

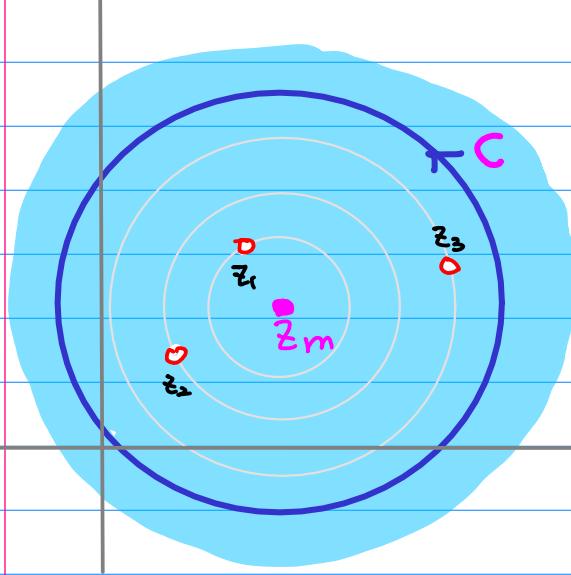
Laurent Series with Applications

20170325

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Series Expansion at z_m



$$f(z) = \sum_{n=n_1}^{\infty} a_n^{(sm)} (z - z_m)^n$$

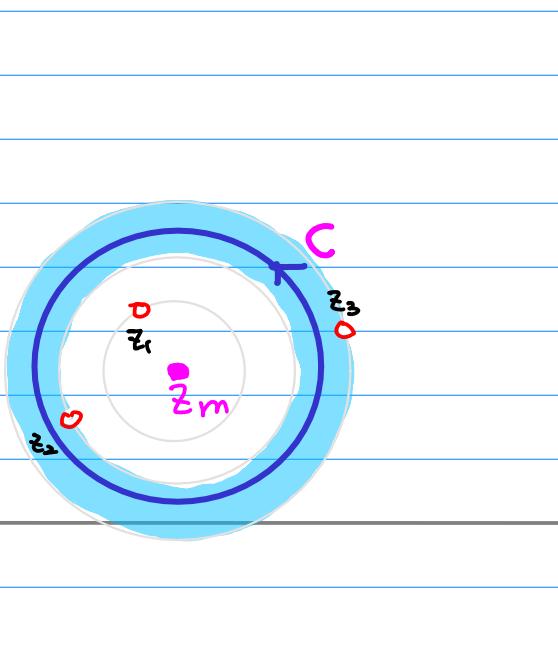
$$\begin{aligned} a_n^{(sm)} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_m)^{n+1}} dz \\ &= \sum_k \text{Res} \left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k \right) \end{aligned}$$

$$\begin{aligned} a_{-1}^{(sm)} &= \frac{1}{2\pi i} \oint_C f(z) dz \\ &= \sum_k \text{Res} (f(z), z_k) \end{aligned}$$

$$a_{-1}^{(sm)} \neq \text{Res} (f(z), z_m)$$

[Annular Region] \Rightarrow Laurent Series

* Even if z_m is a non-singular point of $f(z)$,
 z_m becomes a pole
 in the residue computation.



$$\frac{f(z)}{(z - z_m)^{n+1}} \quad \text{if } n \geq 0$$

$$a_n^{\text{sing}} = \sum_k \text{Res} \left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k \right)$$

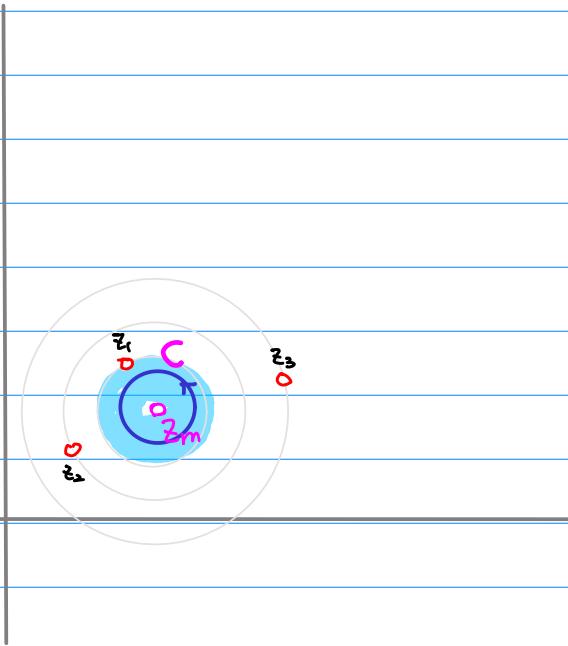
$$a_{-1}^{\text{sing}} \neq \text{Res}(f(z), z_m)$$

\because residue is defined on
 a punctured open disk.

[Annular Region] & [z_m : isolated singularity]

A punctured open disk \Rightarrow Residue.
Laurent Series

- only one pole is enclosed by C



$$f(z) = \sum_{n=n_1}^{\infty} a_n^{(sm)} (z - z_m)^n$$

$$\begin{aligned} a_n^{(sm)} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_m)^{n+1}} dz \\ &= \sum_k \text{Res} \left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k \right) \end{aligned}$$

$$\begin{aligned} a_{-1}^{(sm)} &= \frac{1}{2\pi i} \oint_C f(z) dz \\ &= \sum_k \text{Res} (f(z), z_k) \end{aligned}$$

$$a_{-1}^{(sm)} = \text{Res} (f(z), z_m)$$

$\sum_k z_k = z_m$ ←
the only pole enclosed
by C , is z_m
a punctured open disk

Isolated Singularity

$$z = z_0$$

④ a **singularity** of a complex function f

⑤ an **isolated singularity** of a complex function f

if there exists some **deleted neighborhood**

or **punctured open disk** of z_0

where $f(z)$ is **analytic**

$$0 < |z - z_0| < R$$



⑥ a **non-isolated singularity**

if every neighborhood of z_0 contains

at least one singularity of f other than z_0

the branch point $z=0$

$\ln z$

z_0



Every neighborhood of $z=0$

contains points on the negative real axis

branch cut : non-positive real axis

$\text{Ln } z$

Principal Argument

$$\text{Arg}(z) = \theta \quad -\pi < \theta \leq \pi$$

$$z \neq 0 \quad \theta = \arg z$$

$$\ln z = \log_e |z| + i(\theta + 2n\pi) \quad n=0, \pm 1, \pm 2, \dots$$

Principal Value

$$\ln z = \log_e |z| + i \text{Arg } z$$

$$\text{Arg } z : \text{unique} \rightarrow \ln z : \text{unique} \quad \text{for } z \neq 0$$

$$f(z) = \ln z \quad \text{not continuous at } z=0$$

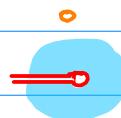
$$f(0) \text{ not defined} \leftarrow \log_e 0 \text{ not defined}$$

$$f(z) = \ln z \quad \text{discontinuous at the negative real axis}$$

$\leftarrow \text{Arg } z \quad \text{discontinuous}$

for $x_0 < 0 \quad \lim_{z \rightarrow z_0} \text{Arg } z = \pm \pi$

the non-positive real axis : the branch cut



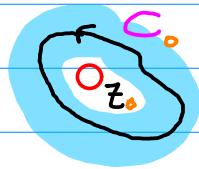
A punctured open disk

if C encloses only one pole z_0 ,

and the expansion at that pole z_0 is assumed,

then

$$a_{-1}^{\{0\}} = \frac{1}{2\pi i} \oint_{C_0} f(z) dz = \text{Res}(f(z), z_0)$$



Let

$$\tilde{a}_{-1}^{\{m\}} = \text{Res}(f(z), z_m)$$

notation \sim

the residue of $f(z)$ at z_m

Using C_m which is in the punctured open disk ROC

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n^{\{m\}} (z - z_m)^n$$

Residues Computation

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s) ds \quad \rightarrow \quad \oint_C f(s) ds = 2\pi i \cdot a_{-1}$$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s) ds = \text{Res}(f(z), z_0)$$

$$= \begin{cases} \lim_{z \rightarrow z_0} (z - z_0) f(z) & (\text{simple}) \\ \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) & (\text{order } n) \end{cases}$$

$$f(z) = \frac{\text{green box}}{(z - z_0) \text{ blue box}}$$

$$\rightarrow \text{Res}(f(z), z_0) = \lim_{z \rightarrow z_0} \frac{\text{green box}}{\text{blue box}}$$

$$f(z) = \frac{\text{green box}}{(z - z_0)^n \text{ blue box}}$$

$$\rightarrow \text{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \left(\frac{d^{n-1}}{dz^{n-1}} \frac{\text{green box}}{\text{blue box}} \right)$$

z_0 : expansion center

$$f(z) = \dots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_1}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

z_0 : Simple pole $\frac{1}{z-z_0}$

z_0 : n -th order pole $\frac{1}{(z-z_0)^n}$

I z_0 : expansion center & simple pole ($a_1 \neq 0$)

$$f(z) = \frac{a_1}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

II z_0 : expansion center & n -th order pole ($a_n \neq 0$)

