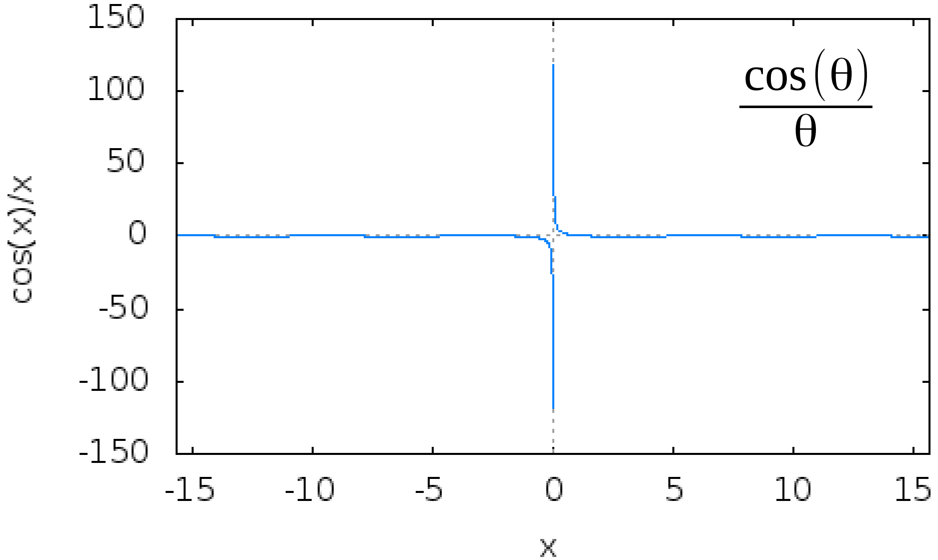
$$= \frac{\sin(\theta)}{\theta}\sin(\theta)\frac{1}{(1+\cos(\theta))}$$

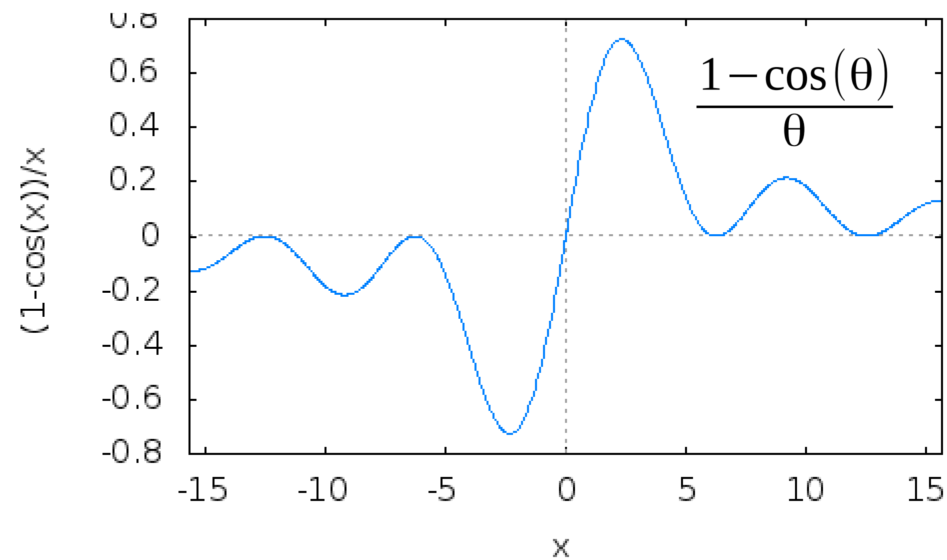
$$\frac{1-\cos(\theta)}{\theta}$$



$$\lim_{\theta \to 0} \frac{1-\cos(\theta)}{\theta} = 0$$


sin(θ) / θ
sin(x)/x
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-15
-10
-5
0
5
10
15
x


cos(θ) / θ
cos(x)/x
150
100
50
0
-50
-100
-150
-15
-10
-5
0
5
10
15
x


1 – cos(θ) / θ
(1-cos(x))/x
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-15
-10
-5
0
5
10
15
x

