

# Laurent Series and z-Transform - Geometric Series Applications



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## 2 formulas

Simple Pole Form

$$\frac{1}{z - p}$$

$$\frac{1}{z^{-1} - p}$$

## 2 representations each

Geometric Series Form

$$\frac{1}{z - p} \begin{cases} \xrightarrow{\text{causal}} \frac{p^{-1}}{1 - p^{-1}z} \triangleq f(z) = \chi(z^{-1}) \\ \xrightarrow{\text{anti-causal}} \frac{z^{-1}}{1 - pz^{-1}} \triangleq \gamma(z) = g(z^{-1}) \end{cases}$$

$$\frac{1}{z^{-1} - p} \begin{cases} \xrightarrow{\text{causal}} \frac{p^{-1}}{1 - p^{-1}z^{-1}} \triangleq \chi(z) = f(z^{-1}) \\ \xrightarrow{\text{anti-causal}} \frac{z}{1 - pz} \triangleq g(z) = \gamma(z^{-1}) \end{cases}$$

Simple Pole Form

Geometric Series Form

## 2 formulas

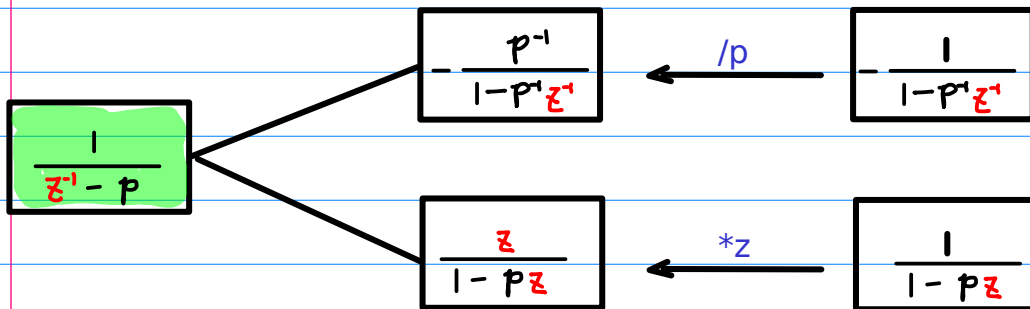
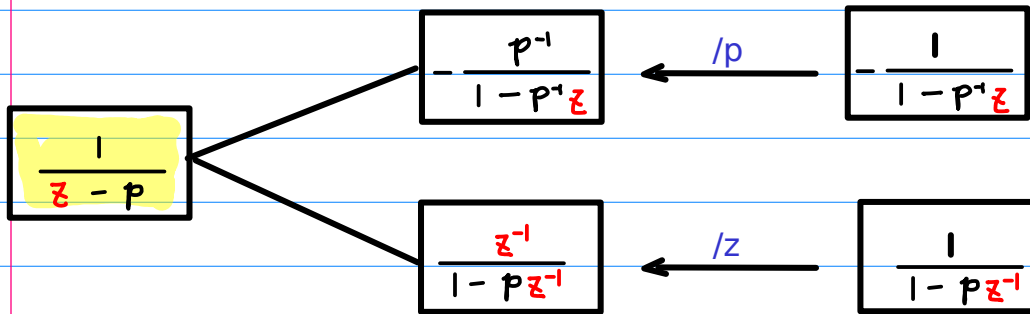
Simple Pole Form

$$\frac{1}{z - p}$$

$$\frac{1}{z^{-1} - p}$$

## 2 representations each

Geometric Series Form



Simple Pole Form

Geometric Series Form

## 2 formulas

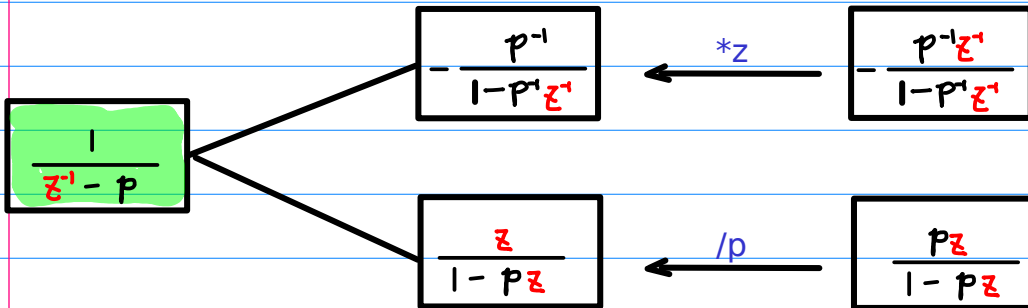
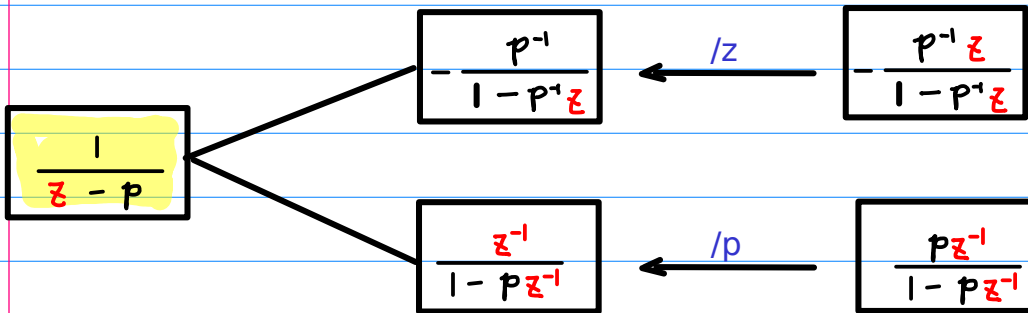
Simple Pole Form

$$\frac{1}{z - p}$$

$$\frac{1}{z^{-1} - p}$$

## 2 representations each

Geometric Series Form



Simple Pole Form

Geometric Series Form

# Geometric Series Form Combinations with a unit start term

$$-\frac{1}{1 - az}$$

$$-\frac{1}{1 - az^{-1}}$$

$$+\frac{1}{1 - a^{-1}z^{-1}}$$

$$+\frac{1}{1 - a^{-1}z}$$

$$-\frac{1}{1 - a^{-1}z}$$

$$-\frac{1}{1 - a^{-1}z^{-1}}$$

$$+\frac{1}{1 - az^{-1}}$$

$$+\frac{1}{1 - az}$$

# Geometric Series Form Combinations with a unit start term

$$-\frac{1}{1 - az}$$

$$-\frac{1}{1 - az^{-1}}$$

$$+\frac{1}{1 - a^{-1}z^{-1}}$$

$$+\frac{1}{1 - a^{-1}z}$$

$$-\frac{1}{1 - a^{-1}z}$$

$$-\frac{1}{1 - a^{-1}z^{-1}}$$

$$+\frac{1}{1 - az^{-1}}$$

$$+\frac{1}{1 - az}$$

# Geometric Series with a unit start term

## Laurent Series

$$a z$$

$$- (a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$a_n = -a^n$$

$$(n \geq 0)$$

$$|z| < a^{-1}$$

$$a z^{-1}$$

$$- (a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \dots)$$

$$- ((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^{-1} + (\frac{1}{a})^2 z^{-2} + \dots)$$

$$a_n = -(\frac{1}{a})^n$$

$$(n < 1)$$

$$|z| > a$$

$$a^{-1} z^{-1}$$

$$(a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \dots)$$

$$a_n = a^n$$

$$(n < 1)$$

$$|z| > a^{-1}$$

$$a^{-1} z$$

$$(a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^1 + (\frac{1}{a})^2 z^2 + \dots)$$

$$a_n = (\frac{1}{a})^n$$

$$(n \geq 0)$$

$$|z| < a$$

$$a^{-1} z$$

$$- (a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$- ((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^1 + (\frac{1}{a})^2 z^2 + \dots)$$

$$a_n = -(\frac{1}{a})^n$$

$$(n \geq 0)$$

$$|z| < a$$

$$a^{-1} z^{-1}$$

$$- (a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \dots)$$

$$a_n = -a^n$$

$$(n < 1)$$

$$|z| > a^{-1}$$

$$a z^{-1}$$

$$(a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \dots)$$

$$((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^{-1} + (\frac{1}{a})^2 z^{-2} + \dots)$$

$$a_n = (\frac{1}{a})^n$$

$$(n < 1)$$

$$|z| > a$$

$$a z$$

$$(a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$a_n = a^n$$

$$(n \geq 0)$$

$$|z| < a^{-1}$$

# Geometric Series with a unit start term

## z-Transform

$$a z$$

$$|z| < a^{-1}$$

$$- (a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$- ((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^1 + (\frac{1}{a})^2 z^2 + \dots)$$