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \dots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_1}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

I) z_0 : expansion center & Simple pole ($a_1 \neq 0$)

$$f(z) = \frac{a_1}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

$$(z-z_0) f(z) = a_1 + a_0(z-z_0) + a_1(z-z_0)^2 + a_2(z-z_0)^3 + \dots$$

$$\lim_{z \rightarrow z_0} (z-z_0) f(z) = a_1$$

II) z_0 : expansion center & n -th order pole ($a_{-n} \neq 0$)

$n=2$

$$f(z) = \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

$$(z-z_0)^2 f(z) = a_{-2} + a_{-1}(z-z_0) + a_0(z-z_0)^2 + a_1(z-z_0)^3 + a_2(z-z_0)^4 + \dots$$

$$\frac{d}{dz} (z-z_0)^2 f(z) = a_{-1} + 2a_0(z-z_0) + 3a_1(z-z_0)^2 + 4a_2(z-z_0)^3 + \dots$$

$$\frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) = a_{-1}$$

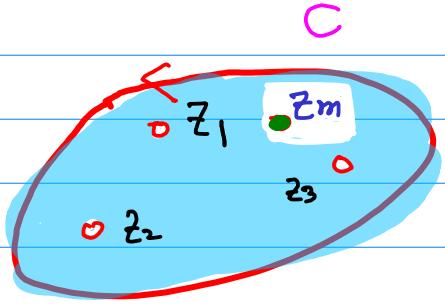
L'Hopital's Theorem

$$\lim_{z \rightarrow z_0} \frac{g(z)}{h(z)} = \frac{g'(z_0)}{h'(z_0)}$$

General Series Expansion

$$f(z) = \sum_{n=0}^{\infty} a_n^{(m)} (z - z_m)^n$$

$$\begin{aligned} a_n^{(m)} &= \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_m)^{n+1}} dz' \\ &= \sum_k \text{Res} \left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k \right) \end{aligned}$$



C is in the same region of analyticity of $f(z)$

typically a circle centered on z_m

non-annular ok

z_k within C : singularities of $\frac{f(z)}{(z - z_m)^{n+1}}$

$n_i = n_{f,m}$ depends on $f(z)$, z_m

$a_n^{(m)}$ depends on $f(z)$, z_m , region of analyticity

Whether $f(z)$ is singular at z_m or not

or singular at other points between z and z_m

We can expand $f(z)$ about any point z_m over powers of $(z - z_m)$.

Whether $f(z)$ is singular at (z_m) or not
 or singular at other points between (z) and (z_m)
 We can expand $f(z)$ about any point (z_m)
 over powers of $(z - z_m)$.

(z)

evaluation point

(z_m)

expansion center

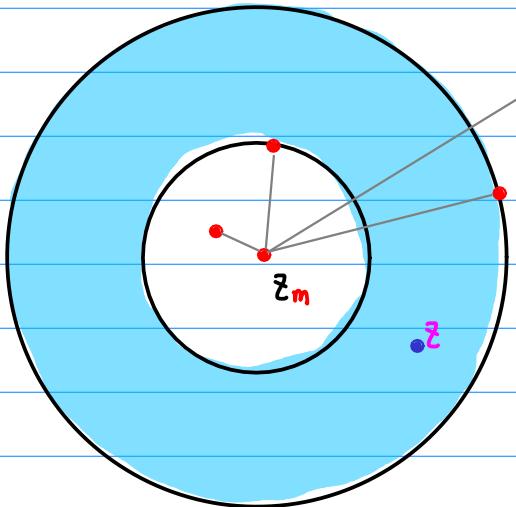
(z_k)

one of K poles of

$$\frac{f(z)}{(z - z_m)^{n+1}}$$

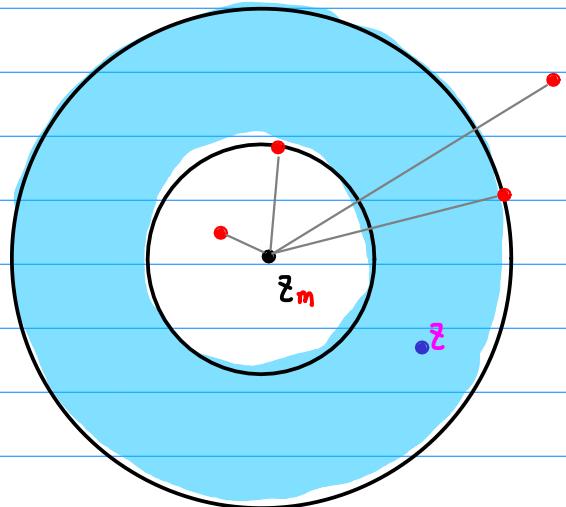
$$= \{ \text{poles of } f(z) \} \cup \{ \text{pole } z_m \text{ depending on } n \}$$

Laurent Series



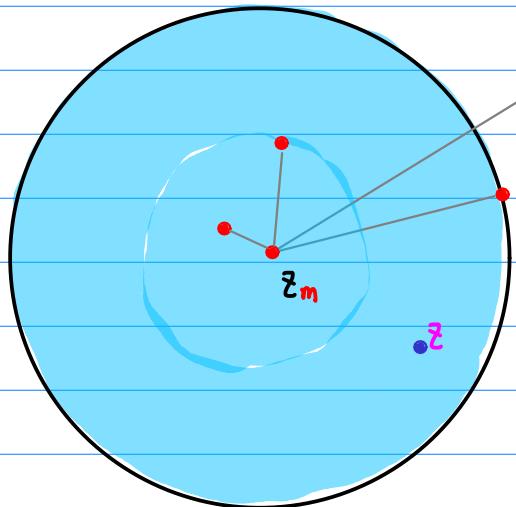
singular z_m

Laurent Series



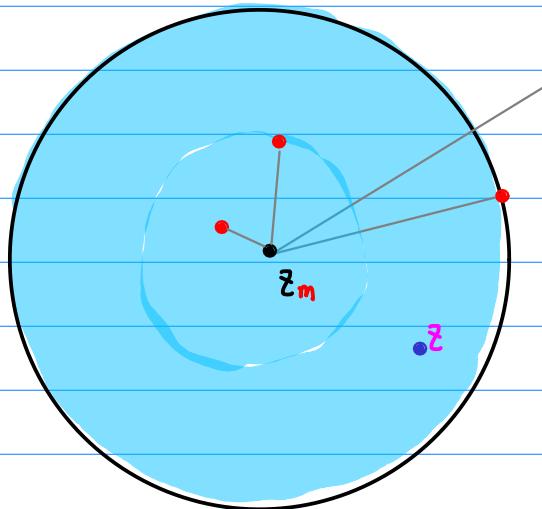
non-singular z_m

Non-Laurent Series

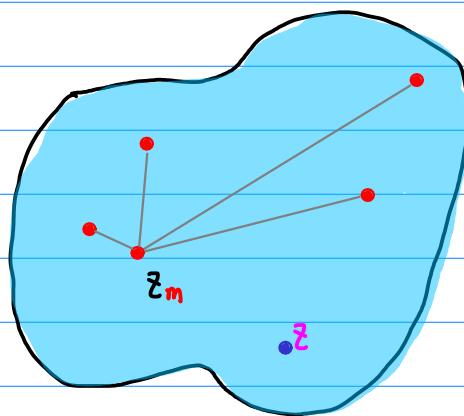


singluar z_m

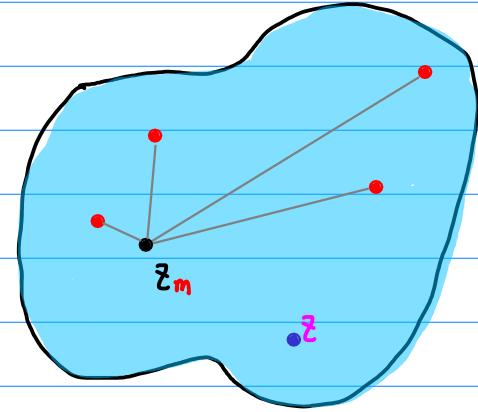
Non-Laurent Series



non-singluar z_m



singluar z_m



non-singluar z_m

Taylor Series

$f(z)$ analytic on and within \mathcal{C}

→ no poles

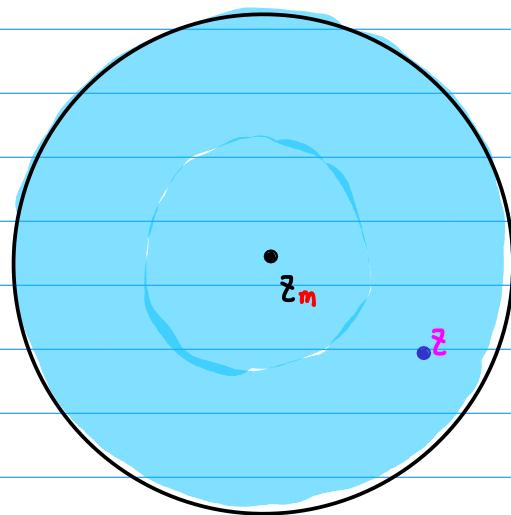
→ z_m becomes the only pole
of the residue of