The Derivative of the Sine Function

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx} \sin(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h}$$

$$= \sin(x) \lim_{h \to 0} \frac{(\cos(h) - 1)}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h}$$

$$= \cos(x)$$

The Derivative of the Cosine Function

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx} \cos(x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x)(\cos(h) - 1) - \sin(x)\sin(h)}{h}$$

$$= \cos(x) \lim_{h \to 0} \frac{(\cos(h) - 1)}{h} - \sin(x) \lim_{h \to 0} \frac{\sin(h)}{h}$$

$$= -\sin(x)$$

The Derivative of the Tangent Function

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx} \tan(x) = \frac{d}{dx} \left(\frac{\sin(x)}{\cos(x)} \right)$$

$$= \frac{[\sin(x)]' \cos(x) - \sin(x) [\cos(x)]'}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos^2(x)} = sec^2(x)$$

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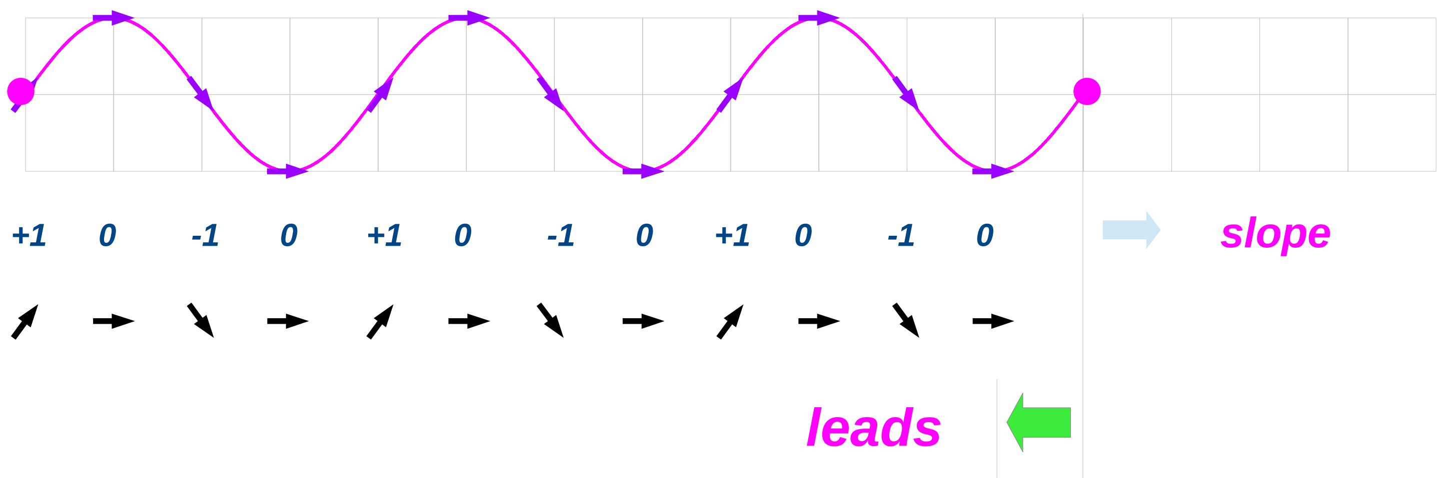
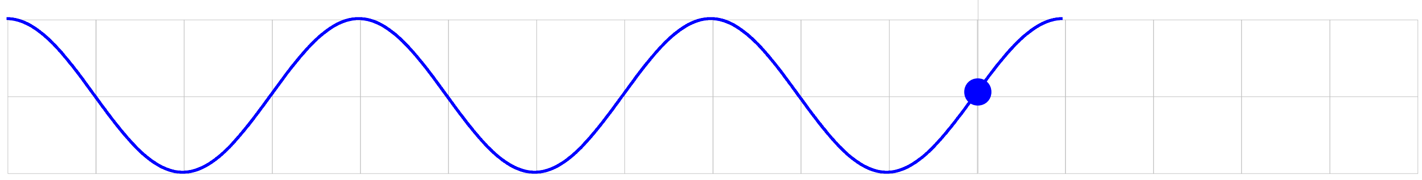
Derivative of sin(x)

