Digital Signal Octave Codes (0A)

- Periodic Conditions

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Based on
M.J. Roberts, Fundamentals of Signals and Systems
S.K. Mitra, Digital Signal Processing : a computer-based approach 2^{nd} ed
S.D. Stearns, Digital Signal Processing with Examples in MATLAB

Sampling and Normalized Frequency

$$\omega_0 t = 2\pi f_0 t$$

$$\omega_0 n T_s = 2\pi f_0 n T_s$$

$$= \frac{2\pi}{T_0} n T_s$$

$$= 2\pi n \frac{T_s}{T_0}$$

$$= 2\pi n F_0$$

$$t = n T_s$$

T_s : sampling period

$$f_0 = \frac{1}{T_0}$$

T_0 : signal period

$$\frac{T_s}{T_0} = \frac{f_0}{f_s}$$

normalization

$$F_0 = f_0 T_s = \frac{f_0}{f_s}$$

normalization

Analog and Digital Frequencies

Analog Signal Frequency

$$\omega_0 t = 2\pi f_0 t$$

$t = nT_s$ $\qquad$ $t = nT_s$

$$n\omega_0 T_s = 2\pi n f_0 T_s$$

$\omega_0 T_s = \Omega_0$ $\qquad$ $f_0 T_s = F_0$

$$\Omega_0 n = 2\pi n F_0$$

Digital Signal Frequency

Multiplying by T_s – Normalization

$$\omega_0 t = 2\pi f_0 t$$

Analog Signal

$\cdot\, T_s$ $\quad$ $\cdot\, T_s$

Normalization

$$\Omega_0 n = 2\pi n F_0$$

Digital Signal

Normalization

$$F_0 = f_0 \cdot T_s$$
$$= f_0 / f_s$$
$$= T_s / T_0$$

$f_0 \cdot T_s$ *Multiplied by T_s*

f_0 / f_s *Divided by f_s*

$$\Omega_0 = 2\pi F_0$$

$f_s > 2 \cdot f_0$

$f_0 / f_s < 0.5$

Sampling Rate Minimum

Normalized Cyclic and Radian Frequencies

Normalized Cyclic Frequency

$$F_0 \; cycles/sample = \frac{f_0 \;\; cycles/second}{f_s \; samples/second}$$

Normalized Radian Frequency

$$\Omega_0 \; cycles/sample = \frac{\omega_0 \;\; cycles/second}{f_s \; samples/second}$$

Periodic Relation : N_0 and F_0

$$e^{j(2\pi(n+N_0)F_0)} = e^{j(2\pi n F_0)}$$

$$e^{j2\pi m} = 1$$

Digital Signal Period N_0
: the smallest integer

$$e^{j2\pi N_0 F_0} \Rightarrow e^{j2\pi m} = 1$$

Periodic Condition
: integer m

$$2\pi N_0 F_0 = 2\pi m$$

$$N_0 F_0 = m$$

Periodic Condition : N_0 and F_0

$$2\pi N_0 F_0 = 2\pi m$$

$$N_0 F_0 = m$$

$$N_0 = \frac{m}{F_0} = m \cdot \frac{T_0}{T_s}$$

Integer * Rational :
must be an Integer

Digital Signal Period N_0
: the smallest integer

Periodic Condition
: the smallest integer m

$$m \neq T_s$$

$$m = p$$

reduced form

Periodic Condition : N_0 and F_0 in a reduced form

Integer

reduced form

$$N_0 = \frac{m}{F_0} = m \cdot \frac{T_0}{T_s} = m \cdot \frac{q}{p}$$

Integer * Rational :
must be an Integer

Rational numbers

N_0 and F_0 in a reduced form : Examples

reduced form

$$F_0 = \frac{p}{q}$$

Rational

$$N_0 = \frac{m}{F_0} = m \cdot \frac{q}{p}$$

integer

$$N_0 \rightarrow q$$
$$m \rightarrow p$$

integers

the smallest integer m

$$\frac{1}{F_0} = \frac{2.678}{4.017} = \frac{2 \cdot 1.339}{3 \cdot 1.339} = \frac{2}{3}$$

$m = 3 \qquad m \neq 4.017$

$N_0 = 2 \qquad N_0 \neq 2.678$

$$\frac{1}{F_0} = \frac{10}{15} = \frac{2 \cdot 5}{3 \cdot 5} = \frac{2}{3}$$

$m = 3 \qquad m \neq 15$

$N_0 = 2 \qquad N_0 \neq 10$

Periodic Relations – Analog and Digital Cases

$$e^{j(2\pi(n+N_0)F_0)} = e^{j(2\pi n F_0)}$$

Digital Signal Period N_0
: the smallest integer

$$N_0 = \frac{m}{F_0} = m \cdot \frac{T_0}{T_s}$$

$$k N_0 F_0 = k \cdot m$$

integer multiple of m
: *some* integers m

$$e^{j(2\pi f_0)(t+T_0)} = e^{j(2\pi f_0)t}$$

Analog Signal Period T_0
: the smallest real number

$$T_0 = \frac{1}{f_0}$$

$$k T_0 f_0 = k \cdot 1$$

all integers

Periodic Conditions – Analog and Digital Cases

$$N F_0 = k \cdot m$$

$$N = \frac{k \cdot m}{F_0}$$

Integer N_0

Rational F_0

***Minimum* Integer N_0**

$$N_0 = q \qquad F_0 = \frac{p}{q}$$

$$m = p$$

reduced form

$$T f_0 = k \cdot 1$$

$$T = \frac{k \cdot 1}{f_0}$$

Real T_0

Real f_0

***Minimum* Real T_0**

$$T_0 = \frac{1}{f_0}$$

$$m = 1$$

$$N_0 = \frac{m}{p/q}$$

Periodic Conditions Examples

$$N F_0 = k \cdot m$$

$$N_0 = \frac{m}{F_0}$$

Integer N

given

$$F_0 = \frac{36}{19}$$

km: multiples of 36

$$N_0 = 36 \cdot \frac{19}{36} \qquad 1 \cdot m = 36$$

$$2 N_0 = 72 \cdot \frac{19}{36} \qquad 2 \cdot m = 72$$

$$3 N_0 = 108 \cdot \frac{19}{36} \qquad 3 \cdot m = 108$$

$$T f_0 = k \cdot 1$$

$$T_0 = \frac{1}{f_0}$$

Real T

given

$$f_0 = \frac{36}{19}$$

k: all integers

$$T_0 = 1 \cdot \frac{19}{36} \qquad 1 \cdot 1 = 1$$

$$2 T_0 = 2 \cdot \frac{19}{36} \qquad 2 \cdot 1 = 2$$

$$3 T_0 = 3 \cdot \frac{19}{36} \qquad 3 \cdot 1 = 3$$

Periodic Condition of a Sampled Signal

$$g(nT_s) = A\cos(2\pi f_0 T_s n + \theta)$$

$$F_0 = f_0 T_s = f_0 / f_s$$

$$g[n] = A\cos(2\pi F_0 n + \theta)$$

$$2\pi F_0 n = 2\pi m \qquad F_0 n = m$$

$$F_0 = \frac{m}{n}$$

integers n, m

Rational Number $F_0 = \frac{m}{n}$

integers n, m

The Smallest Integer n

$$N_0 = min(n) \qquad F_0 = \frac{m}{N_0}$$

M.J. Roberts, Fundamentals of Signals and Systems

F_0 and N_0 of a Sampled Signal

Rational Number F_0

$$F_0 = \frac{m}{n} = \frac{p}{q}$$

integer n, m, p, q

$$F_0 = \frac{f_0}{f_s} = \frac{T_s}{T_0}$$

real f_0, f_s, T_s, T_0

Integer N_0

$$N_0 F_0 = m$$

$$N_0 = \frac{m}{F_0} = m \cdot \frac{T_0}{T_s} = m \cdot \frac{f_s}{f_0} = m \cdot \frac{q}{p}$$