$a_n = -a^{-n}$	$(n \geq 0)$
$a_n = -(\frac{1}{a})^n$	$(n < 1)$

$$a z^{-1}$$

$$|z| > a$$

$$- (a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \dots)$$

$$- ((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^{-1} + (\frac{1}{a})^2 z^{-2} + \dots)$$

$a_n = -(\frac{1}{a})^n$	$(n < 1)$
$a_n = -a^n$	$(n \geq 0)$

$$a^{-1} z^{-1}$$

$$|z| > a^{-1}$$

$$(a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \dots)$$

$$((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^{-1} + (\frac{1}{a})^2 z^{-2} + \dots)$$

$a_n = a^{-n}$	$(n < 1)$
$a_n = (\frac{1}{a})^n$	$(n \geq 0)$

$$a^{-1} z$$

$$|z| < a$$

$$(a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^1 + (\frac{1}{a})^2 z^2 + \dots)$$

$a_n = (\frac{1}{a})^{-n}$	$(n \geq 0)$
$a_n = a^n$	$(n < 1)$

$$a^{-1} z$$

$$|z| < a$$

$$- (a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$- ((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^1 + (\frac{1}{a})^2 z^2 + \dots)$$

$a_n = -(\frac{1}{a})^{-n}$	$(n \geq 0)$
$a_n = -a^n$	$(n < 1)$

$$a^{-1} z^{-1}$$

$$|z| > a^{-1}$$

$$- (a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \dots)$$

$$- ((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^{-1} + (\frac{1}{a})^2 z^{-2} + \dots)$$

$a_n = -a^{-n}$	$(n < 1)$
$a_n = -(\frac{1}{a})^n$	$(n \geq 0)$

$$a z^{-1}$$

$$|z| > a$$

$$(a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \dots)$$

$$((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^{-1} + (\frac{1}{a})^2 z^{-2} + \dots)$$

$a_n = (\frac{1}{a})^{-n}$	$(n < 1)$
$a_n = a^n$	$(n \geq 0)$

$$a z$$

$$|z| < a^{-1}$$

$$(a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^1 + (\frac{1}{a})^2 z^2 + \dots)$$

$a_n = a^{-n}$	$(n \geq 0)$
$a_n = (\frac{1}{a})^n$	$(n < 1)$



# Geometric Series with a unit start term

## Laurent Series vs. z-Transform

$$a z$$

$$|z| < a^{-1}$$

$$a z^{-1}$$

$$|z| > a$$

$$- (a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$- ((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^1 + (\frac{1}{a})^2 z^2 + \dots)$$

$$- (a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \dots)$$

$$- ((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^{-1} + (\frac{1}{a})^2 z^{-2} + \dots)$$

Laurent Series

$$a_n = -a^n \quad (n \geq 0)$$

$$(n \geq 0)$$

$$a_n = -(\frac{1}{a})^n \quad (n < 1)$$

$$(n < 1)$$

z-Transform

$$a_n = -(\frac{1}{a})^n \quad (n < 1)$$

$$(n < 1)$$

$$a_n = -a^n \quad (n \geq 0)$$

$$(n \geq 0)$$

$$a^{-1} z^{-1}$$

$$|z| > a^{-1}$$

$$a^{-1} z$$

$$|z| < a$$

$$(a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \dots)$$

$$((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^{-1} + (\frac{1}{a})^2 z^{-2} + \dots)$$

$$(a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^1 + (\frac{1}{a})^2 z^2 + \dots)$$

Laurent Series

$$a_n = a^n \quad (n < 1)$$

$$(n < 1)$$

$$a_n = (\frac{1}{a})^n \quad (n \geq 0)$$

$$(n \geq 0)$$

z-Transform

$$a_n = (\frac{1}{a})^n \quad (n \geq 0)$$

$$(n \geq 0)$$

$$a_n = a^n \quad (n < 1)$$

$$(n < 1)$$

$$a^{-1} z$$

$$|z| < a$$

$$a^{-1} z^{-1}$$

$$|z| > a^{-1}$$

$$- (a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$- ((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^1 + (\frac{1}{a})^2 z^2 + \dots)$$

$$- (a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \dots)$$

$$- ((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^{-1} + (\frac{1}{a})^2 z^{-2} + \dots)$$

Laurent Series

$$a_n = -(\frac{1}{a})^n \quad (n \geq 0)$$

$$(n \geq 0)$$

$$a_n = -a^n \quad (n < 1)$$

$$(n < 1)$$

z-Transform

$$a_n = -a^n \quad (n < 1)$$

$$(n < 1)$$

$$a_n = -(\frac{1}{a})^n \quad (n \geq 0)$$

$$(n \geq 0)$$

$$a z^{-1}$$

$$|z| > a$$

$$a z$$

$$|z| < a^{-1}$$

$$(a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \dots)$$

$$((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^{-1} + (\frac{1}{a})^2 z^{-2} + \dots)$$

$$(a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$((\frac{1}{a})^0 z^0 + (\frac{1}{a})^1 z^1 + (\frac{1}{a})^2 z^2 + \dots)$$

Laurent Series

$$a_n = (\frac{1}{a})^n \quad (n < 1)$$

$$(n < 1)$$

$$a_n = a^n \quad (n \geq 0)$$

$$(n \geq 0)$$

z-Transform

$$a_n = a^n \quad (n \geq 0)$$

$$(n \geq 0)$$

$$a_n = (\frac{1}{a})^n \quad (n < 1)$$

$$(n < 1)$$

# Geometric Series with a non-unit start term

## Laurent Series

$$a z^{-1}$$

$$|z| > a^{-1}$$

$$(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$a_n = a^n \quad (n < 0)$$

$$a z^{-1}$$

$$|z| < a$$

$$(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$((\frac{1}{a}) z^{-1} + (\frac{1}{a}) z^{-2} + (\frac{1}{a}) z^{-3} + \dots)$$

$$a_n = (\frac{1}{a})^n \quad (n \geq 1)$$

$$a z$$

$$|z| < a^{-1}$$

$$-(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$a_n = -a^n \quad (n \geq 1)$$

$$a z^{-1}$$

$$|z| > a$$

$$-(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$-((\frac{1}{a}) z^{-1} + (\frac{1}{a}) z^{-2} + (\frac{1}{a}) z^{-3} + \dots)$$

$$a_n = -(\frac{1}{a})^n \quad (n < 0)$$

$$a z^{-1}$$

$$|z| > a$$

$$(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$((\frac{1}{a}) z^{-1} + (\frac{1}{a}) z^{-2} + (\frac{1}{a}) z^{-3} + \dots)$$

$$a_n = (\frac{1}{a})^n \quad (n < 0)$$

$$a z$$

$$|z| < a^{-1}$$

$$(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$a_n = a^n \quad (n \geq 1)$$

$$a z^{-1}$$

$$|z| < a$$

$$-(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$-((\frac{1}{a}) z^{-1} + (\frac{1}{a}) z^{-2} + (\frac{1}{a}) z^{-3} + \dots)$$

$$a_n = -(\frac{1}{a})^n \quad (n \geq 1)$$

$$a z^{-1}$$

$$|z| > a^{-1}$$

$$-(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$a_n = -a^n \quad (n < 0)$$

# Geometric Series with a non-unit start term

## z-Transform

$$a^{-1} z^{-1}$$

$$|z| > a^{-1}$$

$$(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$((\frac{1}{a})^{-1} z^{-1} + (\frac{1}{a})^{-2} z^{-2} + (\frac{1}{a})^{-3} z^{-3} + \dots)$$

$a_n = a^{-n}$	$(-n < 0)$
$a_n = (\frac{1}{a})^n$	$(n \geq 1)$

$$a^{-1} z$$

$$|z| < a$$

$$(a^{-1} z^1 + a^{-2} z^2 + a^{-3} z^3 + \dots)$$

$$((\frac{1}{a})^{-1} z^1 + (\frac{1}{a})^{-2} z^2 + (\frac{1}{a})^{-3} z^3 + \dots)$$