$$f(x) = \sin(x)$$



$$\frac{d}{dx} f(x) = \cos(x)$$



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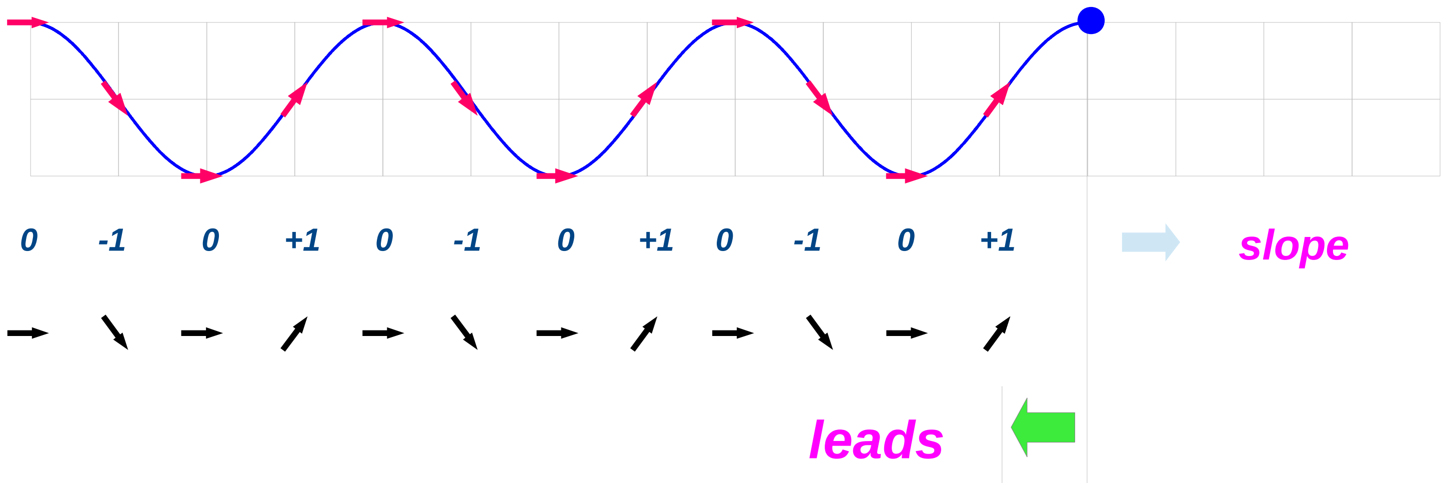
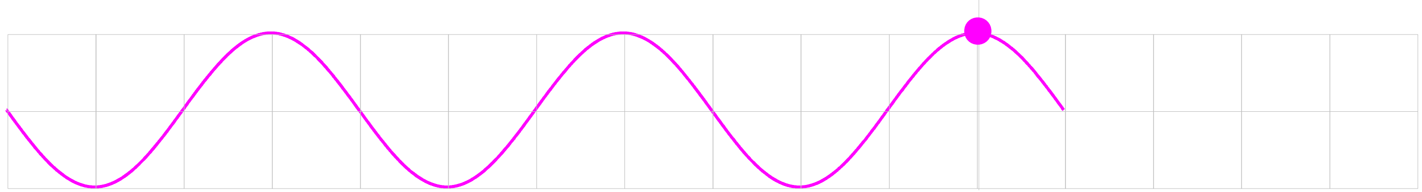
Derivative of cos(x)

$$f(x) = \cos(x)$$



$$\frac{d}{dx} f(x) = -\sin(x)$$



The Derivative of the Inverse Sine Function

$f'(x) = \cos(x)$ $\quad g'(y) = \dfrac{1}{\cos(x)}$ $\quad y = \sin(x)$

$$f'(x)g'(y) = \cos(x)\cdot\frac{1}{\cos(x)} = 1$$

$g'(y) = \dfrac{1}{\cos(x)}$ $\quad x = \sin^{-1}(y)$

$$g'(y) = \frac{1}{\cos(\sin^{-1}(y))}$$

$x \leftarrow y$

$$g'(x) = \frac{1}{\cos(\sin^{-1}(x))}$$

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The Derivative of the Inverse Sine Function