$$\frac{f(z)}{(z - z_m)^{n+1}}$$

when $n \geq 0$

Non-Laurent Series

Taylor Series



non-singular z_m

no singular points
on and within \mathcal{C}

$n \geq 0 \rightarrow$

$$a_n = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{(z - z_m)^{n+1}} dz$$

$$= \text{Res} \left(\frac{f(z)}{(z - z_m)^{n+1}}, z_m \right)$$

$$= \frac{1}{n!} f^{(n)}(z_m)$$

Analytic $f(z) \rightarrow$ Taylor Series

$$n_1 = n_{f,m} \quad \text{depends on } f(z), z_m$$

$f(z)$ analytic on and within C

\rightarrow no poles

$\rightarrow z_m$ becomes the only pole

of the residue of

$$\frac{f(z)}{(z - z_m)^{n+1}}$$

when $n \geq 0$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_m)^{n+1}} dz$$

$$= \begin{cases} 0 & (n < 0) \\ \text{Res} \left(\frac{f(z)}{(z - z_m)^{n+1}}, z_m \right) & (n \geq 0) \end{cases}$$

$$\text{Res} \left(\frac{f(z)}{(z - z_m)^{n+1}}, z_m \right) \quad (n \geq 0)$$

$$n=0 \quad \text{Res} \left(\frac{f(z)}{(z - z_m)^1}, z_m \right) = \lim_{z \rightarrow z_m} (z - z_m) \frac{f(z)}{(z - z_m)} = f(z_m)$$

$$n=1 \quad \text{Res} \left(\frac{f(z)}{(z - z_m)^2}, z_m \right) = \frac{1}{(2-1)!} \lim_{z \rightarrow z_m} \frac{d}{dz} \left((z - z_m)^2 \frac{f(z)}{(z - z_m)^2} \right) = f'(z_m)$$

$$n=2 \quad \text{Res} \left(\frac{f(z)}{(z - z_m)^3}, z_m \right) = \frac{1}{(3-1)!} \lim_{z \rightarrow z_m} \frac{d^2}{dz^2} \left((z - z_m)^3 \frac{f(z)}{(z - z_m)^3} \right) = \frac{1}{2!} f''(z_m)$$

$$n \quad \text{Res} \left(\frac{f(z)}{(z - z_m)^{n+1}}, z_m \right) = \frac{1}{n!} \lim_{z \rightarrow z_m} \frac{d^n}{dz^n} \left((z - z_m)^{n+1} \frac{f(z)}{(z - z_m)^{n+1}} \right) = \frac{1}{n!} f^{(n)}(z_m)$$

$$\text{Res}(G(z), z_0)$$

$$\left\{ \begin{array}{l} \boxed{\lim_{z \rightarrow z_0} (z - z_0) G(z) = a_1} \quad \text{Simple pole } z_0 \\ \boxed{\frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n G(z) = a_1} \quad n\text{-th order pole } z_0 \end{array} \right.$$

$f(z)$ analytic on and within \mathcal{C}

→ no poles

→ z_m becomes the only pole when $n \geq 0$

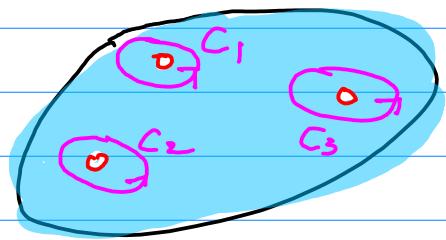
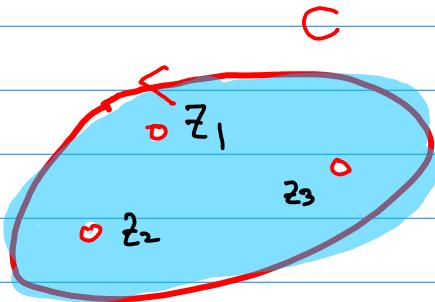
$$\boxed{\frac{f(z)}{(z - z_m)^{n+1}}}$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{(z - z_m)^{n+1}} dz$$

$$= \begin{cases} 0 & (n < 0) \\ \text{Res} \left(\frac{f(z)}{(z - z_m)^{n+1}}, z_m \right) & (n \geq 0) \end{cases}$$

$$= \begin{cases} 0 & (n < 0) \\ \frac{1}{n!} f^{(n)}(z_m) & (n \geq 0) \end{cases}$$

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_m)}{k!} (z - z_m)^k \quad \text{Taylor Series}$$



$$\oint_C f(z) dz = 2\pi j \sum_{k=1}^K \tilde{a}_{-1}^{(k)} = 2\pi j \sum_{k=1}^K \text{Res}(f(z), z_k)$$

residue theorem

$$f(z) = \sum_{n=-\infty}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

general Series

$$a_n^{\{m\}} = \sum_{k=1}^K \text{Res} \left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k \right)$$

general formula

C encloses K poles $(z_1, z_2, \dots, z_k, \dots, z_K)$ of $\frac{f(z)}{(z - z_m)^{n+1}}$

C_k encloses only the k -th pole (z_k)

$\tilde{a}_{-1}^{(k)}$ the residue of the k -th pole z_k enclosed by C_k
 (Laurent series coefficients on a punctured open disk)

Laurent's Theorem

f : analytic within the **annular** domain D

$$r < |z - z_0| < R$$

then

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k,$$

valid for $r < |z - z_0| < R$ (ROC)

The coefficients a_k are given by

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} ds, \quad k=0, \pm 1, \pm 2, \dots$$

C: a simple closed curve
that lies entirely within D
that encloses z_0

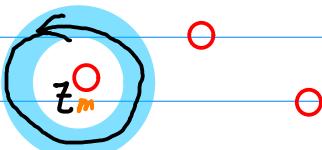
Curve C & Domain D of the Laurent Series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_m)^{n+1}} dz'$$

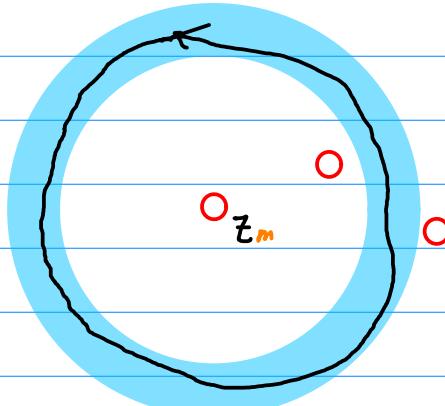
$$= \sum_k \text{Res}\left(\frac{f(z)}{(z - z_m)^{n+1}}, z_m\right)$$