$a_n = (\frac{1}{a})^{-n}$	$(-n \geq 1)$
$a_n = a^n$	$(n < 0)$

$$a z$$

$$|z| < a^{-1}$$

$$-(a^1 z^1 + a^2 z^2 + a^3 z^3 + \dots)$$

$$-((\frac{1}{a})^1 z^1 + (\frac{1}{a})^2 z^2 + (\frac{1}{a})^3 z^3 + \dots)$$

$a_n = -a^n$	$(-n \geq 1)$
$a_n = -(\frac{1}{a})^n$	$(n < 0)$

$$a z^{-1}$$

$$|z| > a$$

$$-(a^1 z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \dots)$$

$$-((\frac{1}{a})^1 z^{-1} + (\frac{1}{a})^2 z^{-2} + (\frac{1}{a})^3 z^{-3} + \dots)$$

$a_n = -(\frac{1}{a})^{-n}$	$(-n < 0)$
$a_n = -a^n$	$(n \geq 1)$

$$a z^{-1}$$

$$|z| > a$$

$$(a^1 z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \dots)$$

$$((\frac{1}{a})^1 z^{-1} + (\frac{1}{a})^2 z^{-2} + (\frac{1}{a})^3 z^{-3} + \dots)$$

$a_n = (\frac{1}{a})^{-n}$	$(-n < 0)$
$a_n = a^n$	$(n \geq 1)$

$$a z$$

$$|z| < a^{-1}$$

$$(a^1 z^1 + a^2 z^2 + a^3 z^3 + \dots)$$

$$((\frac{1}{a})^1 z^1 + (\frac{1}{a})^2 z^2 + (\frac{1}{a})^3 z^3 + \dots)$$

$a_n = a^n$	$(-n \geq 1)$
$a_n = (\frac{1}{a})^n$	$(n < 0)$

$$a^{-1} z$$

$$|z| < a$$

$$-(a^{-1} z^1 + a^{-2} z^2 + a^{-3} z^3 + \dots)$$

$$-((\frac{1}{a})^{-1} z^1 + (\frac{1}{a})^{-2} z^2 + (\frac{1}{a})^{-3} z^3 + \dots)$$

$a_n = -(\frac{1}{a})^{-n}$	$(-n \geq 1)$
$a_n = -a^n$	$(n < 0)$

$$a^{-1} z^{-1}$$

$$|z| > a^{-1}$$

$$-(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$-((\frac{1}{a})^{-1} z^{-1} + (\frac{1}{a})^{-2} z^{-2} + (\frac{1}{a})^{-3} z^{-3} + \dots)$$

$a_n = -a^{-n}$	$(-n < 0)$
$a_n = -(\frac{1}{a})^n$	$(n \geq 1)$

# Geometric Series with a non-unit start term

## Laurent Series vs. z-Transform

$a z^{-1}$	$ z  > a^{-1}$	$a z^{-1}$	$ z  < a$
------------	----------------	------------	-----------

$$(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$((\frac{1}{a})^{-1} z^{-1} + (\frac{1}{a})^{-2} z^{-2} + (\frac{1}{a})^{-3} z^{-3} + \dots)$$

$$(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$((\frac{1}{a})^{-1} z^{-1} + (\frac{1}{a})^{-2} z^{-2} + (\frac{1}{a})^{-3} z^{-3} + \dots)$$

Laurent Series

$$a_n = a^n \quad (n < 0)$$

$$(n < 0)$$

$$a_n = (\frac{1}{a})^n \quad (n \geq 1)$$

$$(n \geq 1)$$

z-Transform

$$a_n = (\frac{1}{a})^n \quad (n \geq 1)$$

$$(n \geq 1)$$

$$a_n = a^n \quad (n < 0)$$

$$(n < 0)$$

$a z$	$ z  < a^{-1}$	$a z^{-1}$	$ z  > a$
-------	----------------	------------	-----------

$$-(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$-((\frac{1}{a})^{-1} z^{-1} + (\frac{1}{a})^{-2} z^{-2} + (\frac{1}{a})^{-3} z^{-3} + \dots)$$

$$-(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$-((\frac{1}{a})^{-1} z^{-1} + (\frac{1}{a})^{-2} z^{-2} + (\frac{1}{a})^{-3} z^{-3} + \dots)$$

Laurent Series

$$a_n = -a^n \quad (n \geq 1)$$

$$(n \geq 1)$$

$$a_n = -(\frac{1}{a})^n \quad (n < 0)$$

$$(n < 0)$$

z-Transform

$$a_n = -(\frac{1}{a})^n \quad (n < 0)$$

$$(n < 0)$$

$$a_n = -a^n \quad (n \geq 1)$$

$$(n \geq 1)$$

$a z^{-1}$	$ z  > a$	$a z$	$ z  < a^{-1}$
------------	-----------	-------	----------------

$$(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$((\frac{1}{a})^{-1} z^{-1} + (\frac{1}{a})^{-2} z^{-2} + (\frac{1}{a})^{-3} z^{-3} + \dots)$$

$$(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$((\frac{1}{a})^{-1} z^{-1} + (\frac{1}{a})^{-2} z^{-2} + (\frac{1}{a})^{-3} z^{-3} + \dots)$$

Laurent Series

$$a_n = (\frac{1}{a})^n \quad (n < 0)$$

$$(n < 0)$$

$$a_n = a^n \quad (n \geq 1)$$

$$(n \geq 1)$$

z-Transform

$$a_n = a^n \quad (n \geq 1)$$

$$(n \geq 1)$$

$$a_n = (\frac{1}{a})^n \quad (n < 0)$$

$$(n < 0)$$

$a z^{-1}$	$ z  < a$	$a z^{-1}$	$ z  > a^{-1}$
------------	-----------	------------	----------------

$$-(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$-((\frac{1}{a})^{-1} z^{-1} + (\frac{1}{a})^{-2} z^{-2} + (\frac{1}{a})^{-3} z^{-3} + \dots)$$

$$-(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$-((\frac{1}{a})^{-1} z^{-1} + (\frac{1}{a})^{-2} z^{-2} + (\frac{1}{a})^{-3} z^{-3} + \dots)$$

Laurent Series

$$a_n = -(\frac{1}{a})^n \quad (n \geq 1)$$

$$(n \geq 1)$$

$$a_n = -a^n \quad (n < 0)$$

$$(n < 0)$$

z-Transform

$$a_n = -a^n \quad (n < 0)$$

$$(n < 0)$$

$$a_n = -(\frac{1}{a})^n \quad (n \geq 1)$$

$$(n \geq 1)$$

# Complement ROC Pairs - Original Geometric Series Form Combinations

unit	$-\frac{1}{1-az} \quad  z  < a^{-1}$	$-a^n \quad (n \geq 0)$	$-\frac{1}{1-a^{-1}z^{-1}} \quad  z  > a$	$-(\frac{1}{a})^n \quad (n < 1)$
non-unit	$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$a^n \quad (n < 0)$	$\frac{a^{-1}z}{1-a^{-1}z} \quad  z  < a$	$(\frac{1}{a})^n \quad (n \geq 1)$

unit	$\frac{1}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$a^n \quad (n < 1)$	$\frac{1}{1-a^{-1}z} \quad  z  < a$	$(\frac{1}{a})^n \quad (n \geq 0)$
non-unit	$-\frac{az}{1-az} \quad  z  < a^{-1}$	$-a^n \quad (n \geq 1)$	$\frac{az^{-1}}{1-az^{-1}} \quad  z  > a$	$-(\frac{1}{a})^n \quad (n < 0)$

unit	$-\frac{1}{1-a^{-1}z} \quad  z  < a$	$-(\frac{1}{a})^n \quad (n \geq 0)$	$-\frac{1}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$-a^n \quad (n < 1)$
non-unit	$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a$	$(\frac{1}{a})^n \quad (n < 0)$	$\frac{az}{1-az} \quad  z  < a^{-1}$	$a^n \quad (n \geq 1)$

unit	$\frac{1}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$(\frac{1}{a})^n \quad (n < 1)$	$\frac{1}{1-az} \quad  z  < a^{-1}$	$a^n \quad (n \geq 0)$
non-unit	$-\frac{az}{1-az} \quad  z  < a$	$-(\frac{1}{a})^n \quad (n \geq 1)$	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$-a^n \quad (n < 0)$

start term

# Complement ROC Pairs - Shifted Geometric Series Form Combinations

$-\frac{a}{1-az} \quad  z  < a^{-1}$	$-a^{n+1} \quad (n \geq 0)$	$-\frac{a}{1-az^{-1}} \quad  z  > a$	$-\left(\frac{1}{a}\right)^{n-1} \quad (n < 1)$
$\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$a^{n+1} \quad (n < 0)$	$\frac{z}{1-a^{-1}z} \quad  z  < a$	$\left(\frac{1}{a}\right)^{n-1} \quad (n \geq 1)$