$$2\pi F_0 n$$

$$2\pi f_0 T_s n$$

M.J. Roberts, Fundamentals of Signals and Systems

A cosine waveform example

```
n= [0:19];
x= cos(2*pi*1*(n/10));
```

$$2\pi F_0 n = 2\pi f_0 T_s n$$

```
n= [0:19];
x= cos(2*pi*(1/10)*n);
```

$$n T_s = n \cdot \frac{1}{10}$$

$$F_0 = f_0 T_s = \frac{f_0}{f_s} = \frac{T_s}{T_0}$$

$$n T_s = n \cdot 1$$

$$2\pi f_0 n T_s$$

$$= 2\pi \cdot 1 \cdot n \cdot \frac{1}{10}$$

$$2\pi f_0 n T_s$$

$$= 2\pi \cdot \frac{1}{10} \cdot n \cdot 1$$

$$T_s = 0.1$$

$$f_0 = 1 \quad (T_0 = 1)$$

$$T_s = 1$$

$$f_0 = 0.1 \quad (T_0 = 10)$$

$$F_0 = f_0 T_s = 0.1$$

$$F_0 = f_0 T_s = 0.1$$

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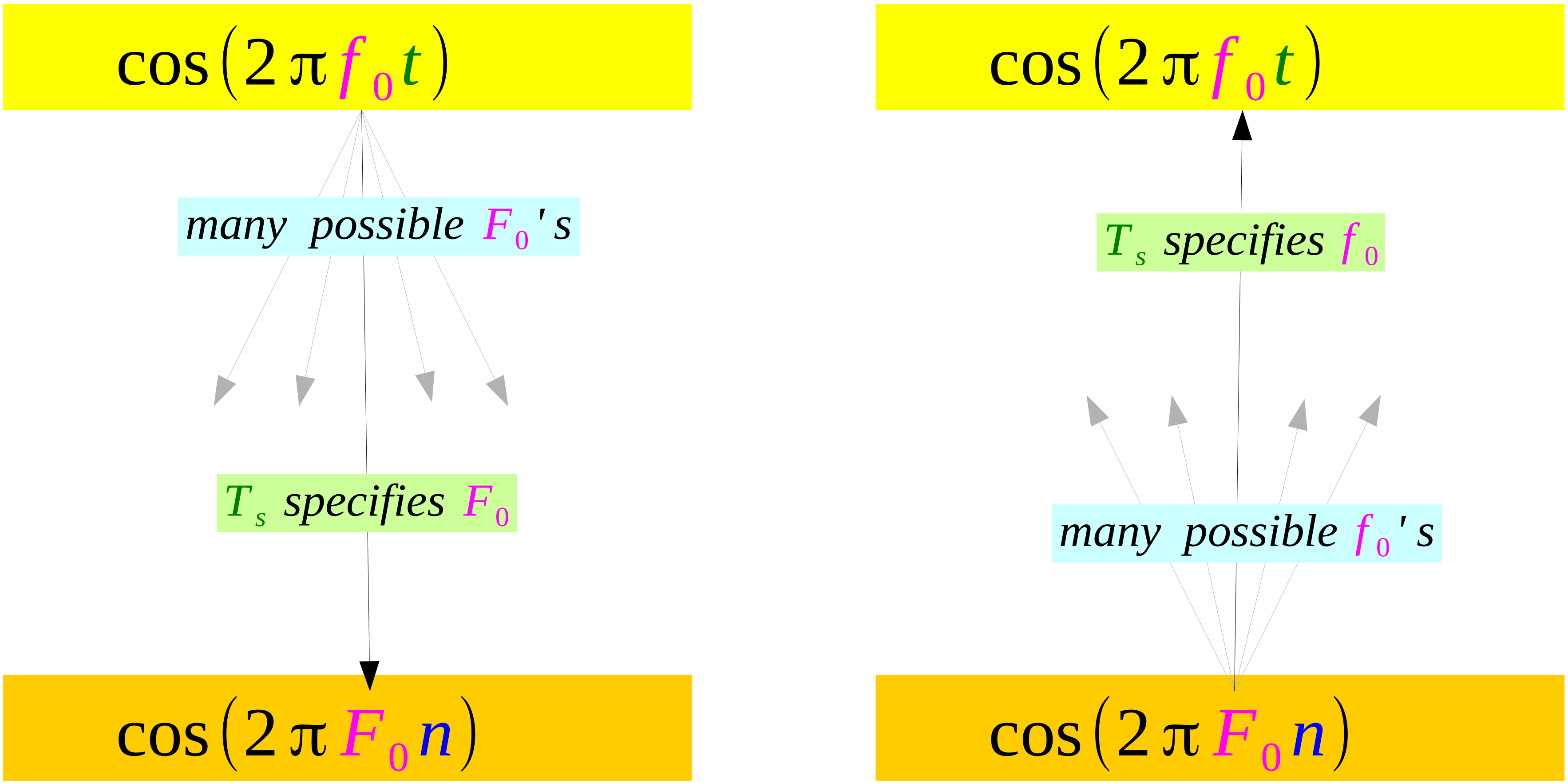
Two cases of the same $F_0 = f_0 T_s$



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The same sampled waveform examples

$\cos(0.2\pi n)$ $\cos(0.2\pi n)$

$2\pi f_0 n T_s$

$2\pi n/10$

$0.1 = f_0 T_s$

$0.1 = F_0$

```
n= [0:19];
x= cos(2*pi*1*0.1*n);
```

$2\pi F_0 n = 2\pi f_0 T_s n$

```
n= [0:19];
x= cos(2*pi*0.1*1*n);
```

… $T_s = 0.1$, $f_0 = 1$ … $T_s = 0.2$, $f_0 = 0.5$ … $T_s = 0.5$, $f_0 = 0.2$ … $T_s = 1$, $f_0 = 0.1$ …

$F_0 = 0.1$ $F_0 = 0.1$ $F_0 = 0.1$ $F_0 = 0.1$

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Many waveforms share the same sampled data

The same sampled data

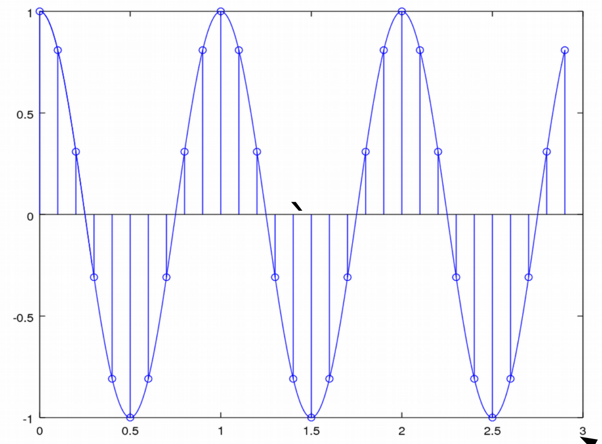
1.00000
0.80902
0.30902
-0.30902
-0.80902
-1.00000
-0.80902
-0.30902
0.30902
0.80902
1.00000
0.80902
0.30902
-0.30902
-0.80902
-1.00000
-0.80902
-0.30902
0.30902
0.80902
1.00000
0.80902
0.30902
-0.30902
-0.80902
-1.00000
-0.80902
-0.30902
0.30902
0.80902

$2\pi n/10$

$2\pi n f_0 T_s$

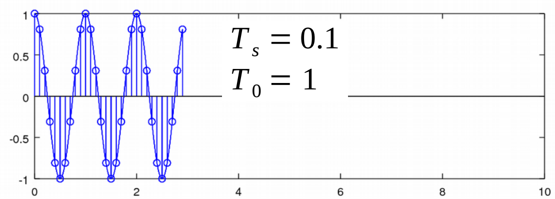
$0.1 = f_0 T_s$

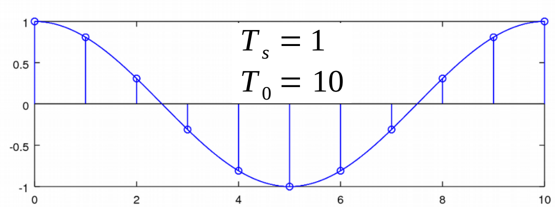
$0.1 = F_0$

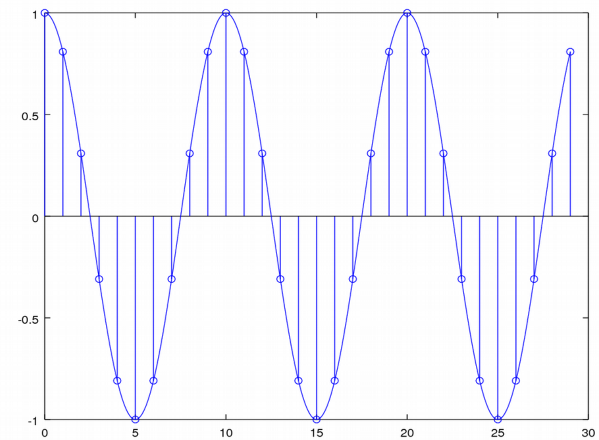


$T_s = 0.1$
$T_0 = 1$

3







$T_s = 1$
$T_0 = 10$

30

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Different number of data points

x= cos(2*pi*n/10);

```
t = [0:19]/10;
y = cos(2*pi*t);
stem(t, y)
hold on
t2 = [0:199]/100;
y2 = cos(2*pi*t2);
plot(t2, y2)
```

[0:19]; $[0, \cdots, 19]$ 20 *data points*

size([0:19], 2) = 20

[0:199]; $[0, \cdots, 199]$ 200 *data points*

size([0:199], 2) = 200

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Normalized data points

x= cos(2*pi*n/10);

```
t = [0:19]/10;
y = cos(2*pi*t);
stem(t, y)
hold on
t2 = [0:199]/100;
y2 = cos(2*pi*t2);
plot(t2, y2)
```

t = [0:19]/10; $[0.0, \cdots, 1.90]$ 20 *data points*

coarse resolution

$[0.0, \cdots, 1.90] \rightarrow 2\ cycles$ $[0, \cdots, 4\pi]$

t2 = [0:199]/100; $[0.0, \cdots, 1.99]$ 200 *data points*

fine resolution

$[0.0, \cdots, 1.99] \rightarrow 2\ cycles$ $[0, \cdots, 4\pi]$

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Different number of data points

`[0:19];` $[0,\cdots,19]$ 20 *data points*

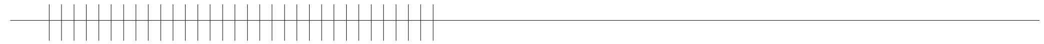
size([0:19], 2) = 20

x= cos(0.2*pi*n);

```
t = [0:19];
y = cos(0.2*pi*t);
stem(t, y)
hold on
t2 = [0:199]/10;
y2 = cos(0.2*pi*t2);
plot(t2, y2)
```

`[0:199];` $[0,\cdots,199]$ 200 *data points*

size([0:199], 2) = 200



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Normalized data points

$x = \cos(0.2*pi*n);$

```
t = [0:19];
y = cos(0.2*pi*t);
stem(t, y)
hold on
t2 = [0:190]/10;
y2 = cos(0.2*pi*t2);
plot(t2, y2)
```

t = [0:19]; $[0.0, \cdots, 19.0]$ 20 *data points*

coarse resolution

$[0.0, \cdots, 19.0] \rightarrow 2$ *cycles* $[0, \cdots, 4\pi]$

t2 = [0:199]/10; $[0.0, \cdots, 19.9]$ 200 *data points*

fine resolution

$[0.0, \cdots, 19.9] \rightarrow 2$ *cycles* $[0, \cdots, 4\pi]$

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Plotting sampled cosine waves

x= cos(2*pi*n/10);

```
t = [0:19]/10;
y = cos(2*pi*t);
stem(t, y)
hold on
t2 = [0:199]/100;
y2 = cos(2*pi*t2);
plot(t2, y2)
```

t = [0:19]/10;	$[0.0, \cdots, 1.9]$	20 *data points*
y = cos(2*pi*t);	stem(t, y)	coarse resolution
t2 = [0:199]/100;	$[0.0, \cdots, 1.99]$	200 *data points*
y = cos(2*pi*t2);	plot(t2, y)	fine resolution

x= cos(0.2*pi*n);

```
t = [0:19];
y = cos(0.2*pi*t);
stem(t, y)
hold on
t2 = [0:190]/10;
y2 = cos(0.2*pi*t2);
plot(t2, y2)
```