$y = \sin(x) \quad \Leftrightarrow \quad x = \sin(y) \qquad -\frac{\pi}{2} < y < +\frac{\pi}{2} \qquad -1 < x < +1$

$y = \sin^{-1}(x)$

$f(x) = \sin(x) \quad \Leftrightarrow \quad g(x) = \sin^{-1}(x)$

$f'(x) = \cos(x)$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(\sin^{-1}(x))}$$

$y = \sin^{-1}(x)$

$$= \frac{1}{\cos(y)}$$

$\cos(y) = \sqrt{1 - \sin^2(y)}$

$$= \frac{1}{\sqrt{1 - \sin^2(y)}}$$

$x = \sin(y)$

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}$$

x

y

x

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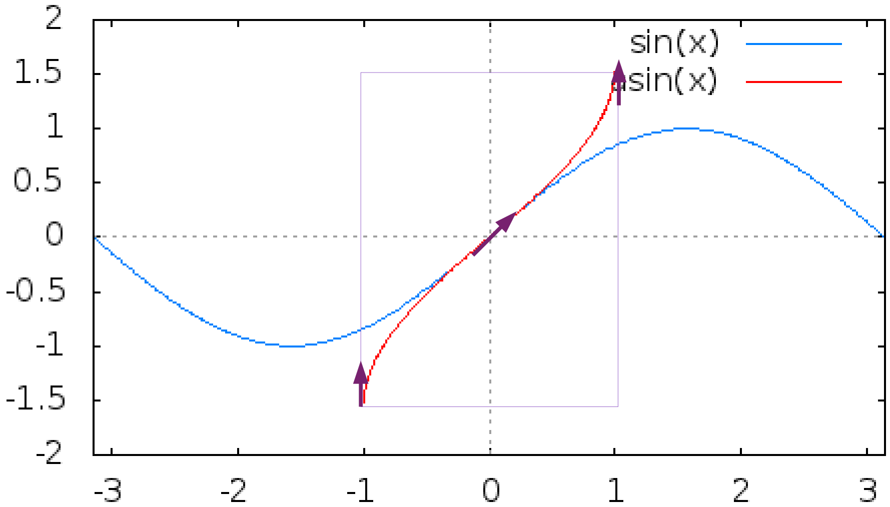
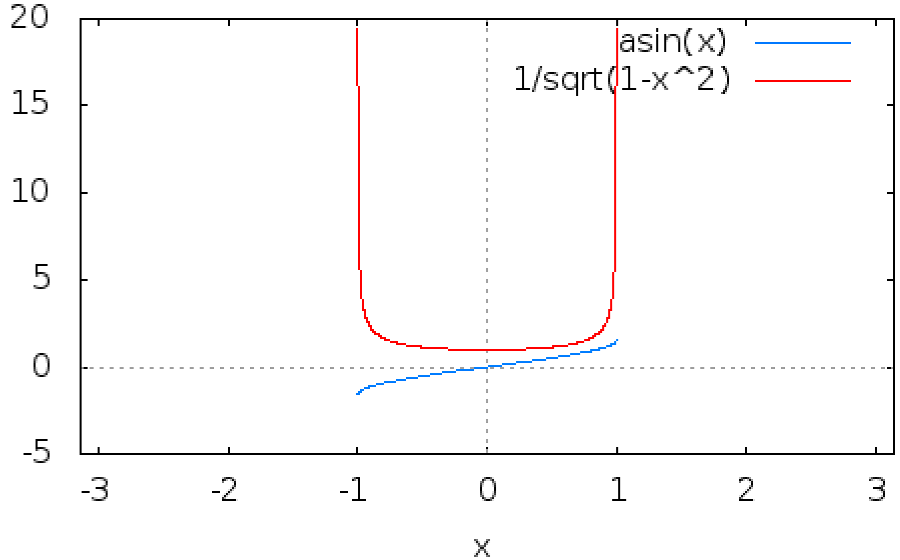
arcsin(x)





$$-\frac{\pi}{2} < y < +\frac{\pi}{2}$$

$$-1 < x < +1$$

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

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The Derivative of the Inverse Cosine Function