$\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$a^{n+1} \quad (n < 0)$	$\frac{z}{1-a^{-1}z} \quad  z  < a$	$\left(\frac{1}{a}\right)^{n-1} \quad (n \geq 1)$
$-\frac{a}{1-az} \quad  z  < a^{-1}$	$-a^{n+1} \quad (n \geq 0)$	$\frac{a}{1-az^{-1}} \quad  z  > a$	$-\left(\frac{1}{a}\right)^{n-1} \quad (n < 1)$

$-\frac{a^{-1}}{1-a^{-1}z} \quad  z  < a$	$-\left(\frac{1}{a}\right)^{n+1} \quad (n \geq 0)$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$-a^{n-1} \quad (n < 1)$
$\frac{z^{-1}}{1-az^{-1}} \quad  z  > a$	$\left(\frac{1}{a}\right)^{n+1} \quad (n < 0)$	$\frac{z}{1-az} \quad  z  < a^{-1}$	$a^{n-1} \quad (n \geq 1)$

$\frac{z^{-1}}{1-az^{-1}} \quad  z  > a$	$\left(\frac{1}{a}\right)^{n+1} \quad (n < 0)$	$\frac{z}{1-az} \quad  z  < a^{-1}$	$a^{n-1} \quad (n \geq 1)$
$-\frac{a^{-1}}{1-a^{-1}z} \quad  z  < a$	$-\left(\frac{1}{a}\right)^{n+1} \quad (n \geq 0)$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$-a^{n-1} \quad (n < 1)$

# Complement ROC Pairs - Reduced Shifted Geometric Series Form Combinations

$-\frac{a}{1-az}$ $ z  < a^{-1}$	$-a^{n+1}$ ( $n \geq 0$ )	$-\frac{a}{1-az^{-1}}$ $ z  > a$	$-(\frac{1}{a})^{n-1}$ ( $n < 1$ )
$\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $ z  > a^{-1}$	$a^{n+1}$ ( $n < 0$ )	$\frac{z}{1-a^{-1}z}$ $ z  < a$	$(\frac{1}{a})^{n-1}$ ( $n \geq 1$ )

$-\frac{a^{-1}}{1-a^{-1}z}$ $ z  < a$	$-(\frac{1}{a})^{n+1}$ ( $n \geq 0$ )	$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $ z  > a^{-1}$	$-a^{n-1}$ ( $n < 1$ )
$\frac{z^{-1}}{1-az^{-1}}$ $ z  > a$	$(\frac{1}{a})^{n+1}$ ( $n < 0$ )	$\frac{z}{1-az}$ $ z  < a^{-1}$	$a^{n-1}$ ( $n \geq 1$ )

$$2z$$

$$2z^{-1}$$

$$2^{-1}z^{-1}$$

$$2^{-1}z$$

$$|z| < 0.5$$

$$|z| > 2$$

$$|z| > 0.5$$

$$|z| < 2$$

$$- \frac{2}{1-2z} \xleftrightarrow{z^{-1}} - \frac{2}{1-2z^{-1}}$$

$$\cdot \frac{(2z)^{-1}}{(2z)^{-1}} \cdot \frac{(2z)}{(2z)}$$

$$\cdot \frac{(2z^{-1})^{-1}}{(2z^{-1})^{-1}} \cdot \frac{(2z^{-1})}{(2z^{-1})}$$

$$+ \frac{z^{-1}}{1-0.5z^{-1}} \xleftrightarrow{z^{-1}} + \frac{z}{1-0.5z}$$



scale(a)

scale(a)

	$-\frac{1}{1-a^2z} \quad  z  < a^{-1}$	$-\frac{a}{1-a^2z} \quad  z  < a^{-1}$	$-\frac{1}{1-a^2z^{-1}} \quad  z  > a$	$-\frac{a}{1-a^2z^{-1}} \quad  z  > a$
Comp.ROC	$\frac{a^2z^{-1}}{1-a^2z^{-1}} \quad  z  > a^{-1}$	$\frac{z^{-1}}{1-a^2z^{-1}} \quad  z  > a^{-1}$	$\frac{a^2z}{1-a^2z} \quad  z  < a$	$\frac{z}{1-a^2z} \quad  z  < a$

scale(1/z)

scale(z)

	$\frac{1}{1-a^2z^{-1}} \quad  z  > a^{-1}$	$\frac{z^{-1}}{1-a^2z^{-1}} \quad  z  > a^{-1}$	$\frac{1}{1-a^2z} \quad  z  < a$	$\frac{z}{1-a^2z} \quad  z  < a$
Comp.ROC	$-\frac{az}{1-az} \quad  z  < a^{-1}$	$-\frac{a}{1-az} \quad  z  < a^{-1}$	$\frac{az^{-1}}{1-az^{-1}} \quad  z  > a$	$\frac{a}{1-az^{-1}} \quad  z  > a$

scale(1/a)

scale(1/a)

	$-\frac{1}{1-a^2z} \quad  z  < a$	$-\frac{a^{-1}}{1-a^2z} \quad  z  < a$	$-\frac{1}{1-a^2z^{-1}} \quad  z  > a^{-1}$	$-\frac{a^{-1}}{1-a^2z^{-1}} \quad  z  > a^{-1}$
Comp.ROC	$\frac{az^{-1}}{1-az^{-1}} \quad  z  > a$	$\frac{z^{-1}}{1-az^{-1}} \quad  z  > a$	$\frac{az}{1-az} \quad  z  < a^{-1}$	$\frac{z}{1-az} \quad  z  < a^{-1}$

scale(1/z)

scale(z)

	$\frac{1}{1-a^2z^{-1}} \quad  z  > a$	$\frac{z^{-1}}{1-a^2z^{-1}} \quad  z  > a$	$\frac{1}{1-a^2z} \quad  z  < a^{-1}$	$\frac{z}{1-a^2z} \quad  z  < a^{-1}$
Comp.ROC	$-\frac{a^2z}{1-a^2z} \quad  z  < a$	$-\frac{a^2}{1-a^2z} \quad  z  < a$	$-\frac{a^2z^{-1}}{1-a^2z^{-1}} \quad  z  > a^{-1}$	$-\frac{a^2}{1-a^2z^{-1}} \quad  z  > a^{-1}$

scale(a)

scale(a)

Comp.ROC

$-\frac{1}{1-az} \quad  z  < a^{-1}$	$-\frac{a}{1-az} \quad  z  < a^{-1}$	$-\frac{1}{1-a^{-1}z^{-1}} \quad  z  > a$	$-\frac{a}{1-a^{-1}z^{-1}} \quad  z  > a$
$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$\frac{a^{-1}z}{1-a^{-1}z} \quad  z  < a$	$\frac{z}{1-a^{-1}z} \quad  z  < a$

scale(1/z)

scale(z)

Comp.ROC

$\frac{1}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$\frac{1}{1-a^{-1}z} \quad  z  < a$	$\frac{z}{1-a^{-1}z} \quad  z  < a$
$-\frac{az}{1-az} \quad  z  < a^{-1}$	$-\frac{a}{1-az} \quad  z  < a^{-1}$	$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a$	$\frac{a}{1-a^{-1}z^{-1}} \quad  z  > a$

scale(1/a)

scale(1/a)