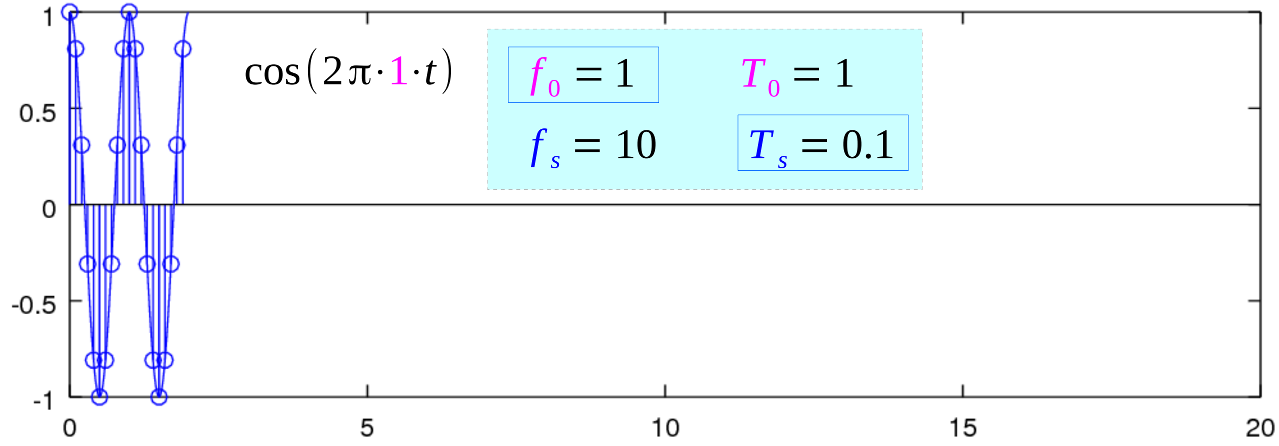
t = [0:19];	$[0.0, \cdots, 1.9]$	20 *data points*
y = cos(0.2*pi*t);	stem(t, y)	coarse resolution
t2 = [0:199]/00;	$[0.0, \cdots, 1.99]$	200 *data points*
y = cos(0.2*pi*t2);	plot(t2, y)	fine resolution

Two waveforms with the same normalized frequency

x= cos(2*pi*n/10);

```
t = [0:19]/10;
y = cos(2*pi*t);
stem(t, y)
hold on
t2 = [0:199]/100;
y2 = cos(2*pi*t2);
plot(t2, y2)
```

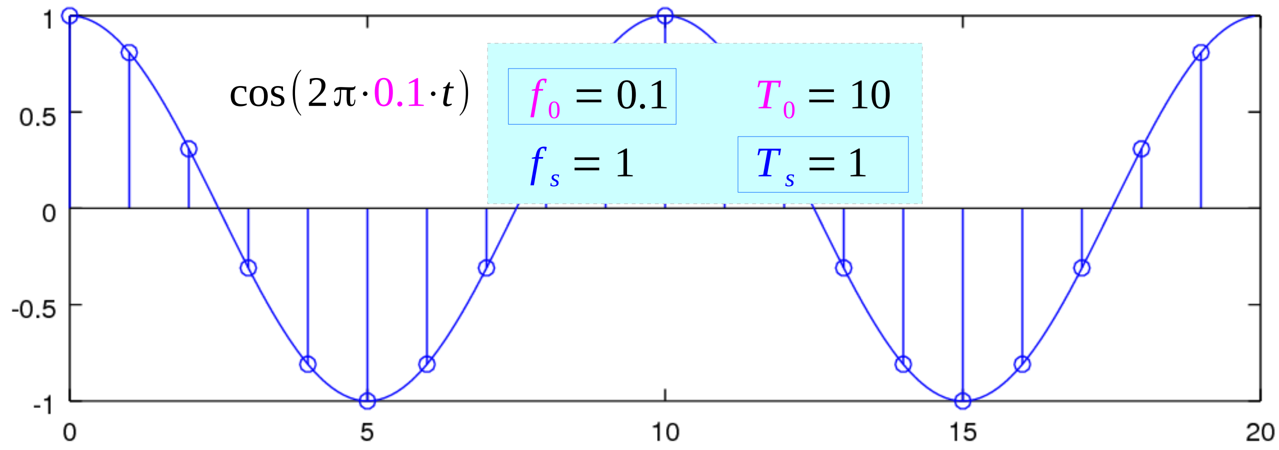
$\cos(2\pi t)$ $[0.0, \cdots, 1.9] \rightarrow$ 2 *cycles* $F_0 = 0.1$



x= cos(0.2*pi*n);

```
t = [0:19];
y = cos(0.2*pi*t);
stem(t, y)
hold on
t2 = [0:190]/10;
y2 = cos(0.2*pi*t2);
plot(t2, y2)
```

$\cos(0.2\pi t)$ $[0., \cdots, 19.] \rightarrow$ 2 *cycles* $F_0 = 0.1$



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Cosine Wave 1

x= cos(2*pi*n/10);

```
t = [0:29]/10;
y = cos(2*pi*t);
stem(t, y)
hold on
t2 = [0:299]/100;
y2 = cos(2*pi*t2);
plot(t2, y2)
```

$f_0 = 1$

$T_s = 0.1$

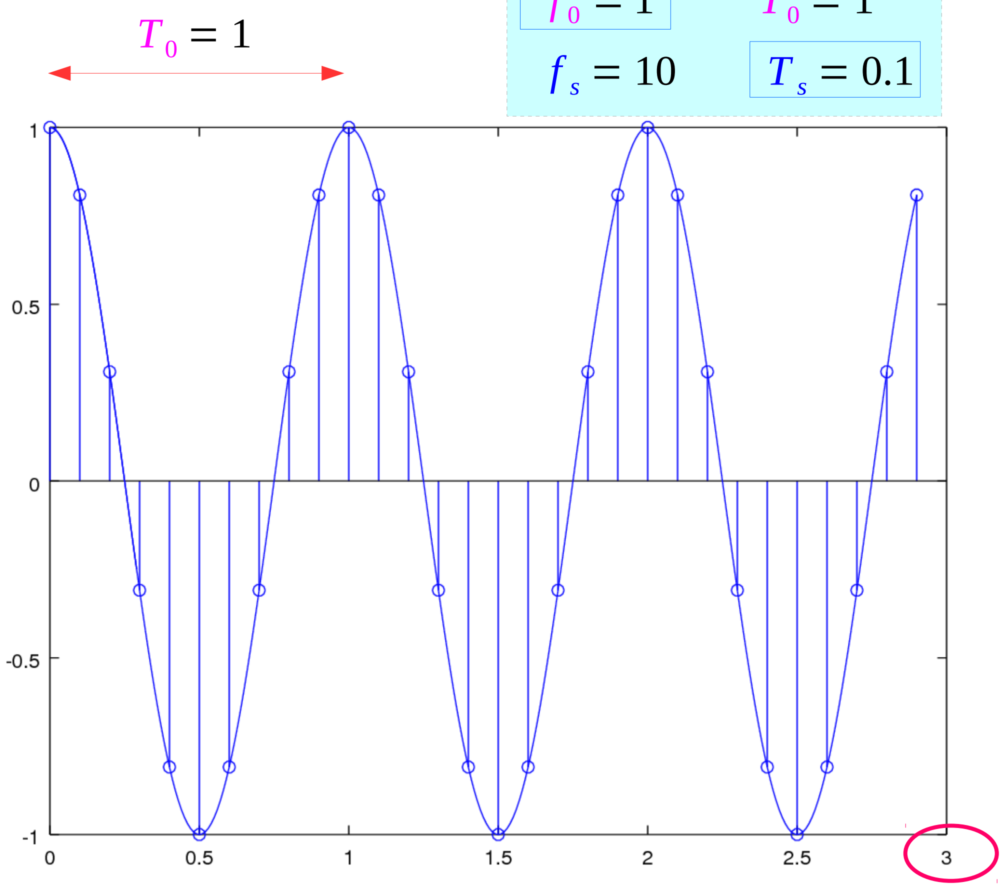
$F_0 = f_0 T_s = 0.1$

$\cos(2\pi t)$

$\cos(2\pi \cdot 1 \cdot t)$

$f_0 = 1$ $T_0 = 1$

$f_s = 10$ $T_s = 0.1$



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Cosine Wave 2

```
x= cos(0.2*pi*n);
```

```
t = [0:29];
y = cos(0.2*pi*t);
stem(t, y)
hold on
t2 = [0:299]/10;
y2 = cos(0.2*pi*t2);
plot(t2, y2)
```

