

Multiple Random Variables

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles, Jr. and B. Shi

Outline

- 1 Central Limit Theorem
 - Unequal Distributions
 - Equal Distributions

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Central Limit Theorem

Definition

the central limit theorem says that the probability distribution function of the sum of large number of random variables approaches a Gaussian distribution.

This theorem is known to apply some cases of statistically independent random variables.

Central Limit Theorem

Unequal Distribution Case

Definition

the sum Y of N independent random variables X_1, X_2, \dots, X_N
Let $Y = X_1 + X_2 + \dots + X_N$, then

$$\bar{Y}_N = \bar{X}_1 + \bar{X}_2 + \dots + \bar{X}_N$$

$$\sigma_{Y_N}^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \dots + \sigma_{X_N}^2$$

the probability distribution of Y asymptotically approaches to Gaussian distribution function as $N \rightarrow \infty$

Sufficient Conditions

Unequal Distribution Case

Definition

$$\sigma_{X_i}^2 > B_1 > 0 \quad i = 1, 2, \dots, N$$

$$E[|X_i - \bar{X}_i|^3] < B_2 \quad i = 1, 2, \dots, N$$

where B_1 and B_2 are positive numbers

these conditions guarantee that no one random variable in the sum dominates

Distribution vs density functions

Unequal Distribution Case

the central limit theorem guarantees

- only that the distribution of the sum of random variables become Gaussian
- the density of the sum of random variables is not always Gaussian
- the sum of continuous random variables :
 - under certain conditions on individual random variables the density of the sum is always Gaussian
- the sum of discrete random variables :
 - the density function may contain impulses and thus is not Gaussian.

Discrete Random Variable Examples

distribution may contain impulses

the sum Y of N independent discrete random variables

$$Y = X_1 + X_2 + \dots + X_N$$

- discrete random variable
- density function may contain impulses
- therefore the density function is not Gaussian
- although the distribution approaches Gaussian

- when the possible discrete values of each random variable are $kb, k = 0, \pm 1, \pm 2, \dots$, where b is a constant
 - the envelope of the impulses in the density of the sum will be Gaussian
 - with the mean Y_N and variance $\sigma_{Y_N}^2$

Central Limit Theorem (1)

Equal Distribution Case

Definition

the sum Y of N independent random variables X_1, X_2, \dots, X_N
assume that X_1, X_2, \dots, X_N have the same distribution function.

Let $Y_N = X_1 + X_2 + \dots + X_N$,

then $W_N = (Y_N - \bar{Y}_N) / \sigma_{Y_N}$ is

the zero-mean, unit-variance random variable

Central Limit Theorem (2)

Equal Distribution Case

Definition

Let $Y_N = X_1 + X_2 + \cdots + X_N$ and $W_N = (Y_N - \bar{Y}_N) / \sigma_{Y_N}$

$$\begin{aligned} W_N &= (Y_N - \bar{Y}_N) / \sigma_{Y_N} \\ &= \sum_{i=1}^N (X_i - \bar{X}_i) / \left[\sum_{i=1}^N \sigma_{\bar{X}_i}^2 \right]^{1/2} \\ &= \frac{1}{\sqrt{N} \sigma_X} \sum_{i=1}^N (X_i - \bar{X}_i) \end{aligned}$$

where $X_i = \bar{X}$ and $\sigma_{\bar{X}_i}^2 = \sigma_{\bar{X}}$

Central Limit Theorem (3)

Equal Distribution Case

$$\begin{aligned}
 W_N &= (Y_N - \bar{Y}_N) / \sigma_{Y_N} \\
 &= \left(X_i - \sum_{i=1}^N \bar{X}_i \right) / \left[\sum_{i=1}^N \sigma_{\bar{X}_i}^2 \right]^{1/2} \\
 &= \sum_{i=1}^N (X_i - \bar{X}_i) / \left[\sum_{i=1}^N \sigma_{\bar{X}_i}^2 \right]^{1/2} \\
 &= \sum_{i=1}^N (X_i - \bar{X}_i) / [N\sigma_X^2]^{1/2} \\
 &= \frac{1}{\sqrt{N}\sigma_X} \sum_{i=1}^N (X_i - \bar{X}_i)
 \end{aligned}$$

where $\bar{X}_i = \bar{X}$ and $\sigma_{\bar{X}_i}^2 = \sigma_{\bar{X}}^2$

$\bar{Y}_N = \bar{X}_1 + \bar{X}_2 + \dots + \bar{X}_N$ and $\sigma_{Y_N}^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \dots + \sigma_{X_N}^2$