$y = \cos(x)$ ⟺ $x = \cos(y)$ $\quad 0 < y < +\pi \quad -1 < x < +1$

$y = \cos^{-1}(x)$

$f(x) = \cos(x)$ ⟺ $g(x) = \cos^{-1}(x)$

$f'(x) = -\sin(x)$

$$g'(x) = \frac{-1}{f'(g(x))} = \frac{-1}{\sin(\cos^{-1}(x))}$$

$y = \cos^{-1}(x)$

$$= \frac{-1}{\sin(y)}$$

$\sin(y) = \sqrt{1 - \cos^2(y)}$

$$= \frac{-1}{\sqrt{1-\cos^2(y)}}$$

$x = \cos(y)$

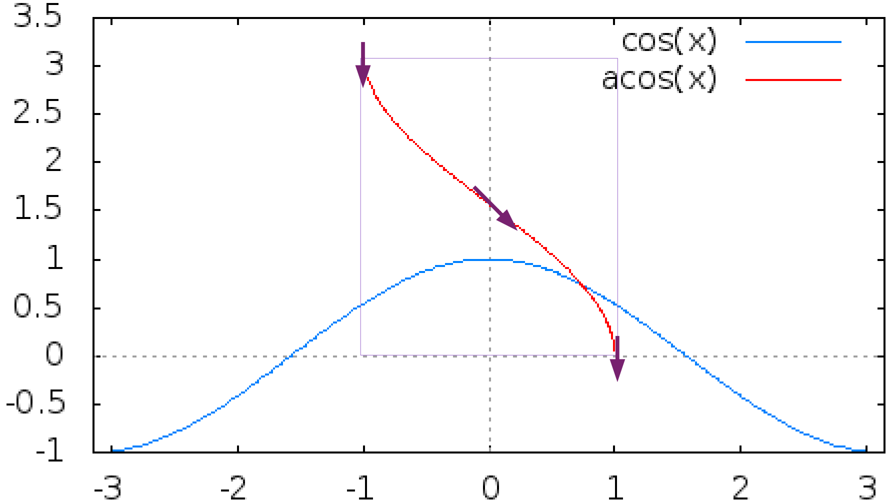
$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$

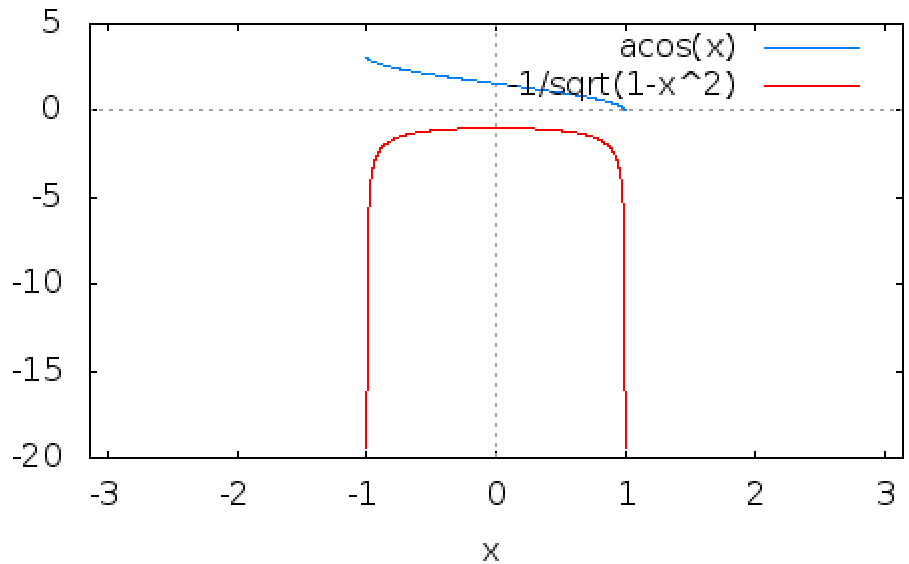
x

y

x

arccos(x)