a punctured open disk



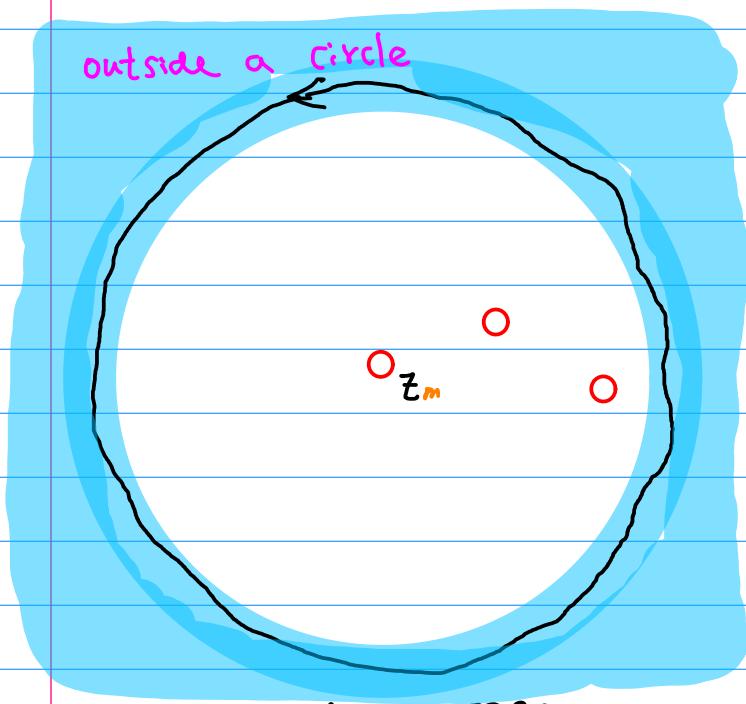
annular region

ring



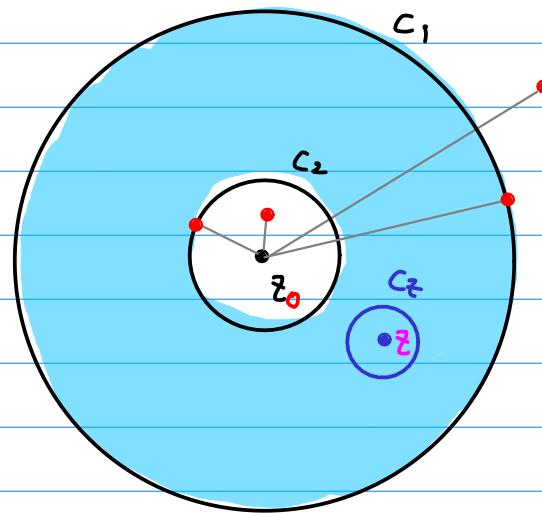
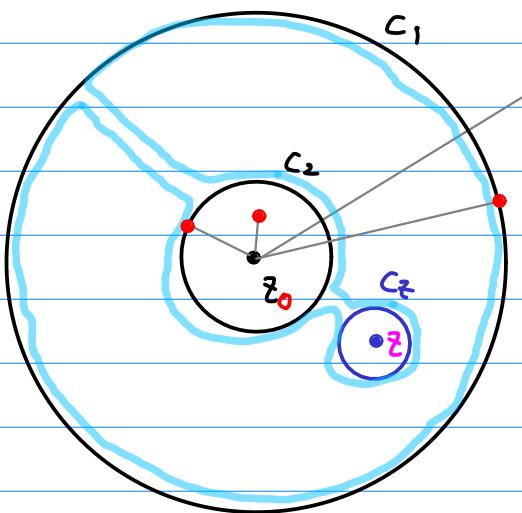
annular region

outside a circle



annular region

Expansion Points and Evaluation Points



(z_0) : expansion point

z : evaluation point (in ROC)

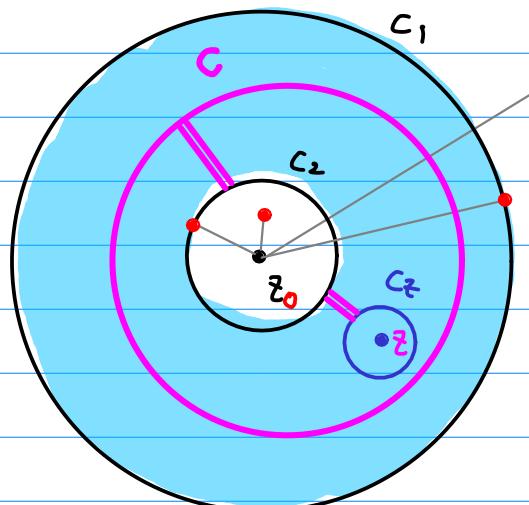
$$\frac{f(z)}{(z - z_0)}$$

is analytic between C_1 & C_2

deformation theorem

$C_1 - C_2$ coincide

common contour C

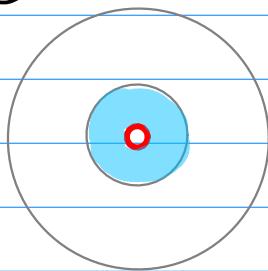


3 types of annular region

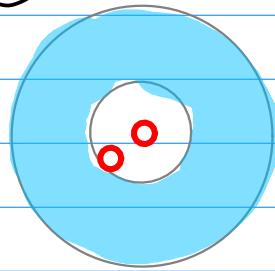
all Laurent Series ROC



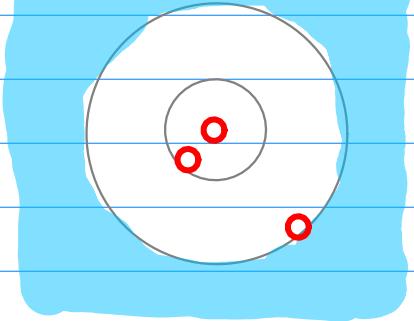
(A) only one pole



(B) more than one pole



(C) all poles



punctured

open disk

ring

outside a circle



Only this region
defines a residue

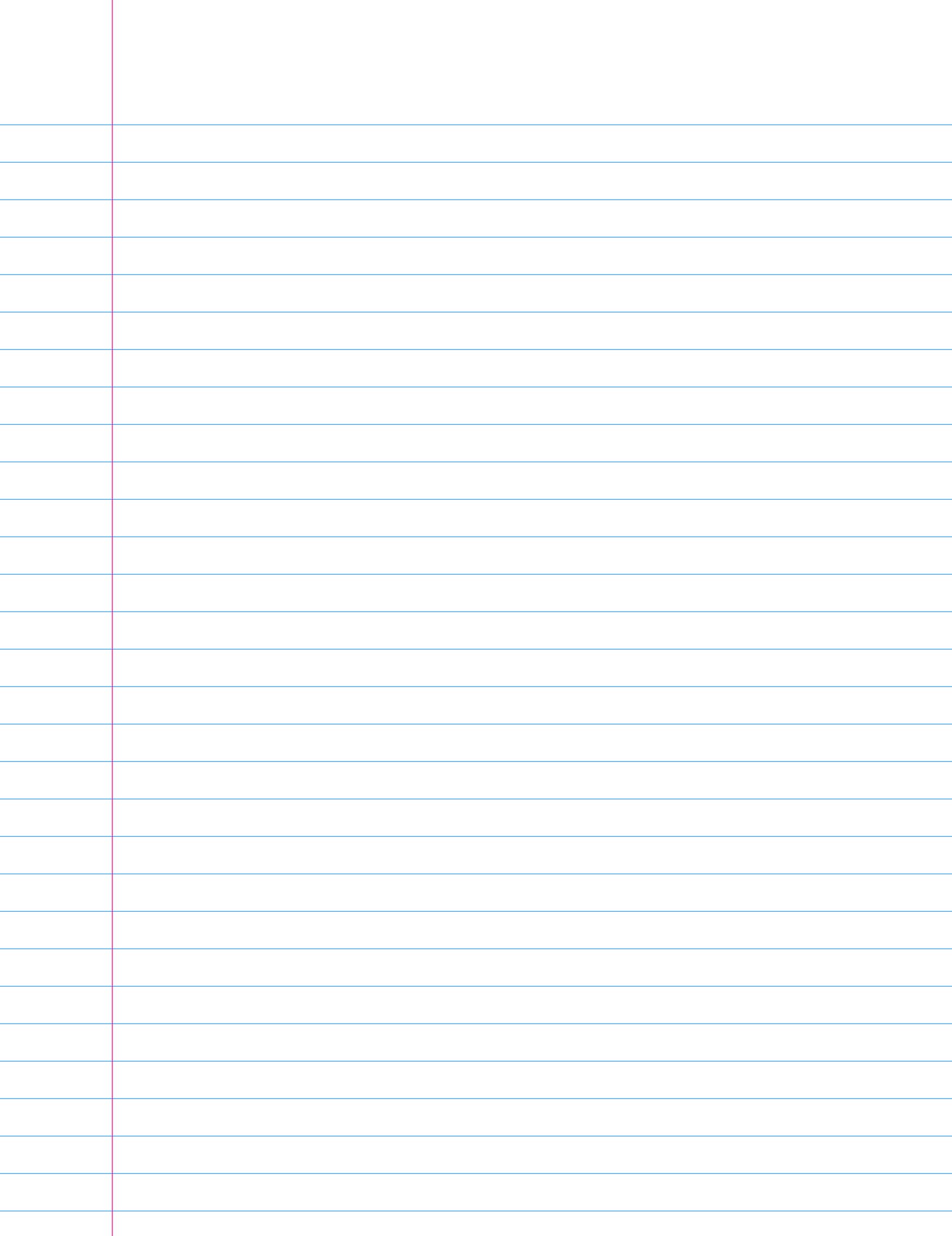
	annular region			non-annular region
	punctured	ring	outside circle	
Singular center z_m				
non-singular center z_m				

$f(z) = \sum_{n=-\infty}^{\infty} a_n^{(m)} (z - z_m)^n$
 $a_n^{(m)} = \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z' - z_m)^{n+1}} dz' = \sum_k \text{Res} \left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k \right)$