$f_0 = 0.1$

$T_s = 1$

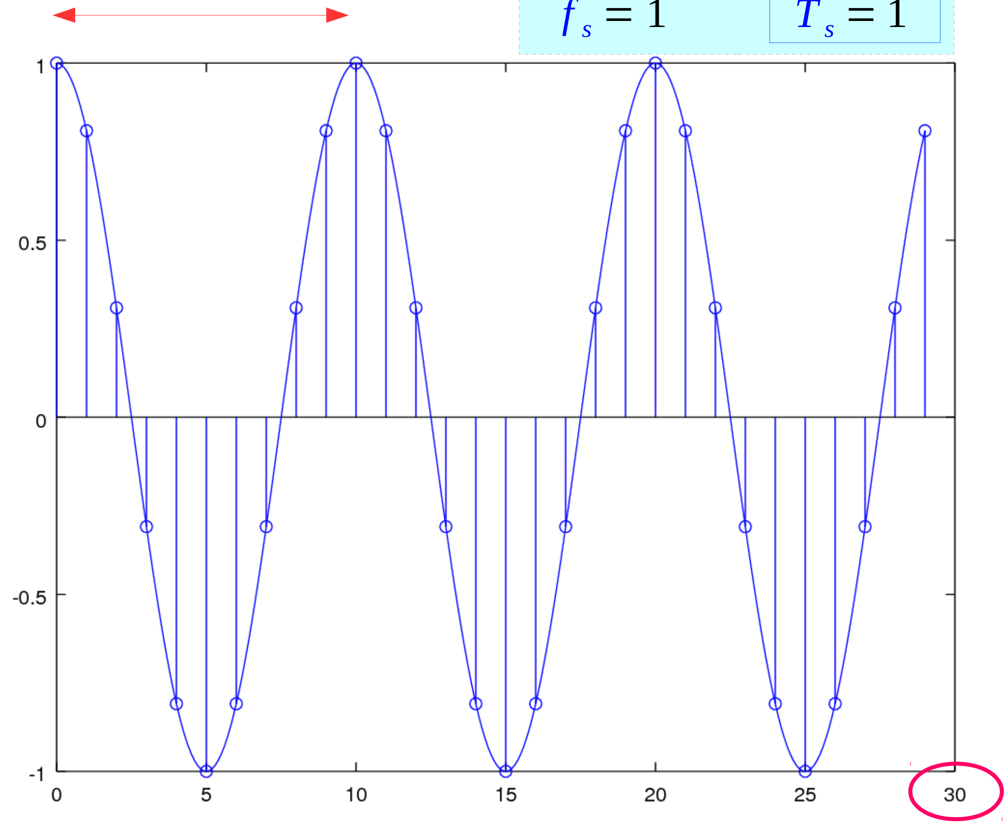
$F_0 = f_0 T_s = 0.1$

$\cos(0.2\pi t)$ $\quad$ $\cos(2\pi \cdot 0.1 \cdot t)$

$f_0 = 0.1$ $\quad$ $T_0 = 10$

$f_s = 1$ $\quad$ $T_s = 1$

$T_0 = 10$



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Sampled Sinusoids

$$g[n] = A\cos(2\pi F_0 n + \theta)$$

$$g[n] = A\cos(2\pi n m/N_0 + \theta)$$

$$g[n] = A\cos(\Omega_0 n + \theta)$$

F_0	$2\pi F_0$
m/N_0	$2\pi m/N_0$
$\Omega_0/2\pi$	Ω_0

$$N_0 = \frac{m}{F_0} \qquad N_0 \neq \frac{1}{F_0}$$

$$g[n] = A e^{\beta n}$$

$$g[n] = A z^{n} \qquad z = e^{\beta}$$

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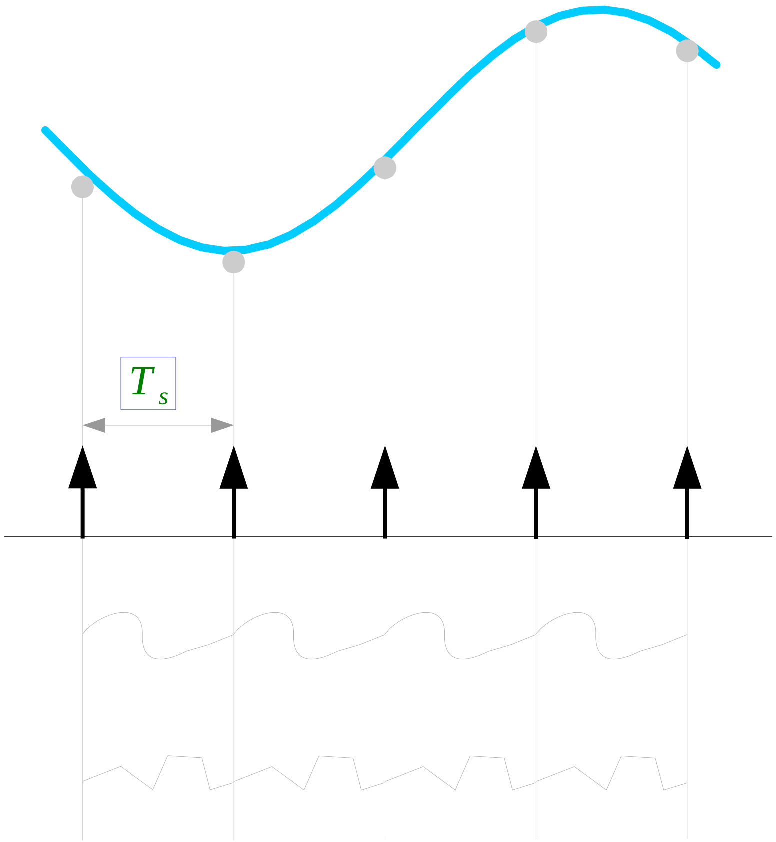
Sampling Period T_s and Frequency f_s

$$g(t) = A\cos(2\pi f_0 t + \theta)$$

$F_0 \leftarrow f_0 \cdot T_s$

$f_0 \leftarrow F_0 \cdot f_s$

$$g[n] = A\cos(2\pi F_0 n + \theta)$$



$T_s = \dfrac{1}{f_s}$ sampling period

$\dfrac{1}{T_s} = f_s$ sampling frequency
sampling rate

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f_0 and F_0

$$g(t) = A\cos(2\pi f_0 t + \theta)$$

$$g[n] = A\cos(2\pi F_0 n + \theta)$$

$$g(t) = 4\cos\left(\frac{72\pi t}{19}\right)$$

$$= 4\cos\left(2\pi\cdot\frac{36}{19}\cdot t\right)$$

$$g[n] = 4\cos\left(\frac{72\pi n}{19}\right)$$

$$= 4\cos\left(2\pi\cdot\frac{36}{19}\cdot n\right)$$

$$f_0 = \frac{36}{19}$$

there are many F_0

$$F_0 = f_0 T_s = \frac{f_0}{f_s}$$

$$T_s = 1 \Rightarrow F_0 = f_0$$

$$F_0 = \frac{36}{19}$$

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T_0 and N_0

$$g(t) = A\cos(2\pi f_0 t + \theta)$$

$$g(t) = 4\cos\left(\frac{72\pi t}{19}\right)$$

$$= 4\cos\left(2\pi\cdot\frac{36}{19}\cdot t\right)$$

$$g(t) = 4\cos\left(2\pi\cdot\frac{36}{19}\cdot(t + T_0)\right)$$

$$T_0 = \frac{19}{36}$$

Fundamental Period of $g(t)$

$$g[n] = A\cos(2\pi F_0 n + \theta)$$

$$g[n] = 4\cos\left(\frac{72\pi n}{19}\right)$$

$$= 4\cos\left(2\pi\cdot\frac{36}{19}\cdot n\right)$$

$$g[n] = 4\cos\left(2\pi\cdot\frac{36}{19}\cdot(n + N_0)\right)$$

there is only one
N_0 for a given F_0

$$N_0 = 19$$

Fundamental Period of $g[n]$

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Real T_0 and Integer N_0

$$g[n] = 4\cos\left(2\pi \cdot \frac{36}{19} \cdot (n + N_0)\right)$$

$\frac{36}{19} \cdot (n + N_0)$	$\frac{1}{19} \cdot N_0$	$N_0 = 19$
integer	integer	integer

$N_0 = 19$ Fundamental period of $g[n]$

$$g(t) = 4\cos\left(2\pi \cdot \frac{36}{19} \cdot (t + T_0)\right)$$

$\frac{36}{19} \cdot (t + T_0)$	$\frac{36}{19} \cdot T_0$	$T_0 = \frac{19}{36}$
integer	integer	~~integer~~

$T_0 = \frac{19}{36}$ Fundamental period of $g(t)$

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Cycles in N_0 samples

$$F_0 = \frac{q}{N_0}$$

the number of cycles in N_0 samples

the smallest integer : fundamental period

$$F_0 N_0 = q$$

$$2\pi F_0 N_0 = 2\pi q$$

q cycles in N_0 samples

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Cycles in T_0 time duration and N_0 samples

$$g(t) = 4\cos\left(2\pi \cdot \frac{36}{19} \cdot (t + T_0)\right)$$

$$T_0 = \frac{19}{36}$$ Fundamental Period of $g(t)$

$$f_0 = \frac{36}{19} = \frac{1}{T_0}$$

$q=1$ cycle in T_0=19/36 time interval

$$g[n] = 4\cos\left(2\pi \cdot \frac{36}{19} \cdot (n + N_0)\right)$$

$$N_0 = 19$$ Fundamental Period of $g[n]$

$$F_0 = \frac{36}{19} = \frac{q}{N_0}$$

$q=36$ cycles in N_0=19 samples

$$N_0 \neq \frac{1}{F_0} \Rightarrow N_0 = \frac{q}{F_0}$$

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Difficult to recognize a discrete-time sinusoid

$$F_0 = \frac{36}{19} = \frac{q}{N_0}$$

⬅ the number of cycles in N_0 samples

⬅ the smallest integer : fundamental period

"When F_0 is not the reciprocal of an integer (q=1),
a discrete-time sinusoid may not be
immediately recognizable from its graph as a sinusoid."

$$F'_0 = \frac{1}{19} = \frac{1}{N_0}$$

M.J. Roberts, Fundamentals of Signals and Systems

Periodic Condition Examples

$$g[n] = 4\cos\left(2\pi\cdot\frac{1}{19}\cdot n\right)$$

1 cycles in N_0=19 samples

$$g[n] = 4\cos\left(2\pi\cdot\frac{2}{19}\cdot n\right)$$

2 cycles in N_0=19 samples

$$g[n] = 4\cos\left(2\pi\cdot\frac{3}{19}\cdot n\right)$$

3 cycles in N_0=19 samples

$$g[n] = 4\cos\left(2\pi\cdot\frac{36}{19}\cdot n\right)$$

36 cycles in N_0=19 samples

```
clf
n = [0:36]; t = [0:3600]/100;
y1 = 4*cos(2*pi*(1/19)*n);
y2 = 4*cos(2*pi*(2/19)*n);
y3 = 4*cos(2*pi*(3/19)*n);
y4 = 4*cos(2*pi*(36/19)*n);
yt1 = 4*cos(2*pi*(1/19)*t);
yt2 = 4*cos(2*pi*(2/19)*t);
yt3 = 4*cos(2*pi*(3/19)*t);
yt4 = 4*cos(2*pi*(36/19)*t);

subplot(4,1,1);
stem(n, y1); hold on;
plot(t, yt1);
subplot(4,1,2);
stem(n, y2); hold on;
plot(t, yt2);
subplot(4,1,3);
stem(n, y3); hold on;
plot(t, yt3);
subplot(4,1,4);
stem(n, y4); hold on;
plot(t, yt4);
```