$$-0 < y < +\pi$$

$$-1 < x < +1$$

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$

The Derivative of the Inverse Tangent Function

$y = \tan(x) \quad \Leftrightarrow \quad x = \tan(y) \qquad +\frac{\pi}{2} < y < -\frac{\pi}{2} \qquad -\infty < x < +\infty$

$y = \tan^{-1}(x)$

$f(x) = \tan(x) \quad \Leftrightarrow \quad g(x) = \tan^{-1}(x)$

$f'(x) = sec^2(x)$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{sec^2(\tan^{-1}(x))}$$

$y = \tan^{-1}(x)$

$$= \frac{1}{sec^2(y)}$$

$sec^2(y) = 1 + \tan^2(y)$

$$= \frac{1}{1+\tan^2(y)}$$

$x = \tan(y)$

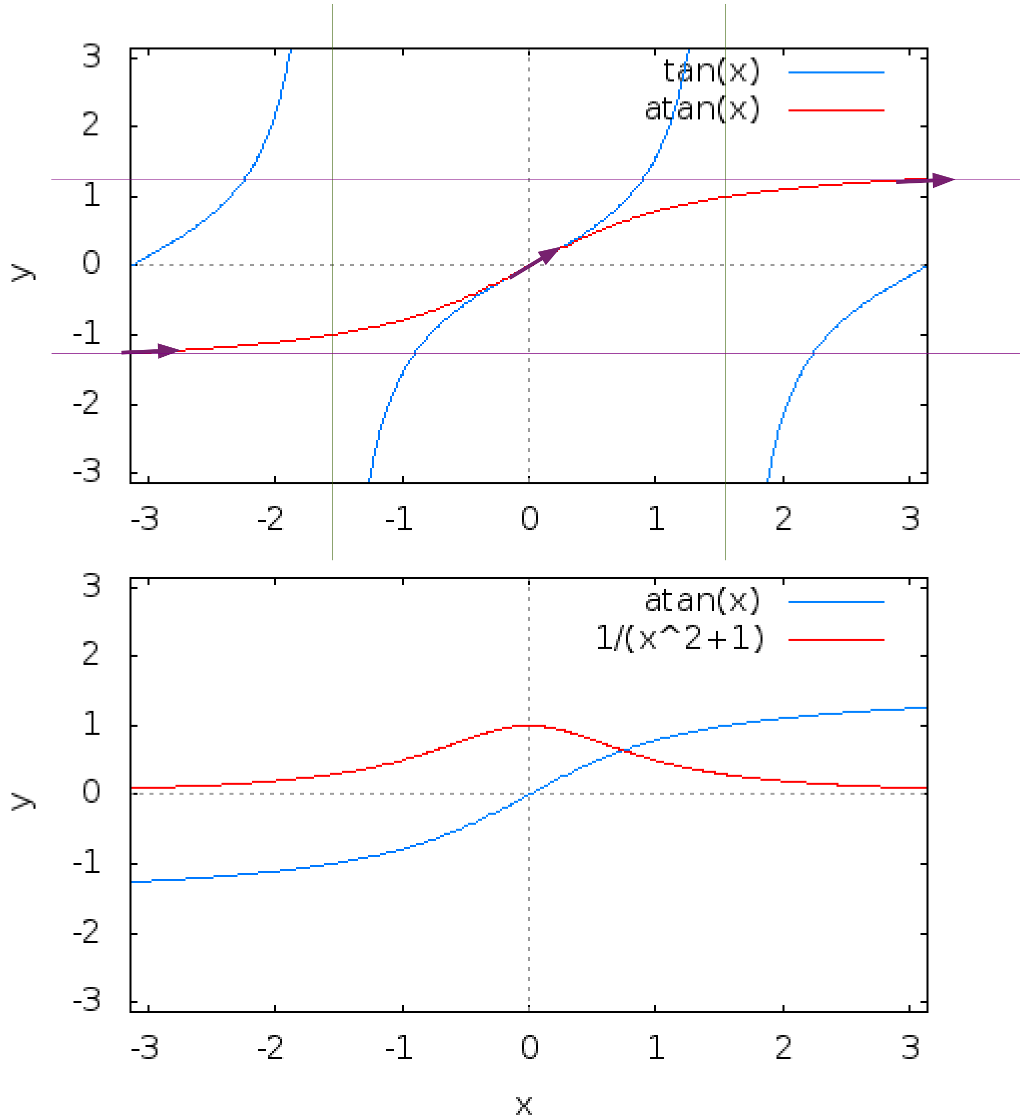
$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