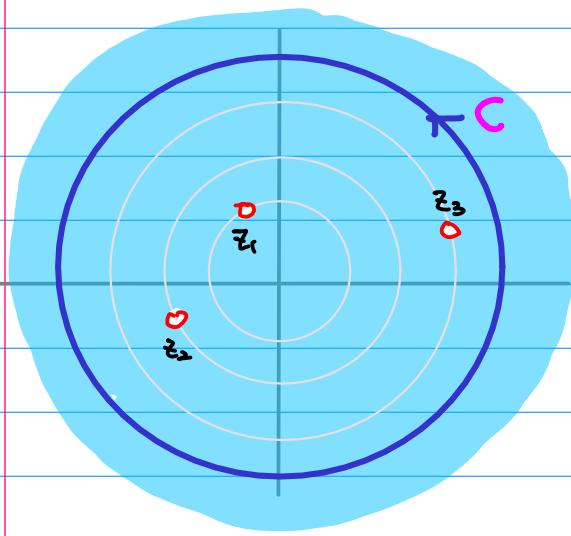
	annular region			non-annular region
	punctured	ring	outside circle	
Singular center z_m				X
non-singular center z_m				X

Laurent Series
 Laurent Series

	annular region			non-annular region
	punctured	ring	outside circle	
Singular center z_m	residue	X	X	X
non-singular center z_m	X	X	X	X



Series Expansion at $z=0$



$$f(z) = \sum_{n=n_1}^{\infty} a_n^{(m)} z^n$$

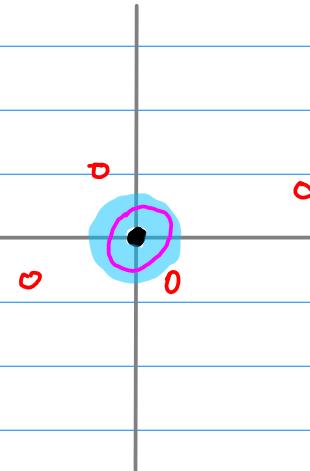
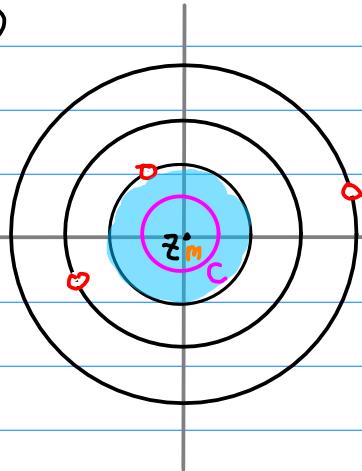
$$\begin{aligned} a_n^{(m)} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} dz \\ &= \sum_k \text{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right) \end{aligned}$$

Poles z_k

$n \geq 0$ $z_1, z_2, z_3, 0$

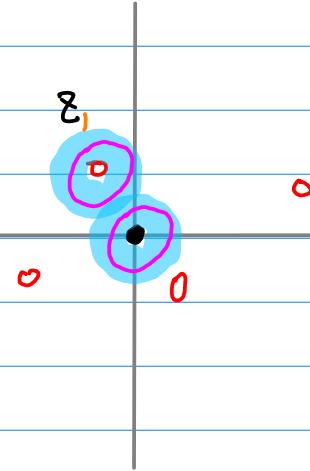
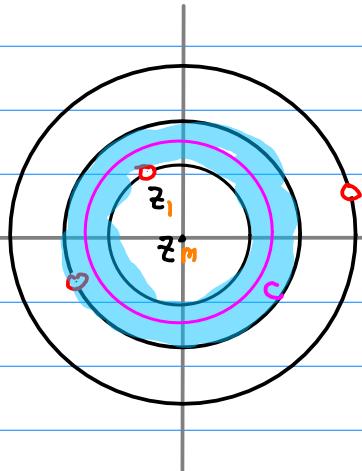
$n < 0$ z_1, z_2, z_3

①



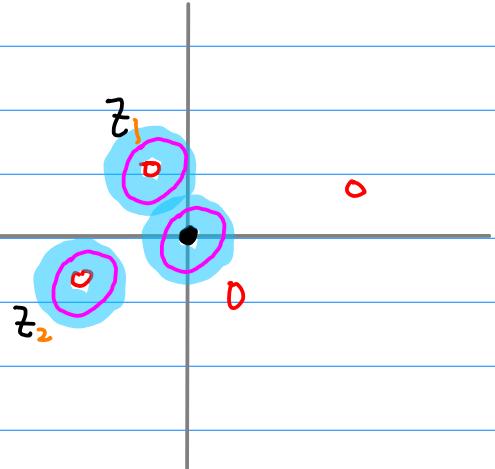
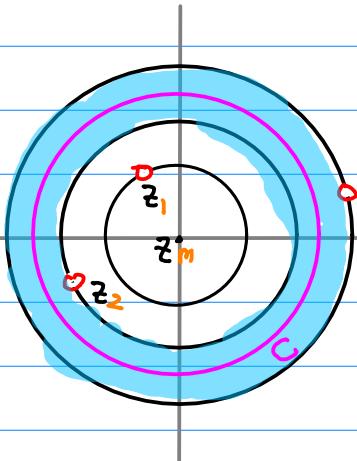
$$\operatorname{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right) \quad (n \geq 0)$$

②



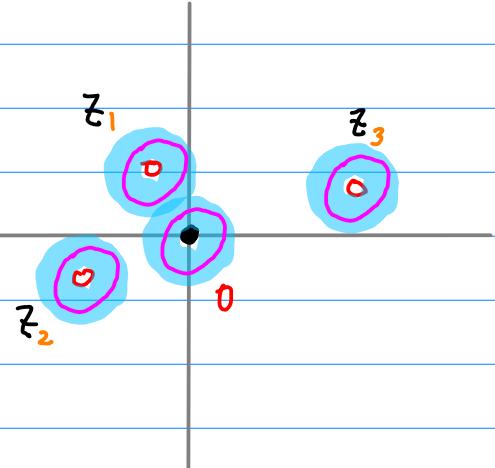
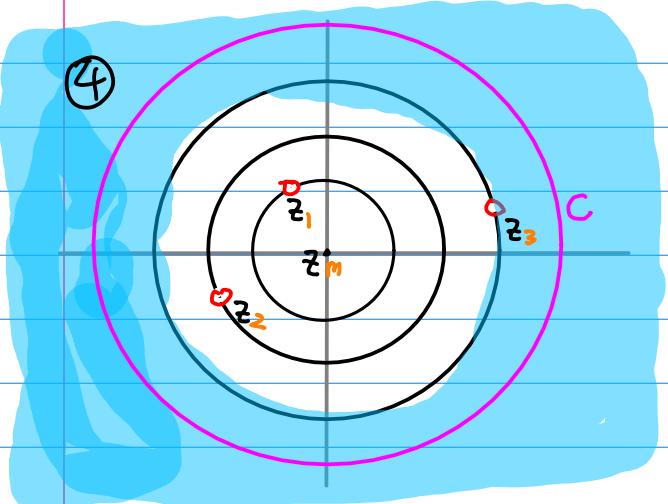
$$\operatorname{Res}\left(\frac{f(z)}{z^{n+1}}, z_1\right) + \operatorname{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right) \quad (n \geq 0)$$

③



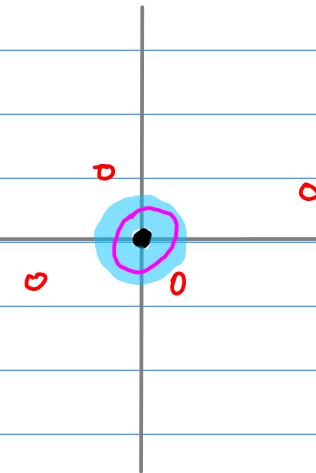
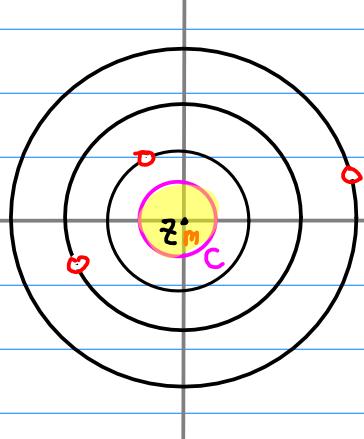
$$\operatorname{Res}\left(\frac{f(z)}{z^n}, z_2\right) + \operatorname{Res}\left(\frac{f(z)}{z^n}, z_1\right) + \operatorname{Res}\left(\frac{f(z)}{z^n}, 0\right) \quad (n \geq 0)$$