M.J. Roberts, Fundamentals of Signals and Systems

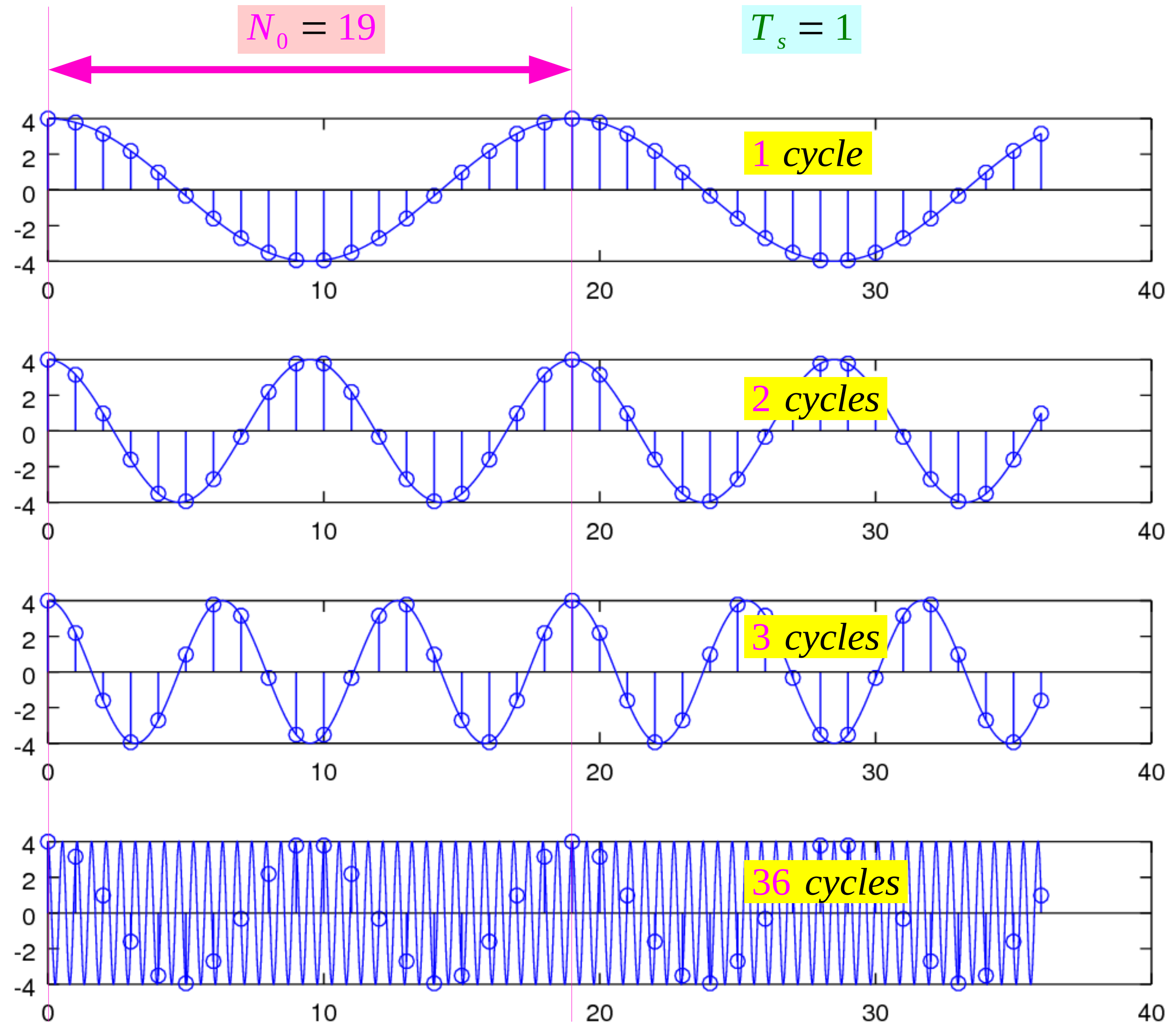
Periodic Condition Examples

$$g[n] = 4\cos\left(2\pi\cdot\frac{1}{19}\cdot n\right)$$

$$g[n] = 4\cos\left(2\pi\cdot\frac{2}{19}\cdot n\right)$$

$$g[n] = 4\cos\left(2\pi\cdot\frac{3}{19}\cdot n\right)$$

$$g[n] = 4\cos\left(2\pi\cdot\frac{36}{19}\cdot n\right)$$



The same digital sequences

$$g(t) = A\cos(2\pi f_0 t + \theta)$$

$$g[n] = A\cos(2\pi F_0 n + \theta)$$

$$k f_0 \cdot n T_s \frac{1}{k}$$

$$f_0 \cdot n T_s$$
$$= 1 \cdot f_0 \cdot n T_s \cdot \frac{1}{1}$$
$$= 2 \cdot f_0 \cdot n T_s \cdot \frac{1}{2}$$
$$= 3 \cdot f_0 \cdot n T_s \cdot \frac{1}{3}$$

M.J. Roberts, Fundamentals of Signals and Systems

The same digital sequence examples

$$g(t) = A\cos(2\pi f_0 t + \theta)$$

$$g[n] = A\cos(2\pi F_0 n + \theta)$$

$$g_1(t) = 4\cos(2\pi \cdot 1 \cdot t) \qquad t \leftarrow nT_1 \qquad g_1[n] = 4\cos(2\pi \cdot 1 \cdot nT_1)$$

$$g_2(t) = 4\cos(2\pi \cdot 2 \cdot t) \qquad t \leftarrow nT_2 \qquad g_2[n] = 4\cos(2\pi \cdot 2 \cdot nT_2)$$

$$g_3(t) = 4\cos(2\pi \cdot 3 \cdot t) \qquad t \leftarrow nT_3 \qquad g_3[n] = 4\cos(2\pi \cdot 3 \cdot nT_3)$$

$$T_1 = \frac{1}{10} \qquad n = 0,\ 1,\ 2,\ 3, \cdots \Rightarrow 1 \cdot nT_1 = 0,\ 0.1,\ 0.2,\ 0.3, \cdots = 1 \cdot t$$

$$T_2 = \frac{1}{20} \qquad n = 0,\ 1,\ 2,\ 3, \cdots \Rightarrow 2 \cdot nT_2 = 0,\ 0.1,\ 0.2,\ 0.3, \cdots = 2 \cdot t$$

$$T_3 = \frac{1}{30} \qquad n = 0,\ 1,\ 2,\ 3, \cdots \Rightarrow 3 \cdot nT_3 = 0,\ 0.1,\ 0.2,\ 0.3, \cdots = 3 \cdot t$$

$$\{\, g_1[n] \,\} \equiv \{\, g_2[n] \,\} \equiv \{\, g_2[n] \,\}$$

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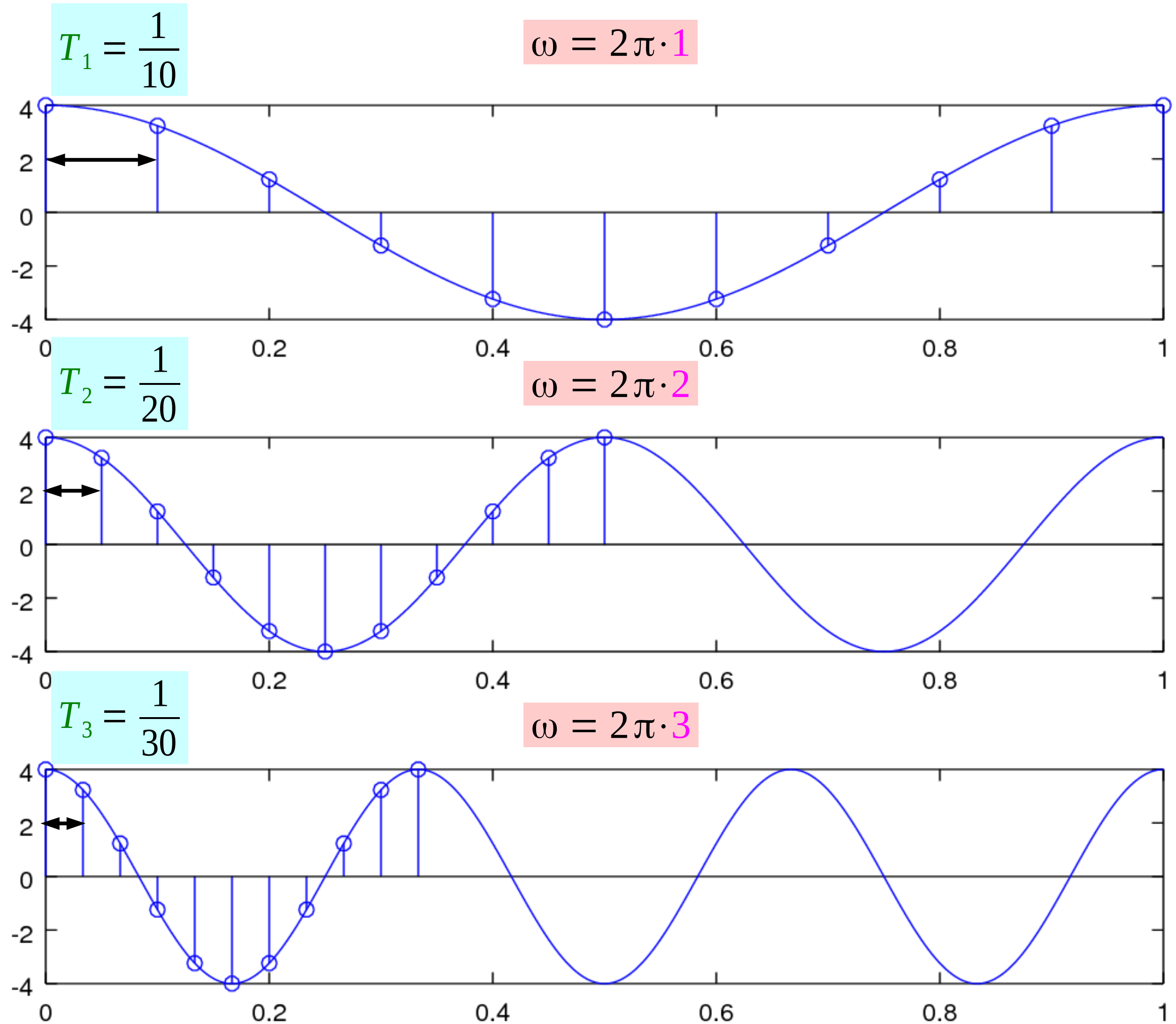
$f_0 n T_s = const$




```
clf
n = [0:10]; t = [0:1000]/1000;
y1 = 4*cos(2*pi*1*n/10);
y2 = 4*cos(2*pi*2*n/20);
y3 = 4*cos(2*pi*3*n/30);
yt1 = 4*cos(2*pi*t);
yt2 = 4*cos(2*pi*2*t);
yt3 = 4*cos(2*pi*3*t);

subplot(3,1,1);
stem(n, y1); hold on;
plot(t, yt1);
subplot(3,1,2);
stem(n/20, y2); hold on;
plot(t, yt2);
subplot(3,1,3);
stem(n/30, y3); hold on;
plot(t, yt3);
```