Comp.ROC

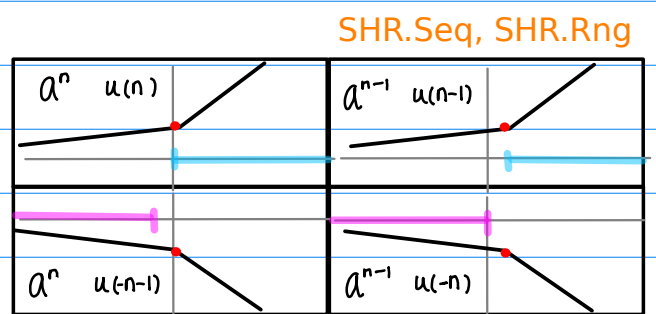
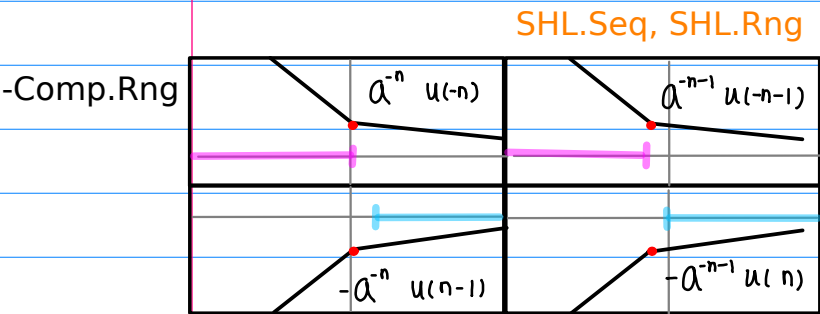
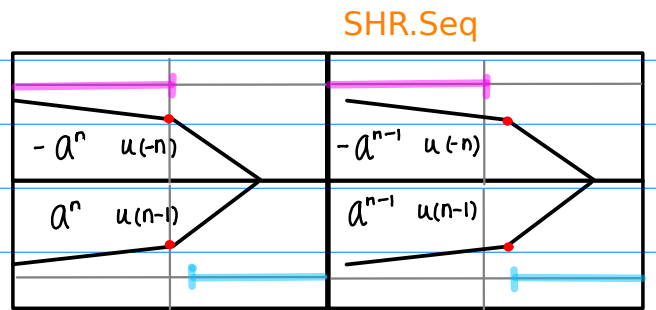
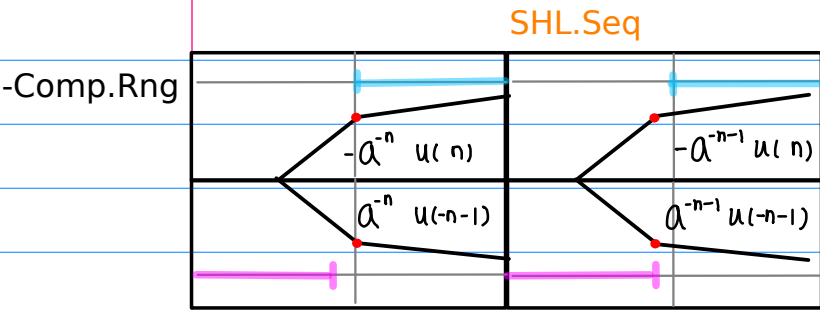
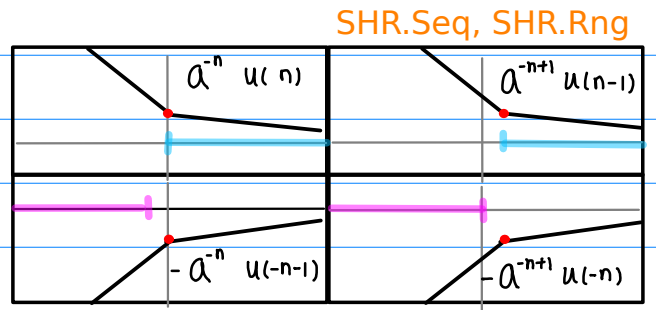
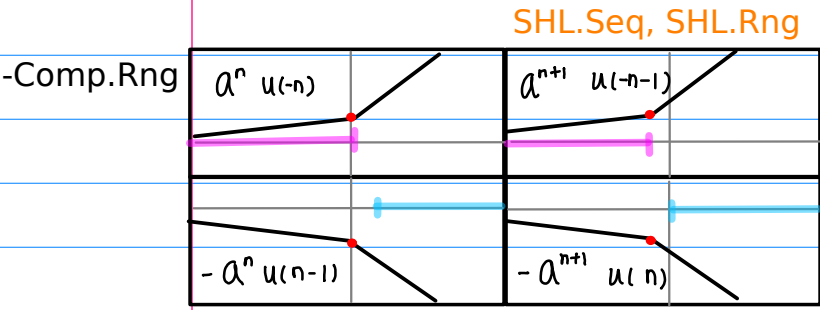
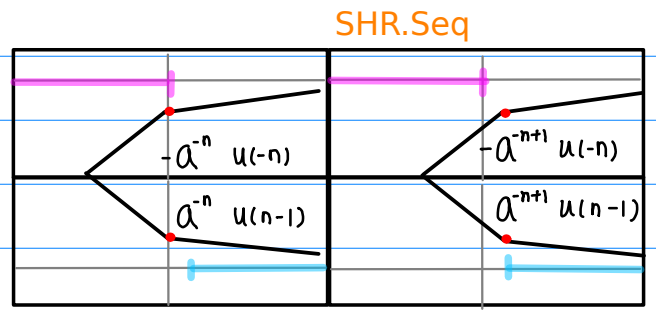
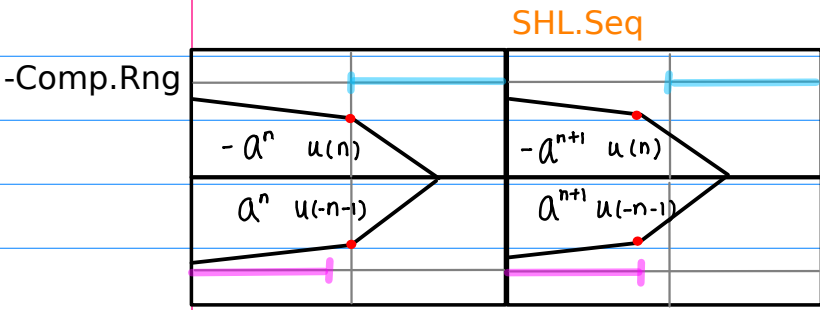
$-\frac{1}{1-a^{-1}z} \quad  z  < a$	$-\frac{a^{-1}}{1-a^{-1}z} \quad  z  < a$	$-\frac{1}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$
$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a$	$\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a$	$\frac{a^{-1}z}{1-a^{-1}z} \quad  z  < a^{-1}$	$\frac{z}{1-a^{-1}z} \quad  z  < a^{-1}$

scale(1/z)

scale(z)

Comp.ROC

$\frac{1}{1-a^{-1}z^{-1}} \quad  z  > a$	$\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a$	$\frac{1}{1-a^{-1}z} \quad  z  < a^{-1}$	$\frac{z}{1-a^{-1}z} \quad  z  < a^{-1}$
$-\frac{a^{-1}z}{1-a^{-1}z} \quad  z  < a$	$-\frac{a^{-1}}{1-a^{-1}z} \quad  z  < a$	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$



Left Shifted  
Sequence

Right Shifted  
Sequence

scale(a)

scale(a)

	$-\frac{1}{1-az} \quad  z  < a^{-1}$	$-\frac{a}{1-az} \quad  z  < a^{-1}$	$-\frac{1}{1-a^{-1}z^{-1}} \quad  z  > a$	$-\frac{a}{1-a^{-1}z^{-1}} \quad  z  > a$
Comp.ROC	$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$\frac{a^{-1}z}{1-a^{-1}z} \quad  z  < a$	$\frac{z}{1-a^{-1}z} \quad  z  < a$

scale(1/z)

scale(z)

	$\frac{1}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$\frac{1}{1-a^{-1}z} \quad  z  < a$	$\frac{z}{1-a^{-1}z} \quad  z  < a$
Comp.ROC	$-\frac{az}{1-az} \quad  z  < a^{-1}$	$-\frac{a}{1-az} \quad  z  < a^{-1}$	$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a$	$\frac{a}{1-a^{-1}z^{-1}} \quad  z  > a$

scale(1/a)

scale(1/a)

	$-\frac{1}{1-a^{-1}z} \quad  z  < a$	$-\frac{a^{-1}}{1-a^{-1}z} \quad  z  < a$	$-\frac{1}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$
Comp.ROC	$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a$	$\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a$	$\frac{az}{1-az} \quad  z  < a^{-1}$	$\frac{z}{1-az} \quad  z  < a^{-1}$

scale(1/z)

scale(z)