Characteristic Function(1)

Equal Distribution Case

the characteristic function of W_N
a zero mean, unit variance Gaussian random variable

$$\Phi_{W_N}(\omega) = \exp(-\omega^2/2)$$

W_N is the density of the Gaussian random variable
Fourier transforms are unique

$$\Phi_{W_N}(\omega) = E[e^{j\omega W_N}]$$

Characteristic Function(2)

Equal Distribution Case

$$W_N = \frac{1}{\sqrt{N}\sigma_X} \sum_{i=1}^N (X_i - \bar{X}_i)$$

$$\begin{aligned}\Phi_{W_N}(\omega) &= E[e^{j\omega W_N}] = E \left[\exp \left(\frac{j\omega}{\sqrt{N}\sigma_X} \sum_{i=1}^N (X_i - \bar{X}) \right) \right] \\ &= E \left[\exp \left(\frac{j\omega}{\sqrt{N}\sigma_X} (X_1 - \bar{X}) \right) \cdots \exp \left(\frac{j\omega}{\sqrt{N}\sigma_X} (X_N - \bar{X}) \right) \right] \\ &= \left\{ E \left[\exp \left(\frac{j\omega}{\sqrt{N}\sigma_X} (X_1 - \bar{X}) \right) \right] \right\}^N\end{aligned}$$

$$E[X_1] = E[X_2] = \cdots = E[X_N]$$

Characteristic Function (3)

Equal Distribution Case

$$\Phi_{W_N}(\omega) = \left\{ E \left[\exp \left(\frac{j\omega}{\sqrt{N}\sigma_X} (X_1 - \bar{X}) \right) \right] \right\}^N$$

$$\ln[\Phi_{W_N}(\omega)] = N \ln \left\{ E \left[\exp \left(\frac{j\omega}{\sqrt{N}\sigma_X} (X_1 - \bar{X}) \right) \right] \right\}$$

$$E \left[\exp \left(\frac{j\omega}{\sqrt{N}\sigma_X} (X_1 - \bar{X}) \right) \right]$$

$$= E \left[1 + \frac{j\omega}{\sqrt{N}\sigma_X} (X_1 - \bar{X}) + \left(\frac{j\omega}{\sqrt{N}\sigma_X} \right)^2 (X_1 - \bar{X})^2 + \frac{R_N}{N} \right]$$

$$= 1 - \frac{\omega^2}{2N} + \frac{E[R_N]}{N}$$

where $E[R_N]$ approaches zero as $N \rightarrow \infty$

Characteristic Function (4)

Equal Distribution Case

$$\ln[\Phi_{W_N}(\omega)] = N \ln \left[1 - \frac{\omega^2}{2N} + \frac{E[R_N]}{N} \right]$$

$$\ln[1 - z] = - \left[z + \frac{z^2}{2} + \frac{z^3}{3} + \dots \right], |z| < 1$$

$$\ln[\Phi_{W_N}(\omega)] = -\frac{\omega^2}{2} + E[R_N] - \frac{N}{2} \left[\frac{\omega^2}{2N} + \frac{E[R_N]}{N} \right]^2 + \dots$$

$$\ln[\Phi_{W_N}(\omega)] = -\frac{\omega^2}{2} + E[R_N] - \frac{N}{2} \left[\frac{\omega^2}{2N} + \frac{E[R_N]}{N} \right]^2 + \dots$$

$$\lim_{N \rightarrow \infty} \ln[\Phi_{W_N}(\omega)] = \ln \left[\lim_{N \rightarrow \infty} \Phi_{W_N}(\omega) \right] = -\frac{\omega^2}{2}$$

$$\lim_{N \rightarrow \infty} \Phi_{W_N}(\omega) = e^{-\frac{\omega^2}{2}}$$