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The same sampled sequences

$$\cos(\omega_1 t_1) = \cos(\omega_1 n_1 T_1)$$

$$||$$

$$\cos(\omega_2 t_2) = \cos(\omega_2 n_2 T_2)$$

$$\omega_1 n_1 T_1 = \omega_2 n_2 T_2 + 2k\pi$$

the general case

The same phase condition

$$\cos(\omega_1 t_1) = \cos(\omega_1 n_1 T_1)$$

$$\cos(\omega_2 t_2) = \cos(\omega_2 n_2 T_2)$$

$$\omega_1 n_1 T_1 = \omega_2 n_2 T_2$$

the special case

Three special cases

$$\cos(\omega_1 t_1) = \cos(\omega_1 n_1 T_1) = \cos(\Omega_1 n_1) = \cos(2\pi F_1 n_1)$$

$$\cos(\omega_2 t_2) = \cos(\omega_2 n_2 T_2) = \cos(\Omega_2 n_2) = \cos(2\pi F_2 n_2)$$

constant $\boldsymbol{n}$

$$(\omega_1 T_1)\boldsymbol{n} = (\omega_2 T_2)\boldsymbol{n} \qquad \Omega_1 \boldsymbol{n} = \Omega_2 \boldsymbol{n}$$

constant $\boldsymbol{\omega}$

$$(\boldsymbol{\omega} T_1) n_1 = (\boldsymbol{\omega} T_2) n_2 \qquad \Omega_1 n_1 = \Omega_2 n_2$$

constant $\boldsymbol{T}$

$$(\omega_1 \boldsymbol{T}) n_1 = (\omega_2 \boldsymbol{T}) n_2 \qquad \Omega_1 n_1 = \Omega_2 n_2$$

Three special cases

$$\cos(\omega_1 t_1) = \cos(\omega_1 n_1 T_1)$$
$$\cos(\omega_2 t_2) = \cos(\omega_2 n_2 T_2)$$

$$\omega_1 n_1 T_1 = \omega_2 n_2 T_2$$

constant n : $\omega_1 n T_1 = \omega_2 n T_2$; $\omega_1 T_1 = \omega_2 T_2$; $\Omega_1 = \Omega_2$

constant ω : $\omega n_1 T_1 = \omega n_2 T_2$; $n_1 T_1 = n_2 T_2$; $T_{p1} = T_{p2}$

constant T : $\omega_1 n_1 T = \omega_2 n_2 T$; $\omega_1 n_1 = \omega_2 n_2$; $\theta_1 = \theta_2$

$$\Omega_1 n_1 = \Omega_2 n_2$$

F_0 and N_0 of a Sampled Signal

constant n	$\Omega_1 n_1 = \Omega_2 n_2$	$\omega_1 T_1 = \omega_2 T_2$	$F_1 = F_2$	$f_1 \neq f_2$
constant ω	$\Omega_1 n_1 = \Omega_2 n_2$	$n_1 T_1 = n_2 T_2$	$F_1 \neq F_2$	$f_1 = f_2$
constant T	$\Omega_1 n_1 = \Omega_2 n_2$	$\omega_1 n_1 = \omega_2 n_2$	$F_1 \neq F_2$	$f_1 \neq f_2$

F_0 and N_0 of a Sampled Signal

constant $\boldsymbol{n}$	$\omega_1/\omega_2 = T_2/T_1$	$\Omega_1 = \Omega_2$
constant $\boldsymbol{\omega}$	$T_1/T_2 = n_2/n_1$	$\Omega_1/\Omega_2 = n_2/n_1$
constant $\boldsymbol{T}$	$\omega_1/\omega_2 = n_2/n_1$	$\Omega_1/\Omega_2 = n_2/n_1$

F_0 and N_0 of a Sampled Signal

constant n	$\omega_1 T_1 n = \omega_2 T_2 n$ $\omega_1/\omega_2 = T_2/T_1$	$\Omega_1 n = \Omega_2 n$ $\Omega_1 = \Omega_2$ the same <u>angle</u> resolution
constant ω	$\omega T_1 n_1 = \omega T_2 n_2$ $T_1/T_2 = n_2/n_1$	$\Omega_1 n_1 = \Omega_2 n_2$ $\Omega_1/\Omega_2 = n_2/n_1$
constant T the same <u>time</u> resolution	$\omega_1 n_1 T = \omega_2 n_2 T$ $\omega_1/\omega_2 = n_2/n_1$	$\Omega_1 n_1 = \Omega_2 n_2$ $\Omega_1/\Omega_2 = n_2/n_1$