	$\frac{1}{1-a^{-1}z^{-1}} \quad  z  > a$	$\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a$	$\frac{1}{1-az} \quad  z  < a^{-1}$	$\frac{z}{1-az} \quad  z  < a^{-1}$
Comp.ROC	$-\frac{a^{-1}z}{1-a^{-1}z} \quad  z  < a$	$-\frac{a^{-1}}{1-a^{-1}z} \quad  z  < a$	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$

## SHL.Seq

$-a^n$ ( $n \geq 0$ ) $-(a^0, a^1, a^2, \dots)$	$-a^{n+1}$ ( $n \geq 0$ ) $-(a^1, a^2, a^3, \dots)$
$a^n$ ( $n < 0$ ) $(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$	$a^{n+1}$ ( $n < 0$ ) $(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$

## SHR.Seq

$-\left(\frac{1}{a}\right)^n$ ( $n < 1$ ) $-(\dots, a^0, a^1, a^2)$	$-\left(\frac{1}{a}\right)^{n-1}$ ( $n < 1$ ) $-(\dots, a^3, a^2, a^1)$
$\left(\frac{1}{a}\right)^n$ ( $n \geq 1$ ) $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$	$\left(\frac{1}{a}\right)^{n-1}$ ( $n \geq 1$ ) $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$

-Comp.Rng

## SHL.Seq, SHL.Rng

$a^n$ ( $n < 1$ ) $(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$	$a^{n+1}$ ( $n < 0$ ) $(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$
$-a^n$ ( $n \geq 1$ ) $-(a^1, a^2, a^3, \dots)$	$-a^{n+1}$ ( $n \geq 0$ ) $-(a^1, a^2, a^3, \dots)$

## SHR.Seq, SHR.Rng

$\left(\frac{1}{a}\right)^n$ ( $n \geq 0$ ) $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$	$\left(\frac{1}{a}\right)^{n-1}$ ( $n \geq 1$ ) $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$
$-\left(\frac{1}{a}\right)^n$ ( $n < 0$ ) $(\dots, a^3, a^2, a^1)$	$-\left(\frac{1}{a}\right)^{n-1}$ ( $n < 1$ ) $(\dots, a^3, a^2, a^1)$

-Comp.Rng

## SHL.Seq

$-\left(\frac{1}{a}\right)^n$ ( $n \geq 0$ ) $-(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$	$-\left(\frac{1}{a}\right)^{n+1}$ ( $n \geq 0$ ) $-(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$
$\left(\frac{1}{a}\right)^n$ ( $n < 0$ ) $(\dots, a^3, a^2, a^1)$	$\left(\frac{1}{a}\right)^{n+1}$ ( $n < 0$ ) $(\dots, a^2, a^1, a^0)$

## SHR.Seq

$-a^n$ ( $n < 1$ ) $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$	$-a^{n-1}$ ( $n < 1$ ) $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$
$a^n$ ( $n \geq 1$ ) $(a^1, a^2, a^3, \dots)$	$a^{n-1}$ ( $n \geq 1$ ) $(a^0, a^1, a^2, \dots)$

-Comp.Rng

## SHL.Seq, SHL.Rng

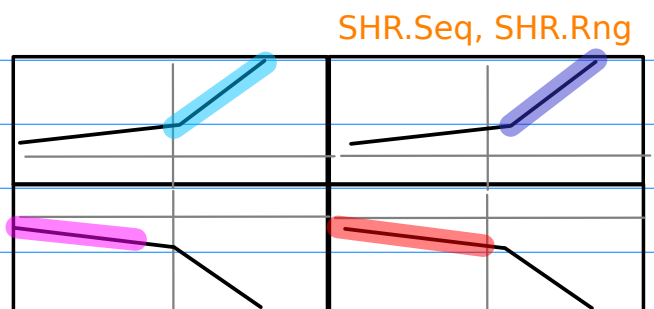
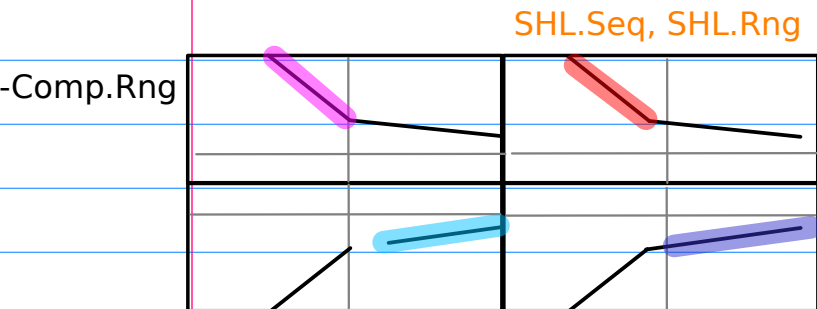
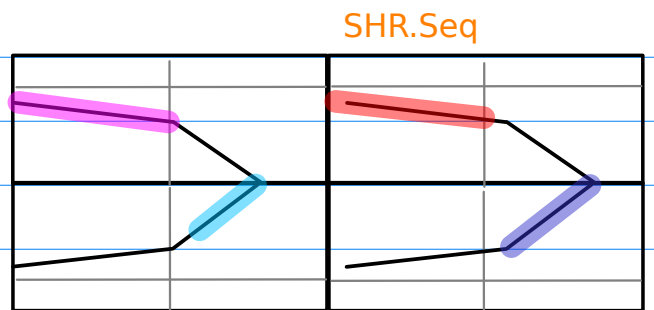
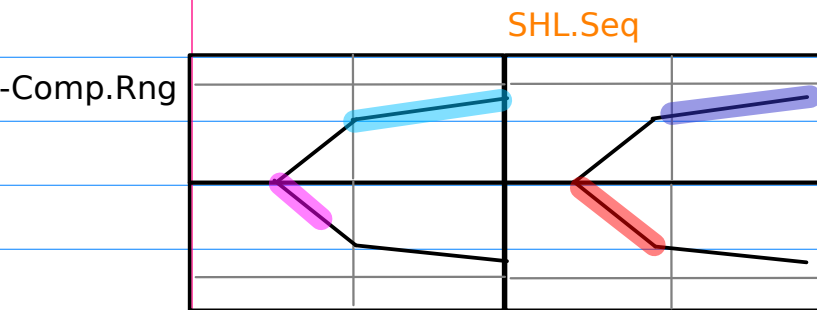
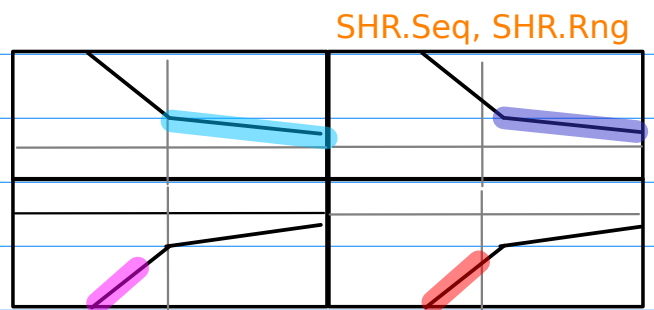
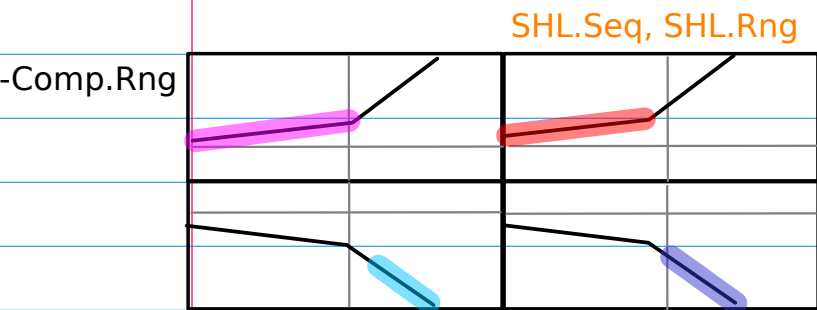
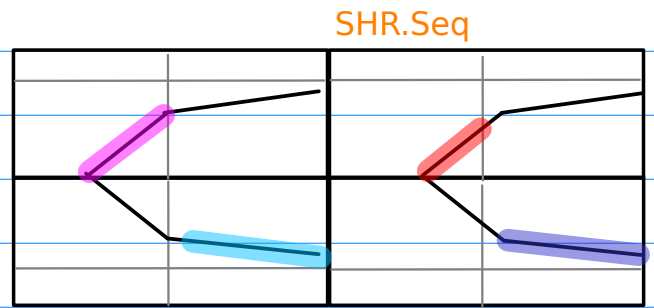
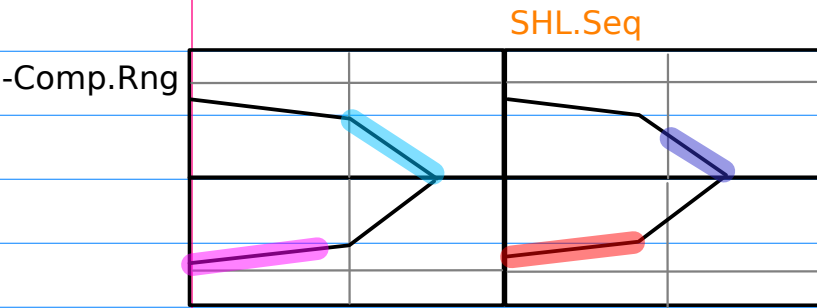
$\left(\frac{1}{a}\right)^n$ ( $n < 1$ ) $(\dots, a^2, a^1, a^0)$	$\left(\frac{1}{a}\right)^{n+1}$ ( $n < 0$ ) $(\dots, a^2, a^1, a^0)$
$-\left(\frac{1}{a}\right)^n$ ( $n \geq 1$ ) $-(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$	$-\left(\frac{1}{a}\right)^{n+1}$ ( $n \geq 0$ ) $-(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$