④



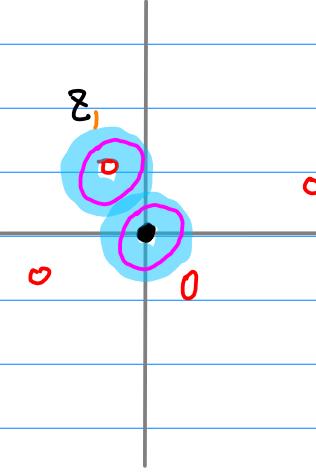
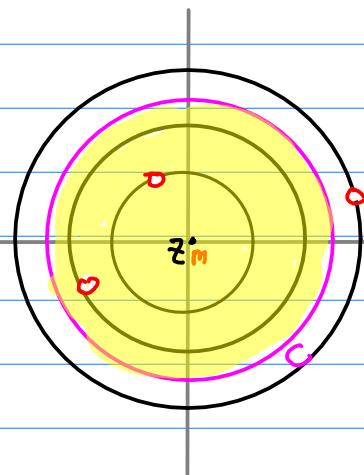
$$\operatorname{Res}\left(\frac{f(z)}{z^n}, z_3\right) + \operatorname{Res}\left(\frac{f(z)}{z^n}, z_2\right) + \operatorname{Res}\left(\frac{f(z)}{z^n}, z_1\right) + \operatorname{Res}\left(\frac{f(z)}{z^n}, 0\right) \quad (n \geq 0)$$

(1)



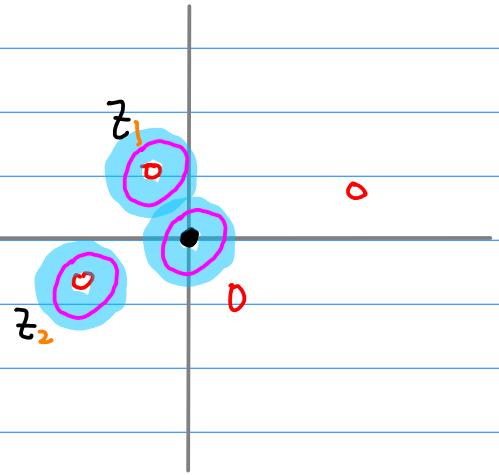
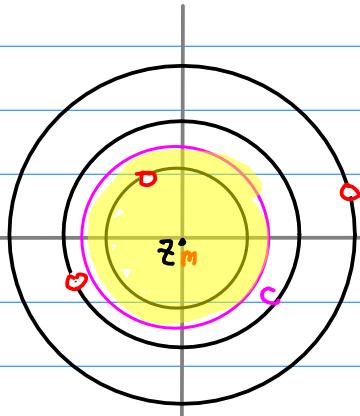
$$\operatorname{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right) \quad (n \geq 0)$$

(2)



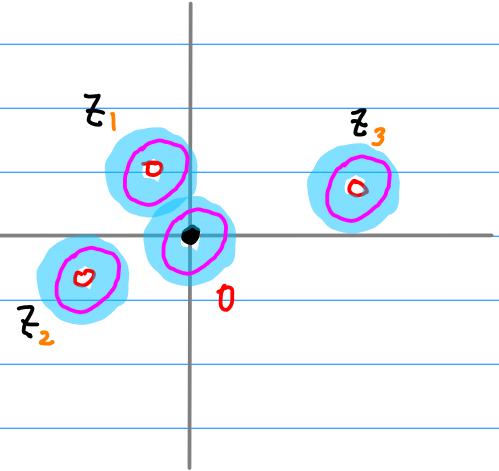
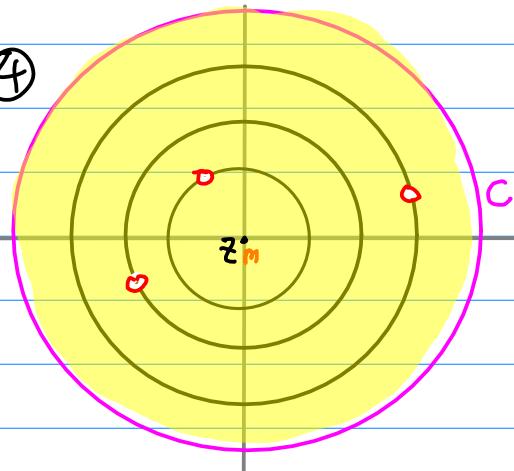
$$\operatorname{Res}\left(\frac{f(z)}{z^{n+1}}, z_1\right) + \operatorname{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right) \quad (n \geq 0)$$

③



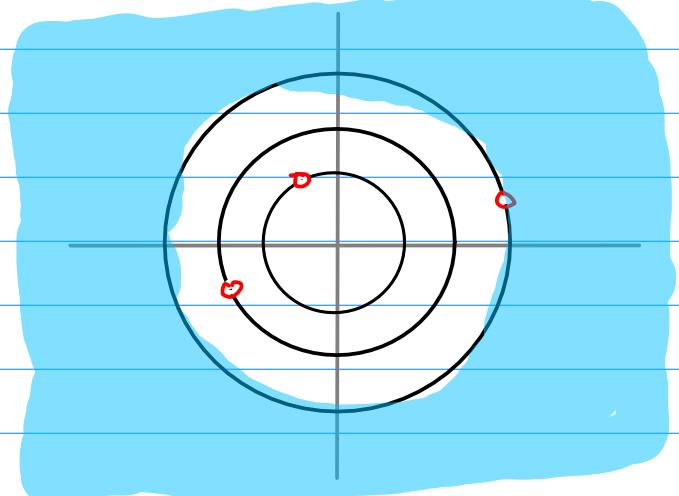
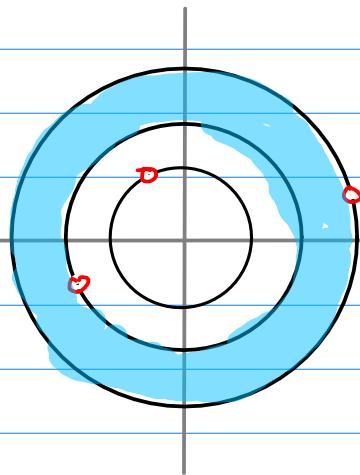
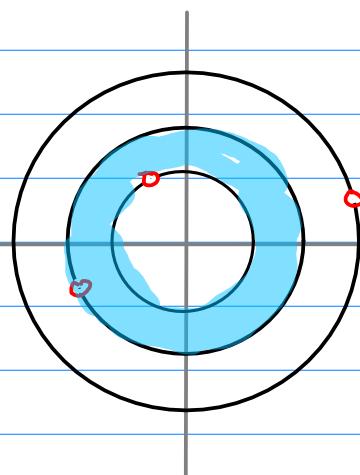
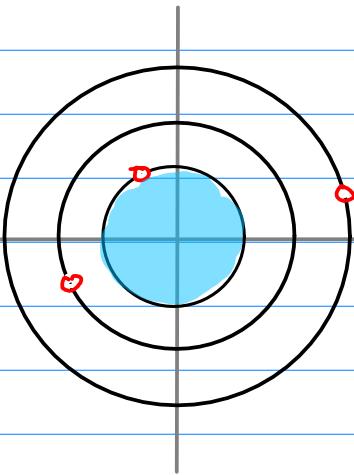
$$\operatorname{Res}\left(\frac{f(z)}{z^n}, z_2\right) + \operatorname{Res}\left(\frac{f(z)}{z^n}, z_1\right) + \operatorname{Res}\left(\frac{f(z)}{z^n}, 0\right) \quad (n > 0)$$

④



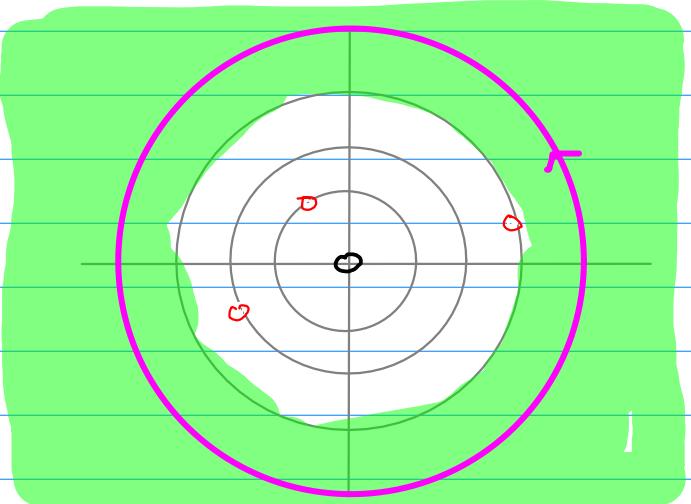
$$\operatorname{Res}\left(\frac{f(z)}{z^n}, z_3\right) + \operatorname{Res}\left(\frac{f(z)}{z^n}, z_2\right) + \operatorname{Res}\left(\frac{f(z)}{z^n}, z_1\right) + \operatorname{Res}\left(\frac{f(z)}{z^n}, 0\right) \quad (n > 0)$$