Normalized data points

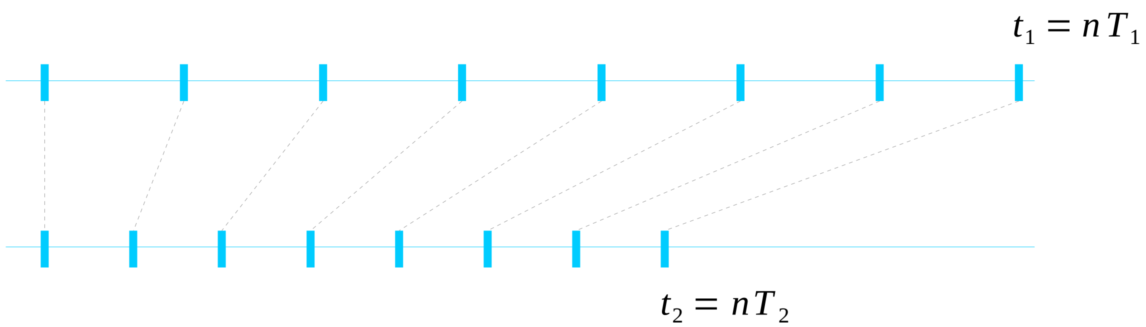
constant n

$$\omega_1 n T_1 = \omega_2 n T_2$$

one-to-one correspondence

shrink / expand
with the same angle resolution

The same number of sample points



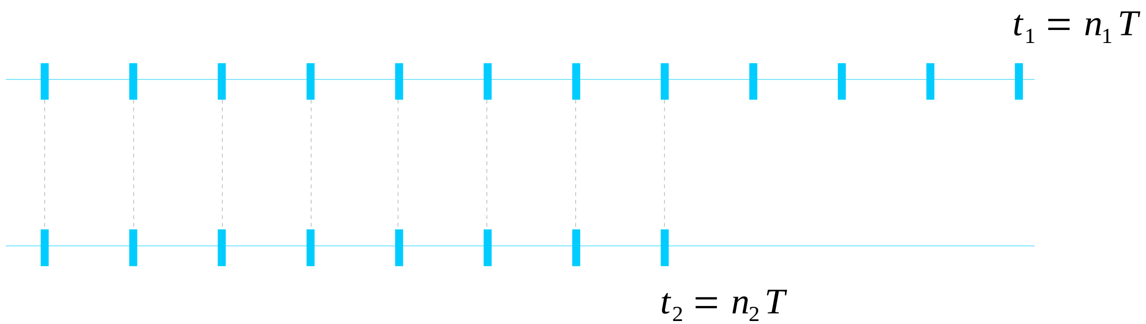
constant T

$$\omega_1 n_1 T = \omega_2 n_2 T$$

fixed sampling period

shrink / expand
with the same time resolution

The same sampling period

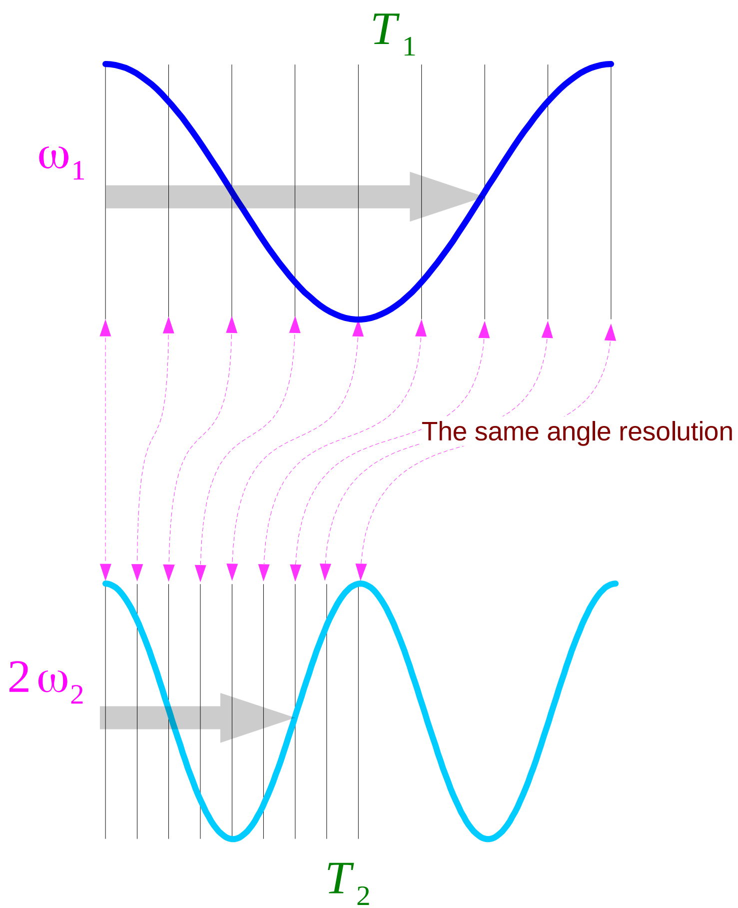


$f_0 T_s$ = const

constant n

$$\omega_1 T_1 = \omega_2 T_2 = \Omega$$

Always the same angles
at the same angle resolution Ω



$$\theta_1 = \omega_1 t_1 = \omega_1 n T_1 = \Omega_1 n$$

$$\omega_1 : \omega_2 = T_2 : T_1 = 1 : 2$$

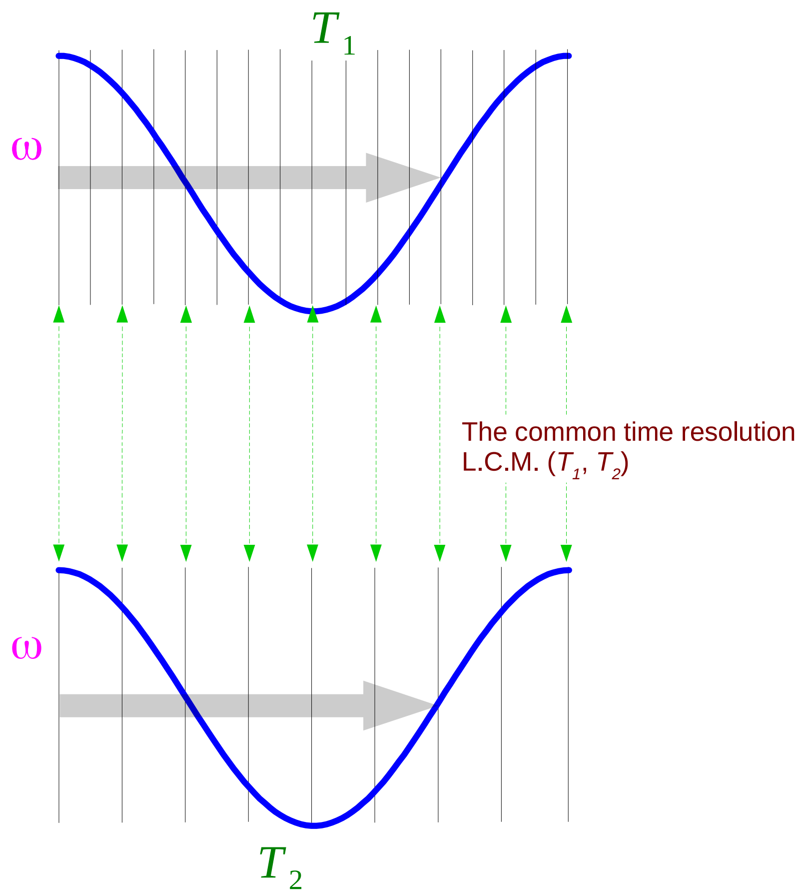
$$\theta_2 = \omega_2 t_2 = \omega_2 n T_2 = \Omega_2 n$$

nT_s = const

constant ω

$$n_1 T_1 = n_2 T_2 = T_p$$

Always the same angles
at the common time resolution $lcm(T_1, T_2)$



$$\theta_1 = \omega_1 t_1 = \omega n_1 T_1$$

$$T_1 : T_2 = n_2 : n_1 = 1 : 2$$

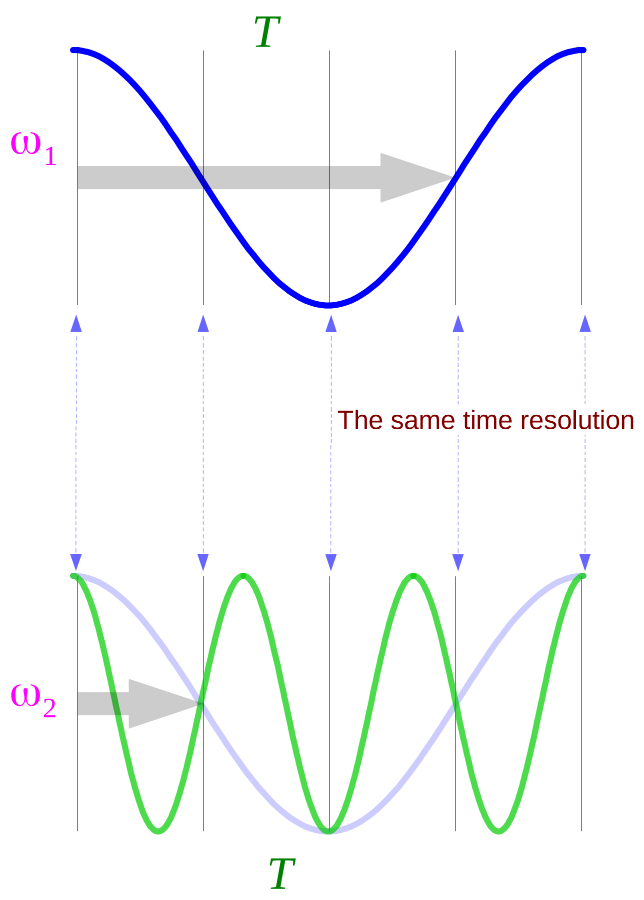
$$\theta_2 = \omega_2 t_2 = \omega n_2 T_2$$

$f_0 n$ = const

constant T

$$\Omega_1 n_1 = \Omega_2 n_2 = \theta$$

possible same angles
at the same time resolution T



$$\theta_1 = \omega_1 t_1 = \omega_1 n_1 T$$

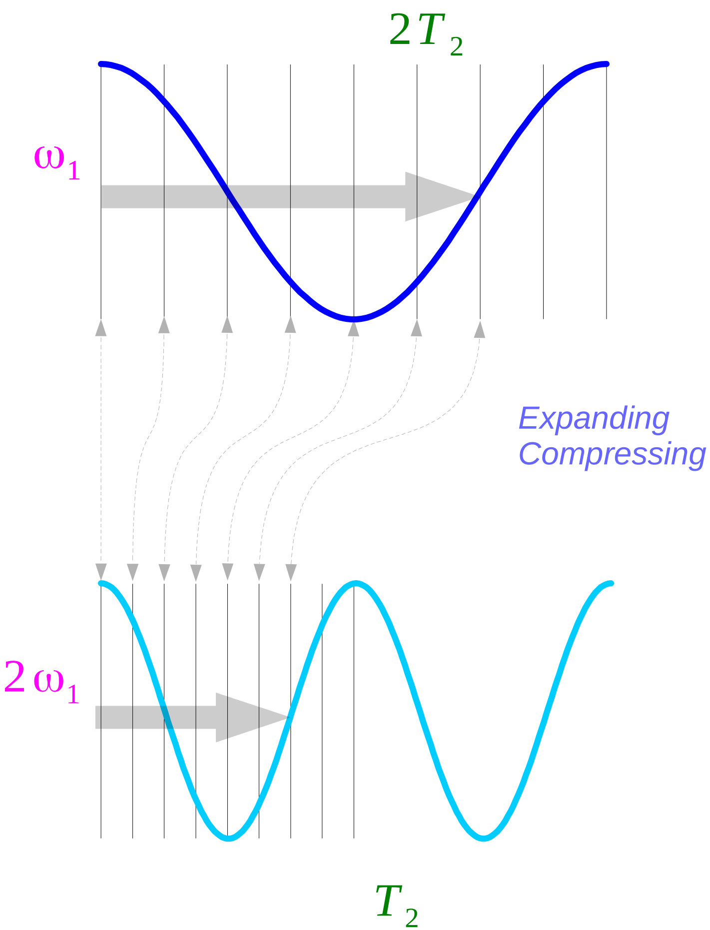
$$\omega_1 : \omega_2 = n_2 : n_1 = 1 : 3$$

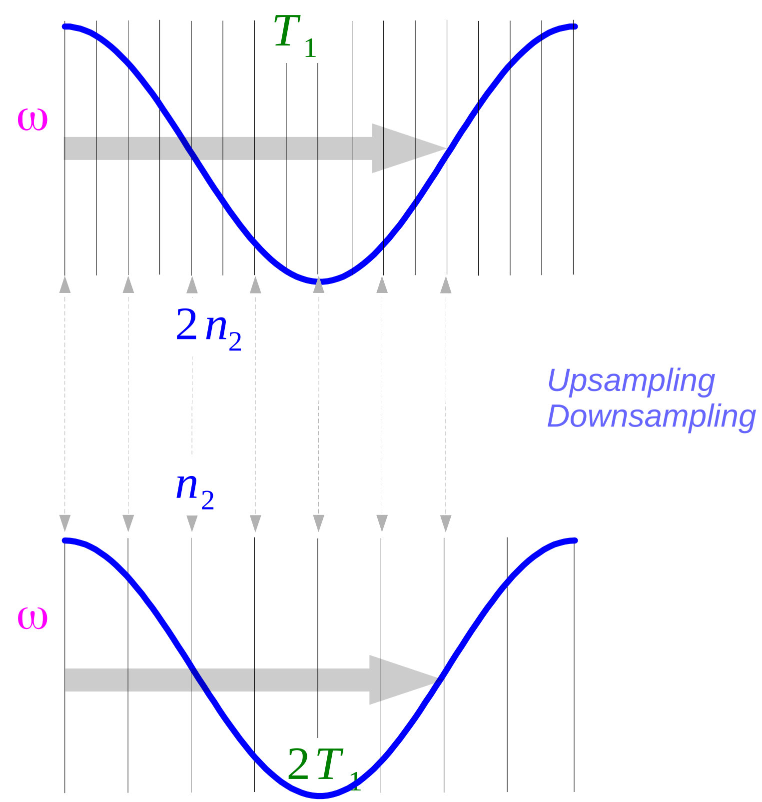
$$\theta_2 = \omega_2 t_2 = \omega_2 n_2 T$$