## SHR.Seq, SHR.Rng

$a^n$ ( $n \geq 0$ ) $(a^0, a^1, a^2, \dots)$	$a^{n-1}$ ( $n \geq 1$ ) $(a^0, a^1, a^2, \dots)$
$-a^n$ ( $n < 0$ ) $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$	$-a^{n-1}$ ( $n < 1$ ) $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$

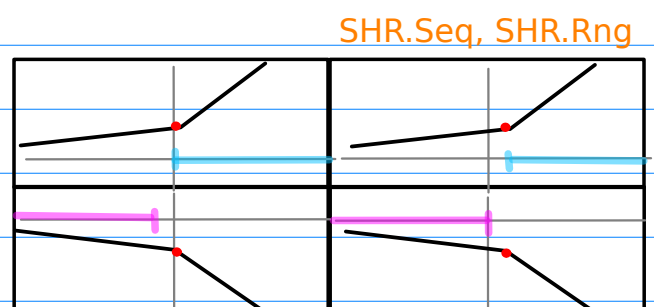
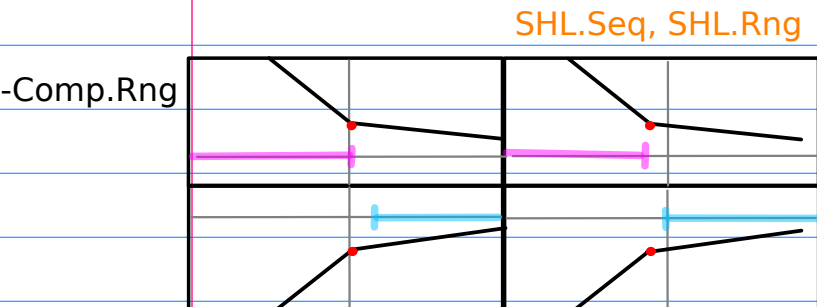
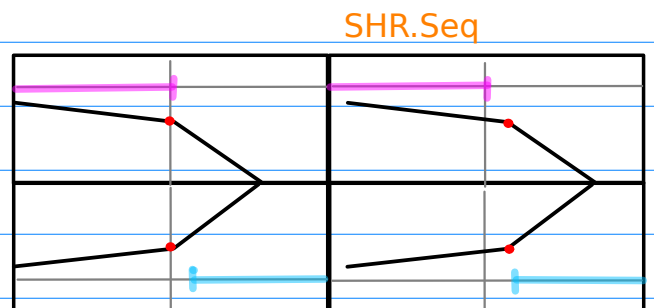
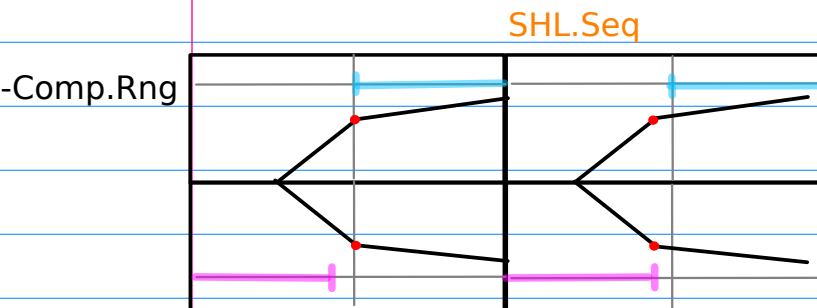
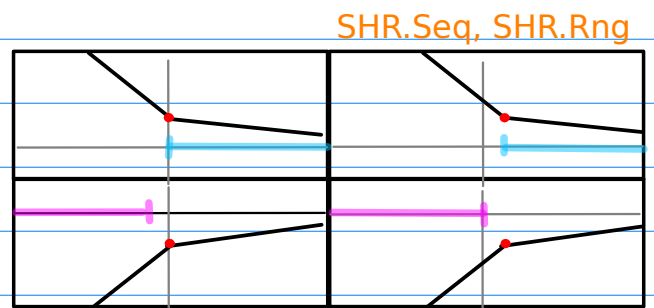
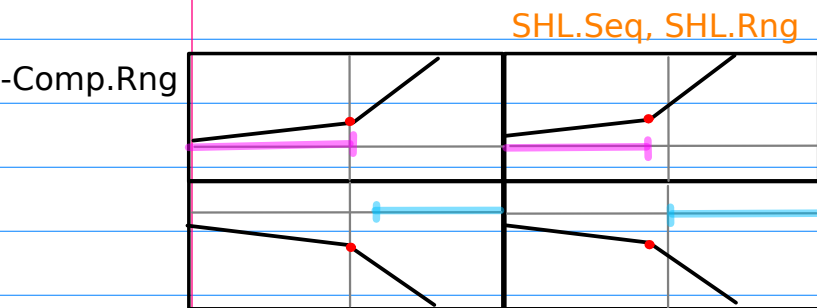
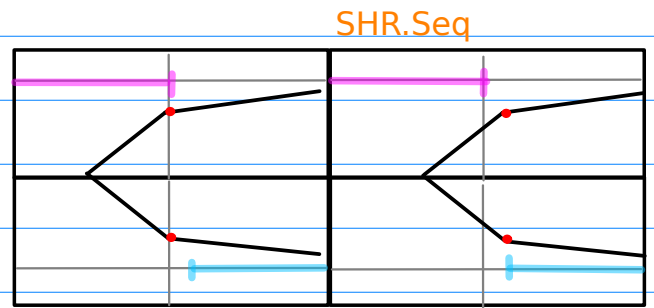
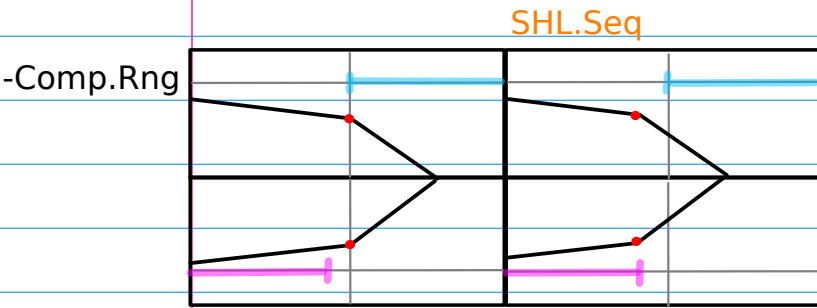
-Comp.Rng

Left Shifted  
SequenceRight Shifted  
Sequence



Left Shifted  
Sequence

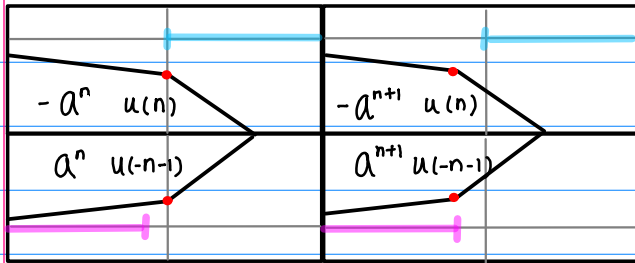
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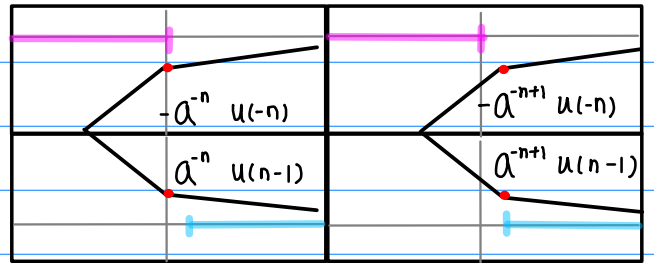
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Sequence