Different D, Different Laurent Series



$$x[n] = \frac{1}{2\pi i} \int_C X(z) z^{n-1} dz$$

$$= \sum_k \text{Res}(X(z) z^{n-1}, z_k)$$



z -transform

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

Complex Variables and App
Brown & Churchill

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

$$D_1 : |z| < 1$$

$$D_2 : 1 < |z| < 2$$

$$D_3 : |z| > 2$$

$$\textcircled{1} \quad D_1 \quad |z| < 1, \quad |\frac{z}{2}| < 1$$

$$\begin{aligned} f(z) &= \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{2} \frac{1}{1-\left(\frac{z}{2}\right)} \\ &= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n \quad |z| < 1 \end{aligned}$$

$$\textcircled{2} \quad D_2 \quad 1 < |z| < 2 \Rightarrow |\frac{1}{z}| < 1, \quad |\frac{z}{2}| < 1$$

$$\begin{aligned} f(z) &= \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\left(\frac{1}{z}\right)} + \frac{1}{2} \frac{1}{1-\left(\frac{z}{2}\right)} \\ &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \end{aligned}$$

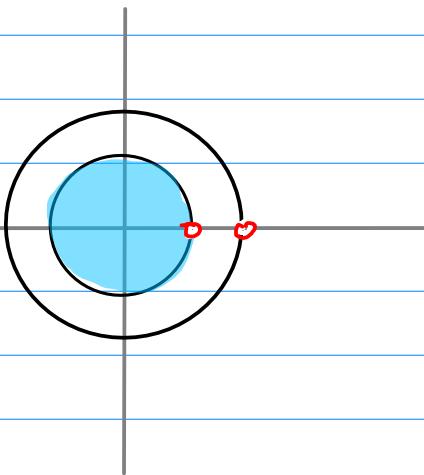
$$\textcircled{3} \quad D_3 \quad |z| > 2 \quad |\frac{2}{z}| < 1 \quad |\frac{1}{z}| < 1$$

$$\begin{aligned} f(z) &= \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z} \frac{1}{1-\left(\frac{2}{z}\right)} \\ &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{1-2^n}{z^n} \end{aligned}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

D₁ $|z| < 1$, $|z| < 1$

$$\frac{f(z)}{z^{n+1}} = \frac{-1}{(z-1)(z-2)z^{n+1}}$$

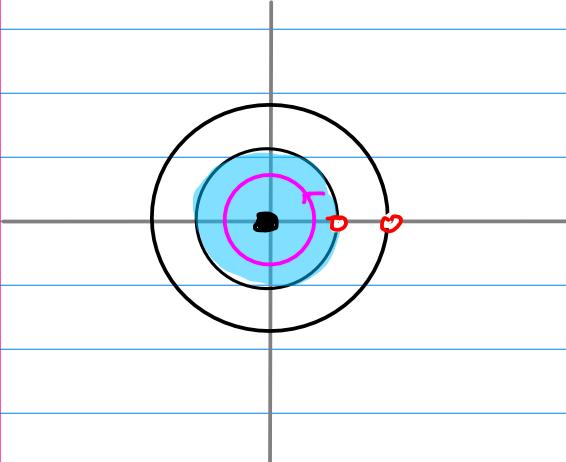


$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{2} \frac{1}{1-(\frac{z}{2})}$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} (2^{-n-1} - 1)z^n \quad |z| < 1$$

$$a_n = \frac{f(z)}{z^{n+1}} = \frac{1}{(z-1)(z-2)z^{n+1}} \quad \frac{1}{z-1} - \frac{1}{z-2}$$

$$a_n = \sum_{k=1}^M \text{Res} \left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k \right) = \text{Res} \left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0 \right)$$



$$a_n = \sum_{k=1}^{\infty} \operatorname{Res} \left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k \right) = \operatorname{Res} \left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0 \right)$$

$n \geq 0$ then the pole $z=0$

$$\frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \text{ (order } n)$$

$$\frac{d}{dz} ((z-1)^{-1} - (z-2)^{-1}) = (-1) ((z-1)^{-2} - (z-2)^{-2})$$

$$\frac{d^2}{dz^2} ((z-1)^{-1} - (z-2)^{-1}) = (-1)(-2) ((z-1)^{-3} - (z-2)^{-3})$$

$$\frac{d^3}{dz^3} ((z-1)^{-1} - (z-2)^{-1}) = (-1)(-2)(-3) ((z-1)^{-4} - (z-2)^{-4})$$

$$\frac{d^n}{dz^n} ((z-1)^{-1} - (z-2)^{-1}) = (-1)^n n! ((z-1)^{-n-1} - (z-2)^{-n-1})$$

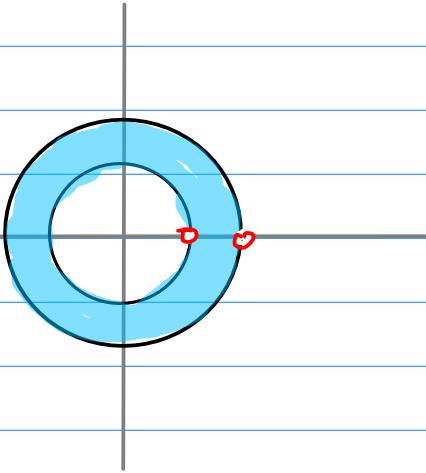
$$\begin{aligned} \frac{1}{n!} \lim_{z \rightarrow 0} \frac{d^n}{dz^n} ((z-1)^{-1} - (z-2)^{-1}) &= (-1)^n \lim_{z \rightarrow 0} ((z-1)^{-n-1} - (z-2)^{-n-1}) \\ &= (-1)^n ((-1)^{-n-1} - (-2)^{-n-1}) \\ &= -1 + 2^{-n-1} \end{aligned}$$

$$a_n = -1 + 2^{-n-1} \quad (n \geq 0)$$

$$f(z) = \sum_{n=1}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n$$

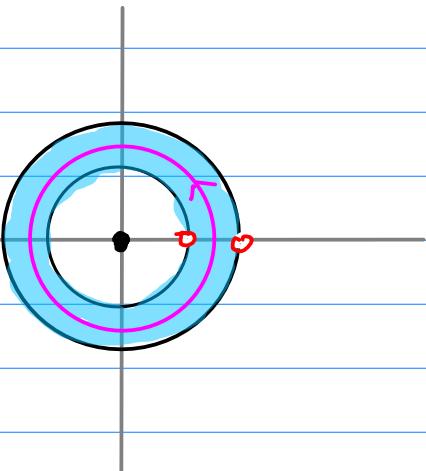
$$f(z) = \frac{-1}{(z-1)(z-2)}$$

② D_2 $1 < |z| < 2 \Rightarrow |\frac{1}{z}| < 1, \quad |\frac{z}{2}| < 1$



$$\begin{aligned} f(z) &= \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\left(\frac{1}{z}\right)} + \frac{1}{2} \frac{1}{1-\left(\frac{2}{z}\right)} \\ &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \end{aligned}$$

$$\begin{aligned} a_n &= \sum_{k=1}^M \operatorname{Res} \left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k \right) = \operatorname{Res} \left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0 \right) \\ &\quad + \operatorname{Res} \left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1 \right) \end{aligned}$$



$$a_n = \sum_{k=1}^{\infty} \operatorname{Res} \left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k \right) = \operatorname{Res} \left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0 \right) + \operatorname{Res} \left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1 \right)$$

$$\frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) \quad (\text{order } n)$$

$$\begin{aligned} \frac{1}{n!} \lim_{z \rightarrow 0} \frac{d^n}{dz^n} ((z-1)^{-1} - (z-2)^{-1}) &= (-1)^n \lim_{z \rightarrow 0} ((z-1)^{-n-1} - (z-2)^{-n-1}) \\ &= (-1)^n ((-1)^{-n-1} - (-2)^{-n-1}) \\ &= -1 + 2^{-n-1} \end{aligned}$$

$$\operatorname{Res} \left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0 \right) = -1 + 2^{-n-1} \quad (n \geq 0)$$

$$\operatorname{Res} \left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1 \right) = \lim_{z \rightarrow 1} (z-1) \frac{-1}{(z-1)(z-2)z^{n+1}} = 1$$