$f_0 T_s$ = const, $n T_s$ = const

$\omega_1 T_1 = \omega_2 T_2$ *constant* n

$n_1 T_1 = n_2 T_2$ *constant* ω

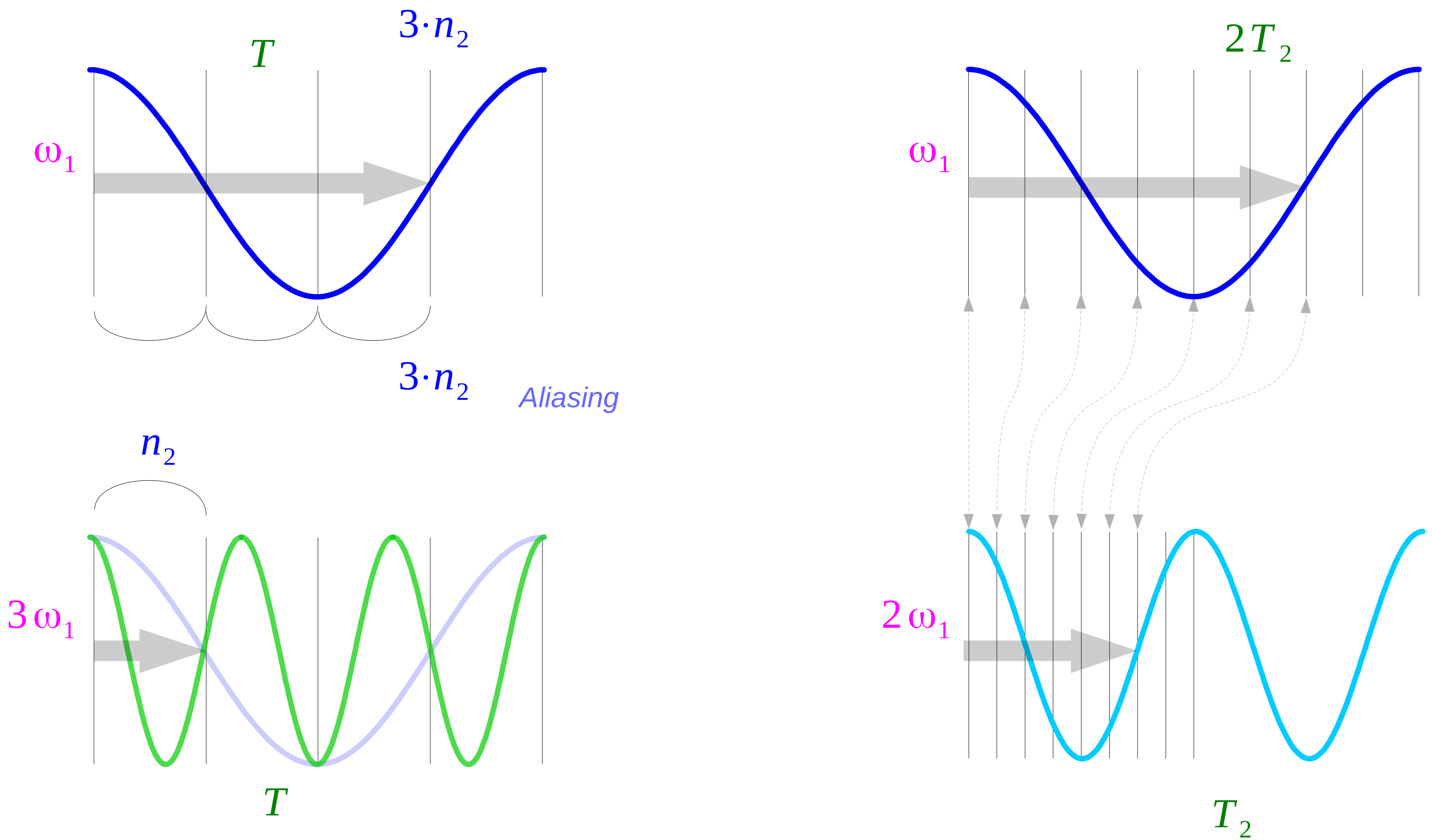




$f_0 n$ = const, $f_0 T_s$ = const

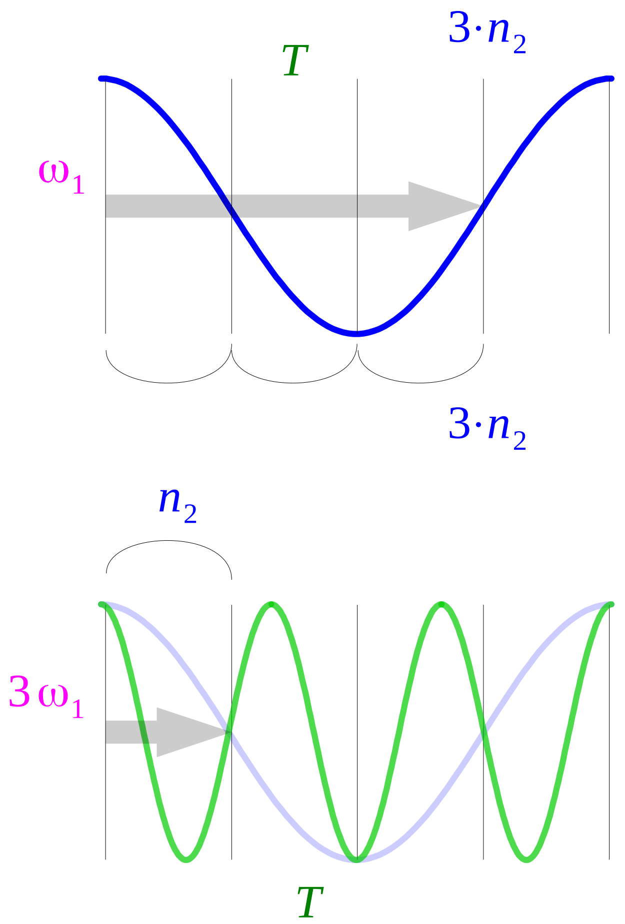
$\omega_1 n_1 = \omega_2 n_2$ *constant* T

$\omega_1 T_1 = \omega_2 T_2$ *constant* n

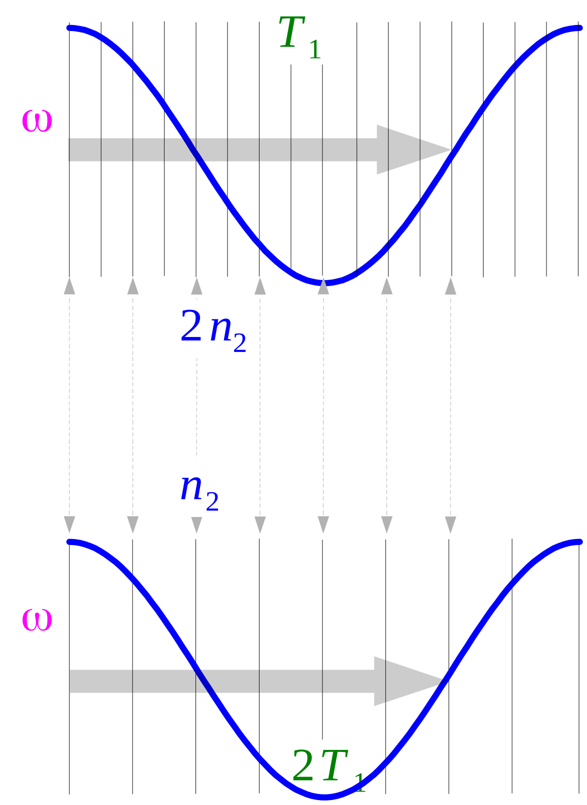


$f_0 n$ = const, nT_s = const

$\omega_1 n_1 = \omega_2 n_2$ *constant* T



$n_1 T_1 = n_2 T_2$ *constant* ω



Periodic Condition Examples

$$\cos(\omega_1 t_1) = \cos(\omega_1 n T_1)$$

$$\cos(\omega n_1 T_1)$$

$$\omega_1 t_1 = \omega_2 t_2$$

$$\omega t_1 = \omega t_2$$

$$\omega_1 n T_1 = \omega_2 n T_2$$

$$\omega n_1 T_1 = \omega n_2 T_2$$

constant n

constant ω

$$\omega_1 (2 T_2) = (2 \omega_1) T_2$$

$$(2 n_2) T_1 = n_2 (2 T_1)$$

$$\cos(\omega_2 t_2) = \cos(\omega_2 n T_2)$$

$$\cos(\omega n_2 T_2)$$

Periodic Condition Examples

$\omega_1 n T_1 = \omega_2 n T_2$

constant n

$\omega \uparrow \quad T_s \downarrow$

$\omega \downarrow \quad T_s \uparrow$

$\omega n_1 T_1 = \omega n_2 T_2$

constant ω

$n \uparrow \quad T_s \downarrow$

$n \downarrow \quad T_s \uparrow$

References

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[3] M.J. Roberts, Fundamentals of Signals and Systems
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[5] K. Shin, et al., Fundamentals of Signal Processing for Sound and Vibration Engineerings

[6] A “graphical interpretation” of the DFT and FFT, by Steve Mann