x

y

x

arctan(x)



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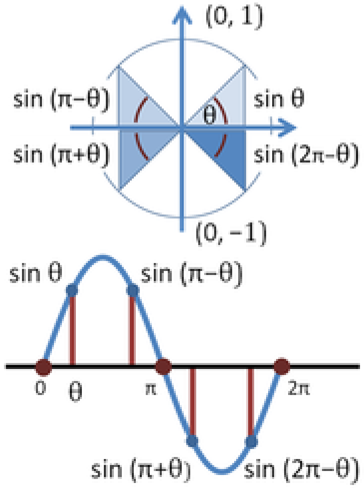
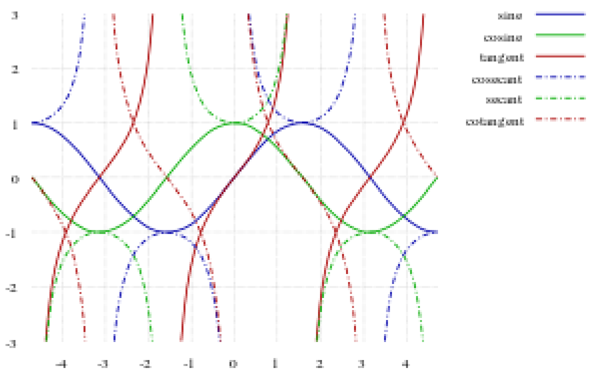
Integration

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$





http://en.wikipedia.org/wiki/Derivative

Derivatives of inverse trigonometric functions

$$y = \arcsin x$$
$$-\tfrac{\pi}{2} \le y < +\tfrac{\pi}{2}$$
$$\sin y = x$$
$$\frac{d}{dx}(\sin y) = \frac{d}{dx} x$$
$$(\cos y) \cdot y' = 1$$
$$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$
$$\frac{dy}{dx} = \frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}$$

$$y = \arccos x$$
$$0 \le y < +\pi$$
$$\cos y = x$$
$$\frac{d}{dx}(\cos y) = \frac{d}{dx} x$$
$$(-\sin y) \cdot y' = 1$$
$$y' = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1 - x^2}}$$
$$\frac{dy}{dx} = \frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1 - x^2}}$$

$$y = \arctan x$$
$$-\tfrac{\pi}{2} \le y < +\tfrac{\pi}{2}$$
$$\tan y = x$$
$$\frac{d}{dx}\left(\frac{\sin y}{\cos y}\right) = \frac{d}{dx} x$$
$$\frac{\cos^2 y \cdot y' + \sin^2 y \cdot y'}{\cos^2 y} = 1$$
$$(1 + \tan^2 y) \cdot y' = 1$$
$$\frac{dy}{dx} = \frac{d}{dx}(\arctan x) = \frac{1}{1 + x^2}$$

References

[1] http://en.wikipedia.org/
[2] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[3] E. Kreyszig, “Advanced Engineering Mathematics”
[4] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”