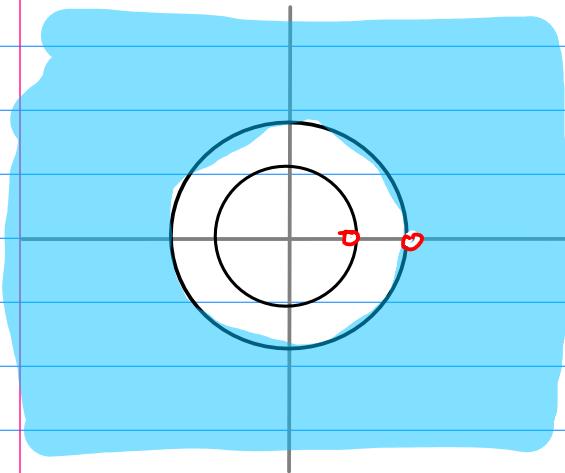
$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$\operatorname{Res} \left(\frac{f(z)}{z^{n+1}}, 0 \right)$
0	0	0	$-1 + 2^{-1}$	$-1 + 2^{-2}$	$-1 + 2^{-3}$	$\operatorname{Res} \left(\frac{f(z)}{z^{n+1}}, 0 \right)$
1	1	1	1	1	1	$\operatorname{Res} \left(\frac{f(z)}{z^{n+1}}, 1 \right)$
1	1	1	2^{-1}	2^{-2}	2^{-3}	

$$\begin{cases} a_n = 2^{-n-1} & n \geq 0 \\ a_n = 1 & n < 0 \end{cases} \quad \begin{cases} 2^{-n-1} z^n \\ z^{-n} \end{cases}$$

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

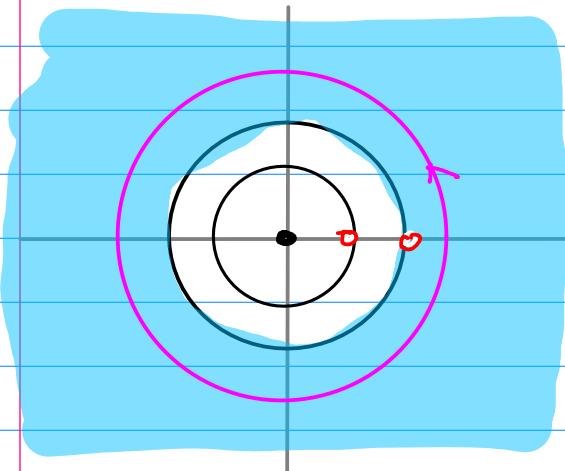
$$f(z) = \frac{-1}{(z-1)(z-2)}$$

③ $D_3 \quad 2 < |z| \quad |\frac{2}{z}| < 1 \quad |\frac{1}{z}| < 1$



$$\begin{aligned} f(z) &= \frac{1}{z-1} - \frac{1}{z-2} = \frac{\frac{1}{z}}{1-\left(\frac{1}{z}\right)} - \frac{\frac{1}{z}}{1-\left(\frac{2}{z}\right)} \\ &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n} \end{aligned}$$

$$\begin{aligned} a_n &= \sum_{k=1}^M \operatorname{Res} \left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k \right) = \operatorname{Res} \left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0 \right) \\ &\quad + \operatorname{Res} \left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1 \right) \\ &\quad + \operatorname{Res} \left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 2 \right) \end{aligned}$$



$$\text{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) = -1 + 2^{-n-1} \quad (n \geq 0)$$

$$\text{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) = \lim_{z \rightarrow 1} (z-1) \frac{-1}{(z-1)(z-2)z^{n+1}} = 1$$

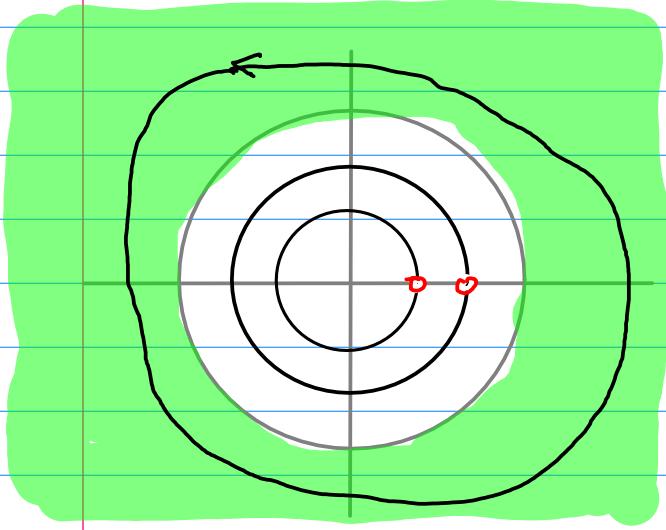
$$\text{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 2\right) = \lim_{z \rightarrow 2} (z-2) \frac{-1}{(z-1)(z-2)z^{n+1}} = -\frac{1}{2^{n+1}}$$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$\text{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$
0	0	0	$-1 + 2^{-1}$	$-1 + 2^{-2}$	$-1 + 2^{-3}$	$\text{Res}\left(\frac{f(z)}{z^{n+1}}, 1\right)$
1	1	1	1	1	1	$\text{Res}\left(\frac{f(z)}{z^{n+1}}, 2\right)$
-2^{-2}	-2	-1	-2^{-1}	-2^{-2}	-2^{-3}	
$1-2^{-2}$	$1-2$	0	0	0	0	

$$a_n = 1 - 2^{-n+1} \quad n < 0 \quad = \sum_{n=1}^{\infty} \frac{1-2^{-n}}{z^n}$$

$$f(z) = \sum_{n=-1}^{-\infty} (1-2^{-n+1}) z^n = \sum_{n=1}^{\infty} \frac{1-2^{-n-1}}{z^n}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$



$x[n]$

$$= \frac{1}{2\pi i} \int_C [X(z) z^{n-1}] dz$$

$$= \sum_{j=1}^k \text{Res} ([X(z) z^{n-1}], z_j)$$

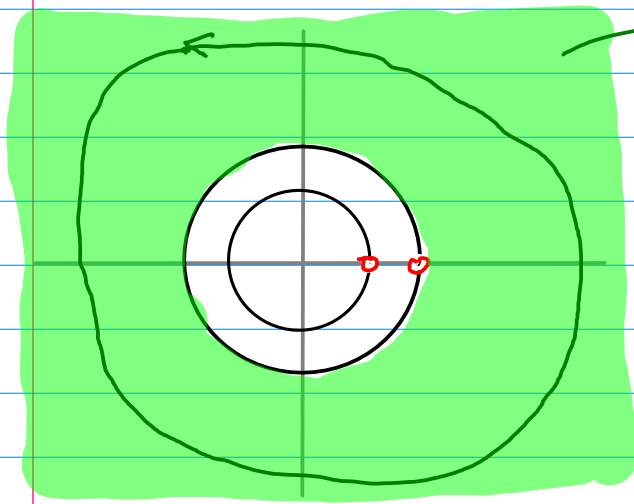
$$X(z) = \frac{-1}{(z-1)(z-2)}$$

$$X(z) z^{n-1} = \frac{-1}{(z-1)(z-2)} z^{n-1}$$

$$\text{Res} ([X(z) z^{n-1}], 1) = (z-1) \frac{-1}{(z-1)(z-2)} z^{n-1} \Big|_{z=1} = 1$$

$$\text{Res} ([X(z) z^{n-1}], 2) = (z-2) \frac{-1}{(z-1)(z-2)} z^{n-1} \Big|_{z=2} = -2^{n-1}$$

$$x[n] = 1 - 2^{n-1}$$



ROC (Region of Convergence)

$$|z| > 2 \Rightarrow \frac{2}{|z|} < 1$$

$$\left(\frac{2}{z}\right)^0 + \left(\frac{2}{z}\right)^1 + \left(\frac{2}{z}\right)^2 + \dots \xrightarrow{\text{converge}} \frac{1}{1 - \frac{2}{z}}$$

$$|z| > 2 \Rightarrow \frac{1}{|z|} < 1$$

$$\left(\frac{1}{z}\right)^0 + \left(\frac{1}{z}\right)^1 + \left(\frac{1}{z}\right)^2 + \dots \xrightarrow{\text{converge}} \frac{1}{1 - \frac{1}{z}}$$

$$\begin{aligned} f(z) &= \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z} \frac{1}{1-\left(\frac{2}{z}\right)} \\ &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{1-2^n}{z^n} \end{aligned}$$

$$\begin{aligned} \left(\frac{1}{z}\right)^0 + \left(\frac{1}{z}\right)^1 + \left(\frac{1}{z}\right)^2 + \dots \\ + \frac{1}{z} \left\{ \left(\frac{2}{z}\right)^0 + \left(\frac{2}{z}\right)^1 + \left(\frac{2}{z}\right)^2 + \dots \right\} \end{aligned} \xrightarrow{\text{converge}} \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{(z-1)(z-2)}$$

$$(1-2^0)z^0 + (1-2^1)z^1 + (1-2^2)z^2 + \dots \xrightarrow{\text{converge}} \frac{-1}{(z-1)(z-2)} \quad (|z| > 2)$$

$$x[n] = 1 - 2^n \quad \longleftrightarrow \quad X(z) = \frac{-1}{(z-1)(z-2)} \quad (|z| > 2)$$