Right Shifted  
Sequence

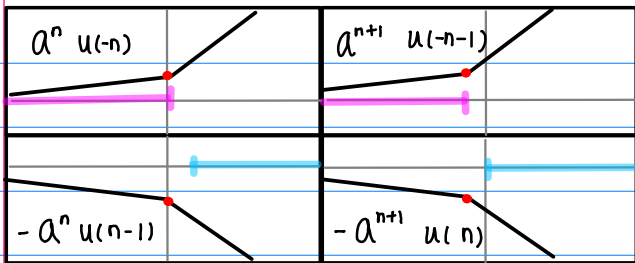
SHL.Seq



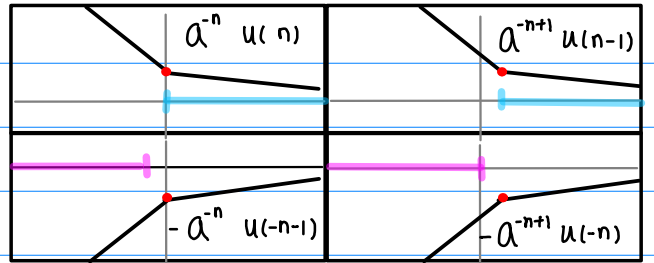
SHR.Seq



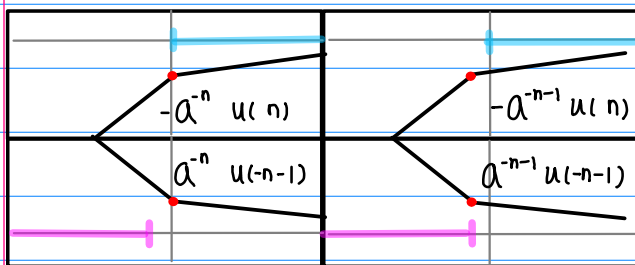
SHL.Seq, SHL.Rng



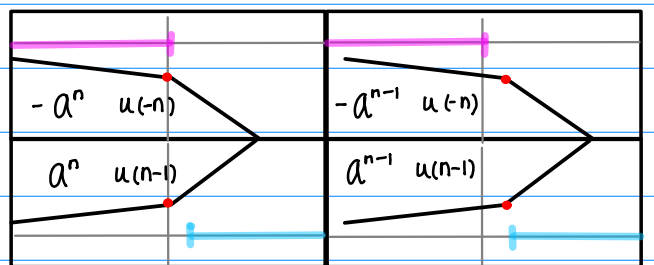
SHR.Seq, SHR.Rng



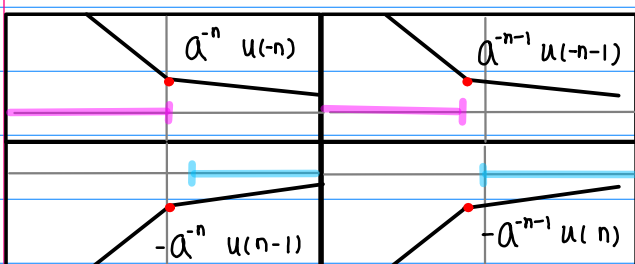
SHL.Seq



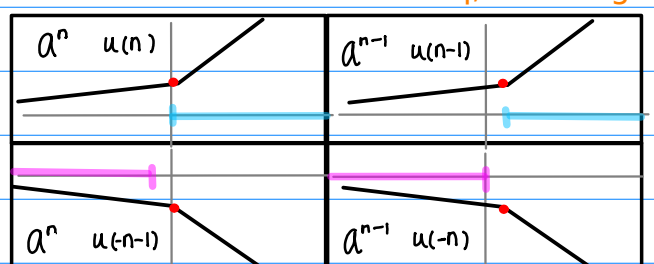
SHR.Seq



SHL.Seq, SHL.Rng



SHR.Seq, SHR.Rng

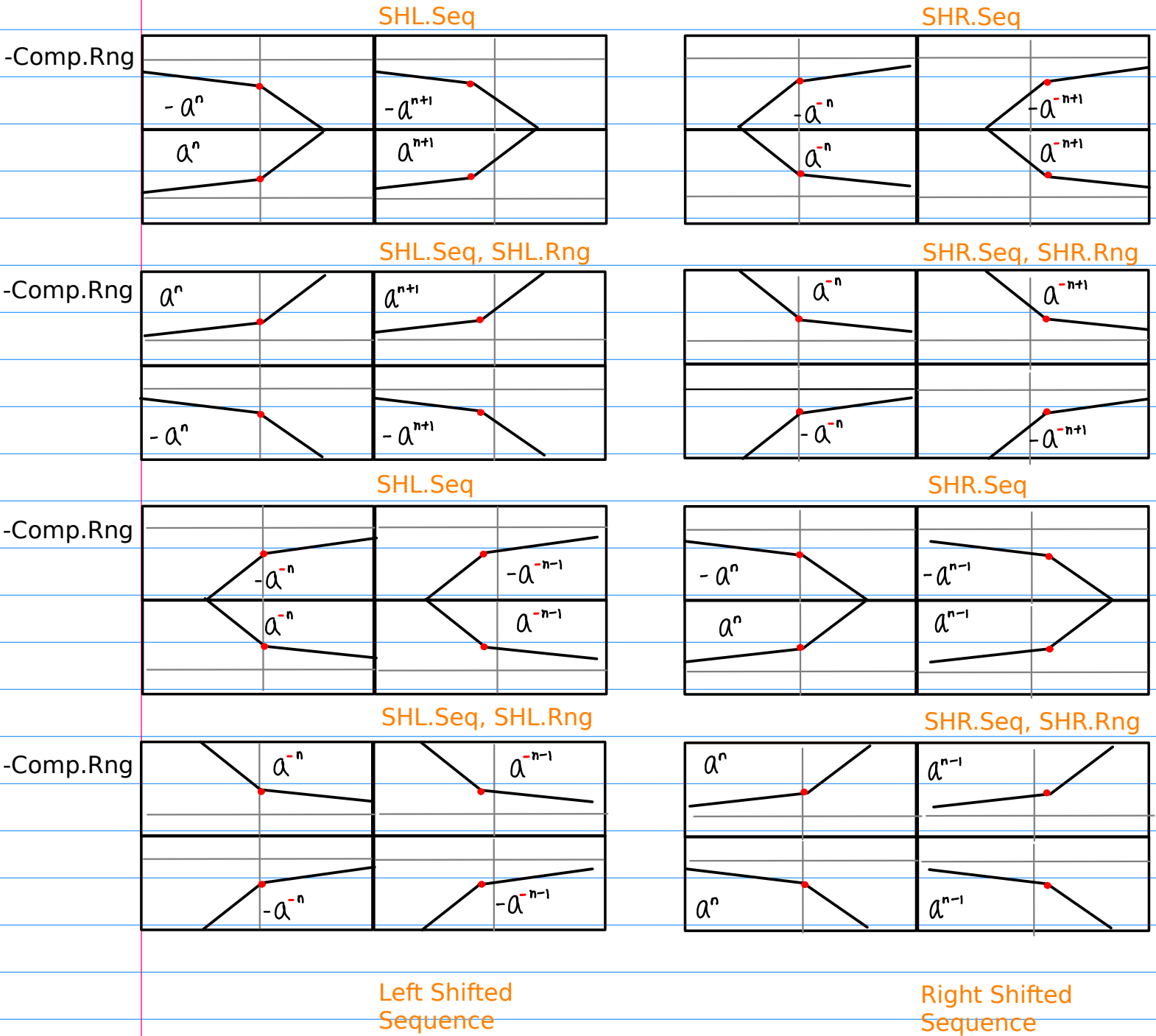


Left Shifted Sequence

Right Shifted Sequence



# a Sequence Function



# Range of a Sequence

