Temporal Characteristics of Random Processes

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Based on
Probability, Random Variables and Random Signal Principles,
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Outline

Random Variable Definition

A random variable

a **function** over a **sample space** $S = \{s_1, s_2, s_3, ..., s_n\}$

$$s \rightarrow X(s)$$

$$x = X(s)$$

a **function** of a possible **outcome** s of an **experiment**

Random Variable Definition

A random variable

- a **random variable** : a capital letter X
- a particular value : a lowercase letter x
- a **sample space** $S = \{s_1, s_2, s_3, ..., s_n\}$
- an **outcome** (an element of S) : s

$s \rightarrow X(s)$

$x = X(s)$

$s \rightarrow x$

Understanding Random Variables (1)

- **random variables** are used to quantify outcomes of a random occurrence, and therefore, can take on many values.
- **random variables** are required to be measurable and are typically real numbers.

for example, the letter X may be designated
to represent the *sum* of the resulting numbers
after *three dice* are rolled.
therefore, X could be 3 (1 + 1+ 1), 18 (6 + 6 + 6),
or somewhere between 3 and 18

https://www.investopedia.com/terms/r/random-variable.asp

Understanding Random Variables (2)

- A **random variable** is different from an algebraic variable.
  - The variable in an algebraic equation is an unknown value that can be calculated.
  - The equation $10 + x = 13$ shows that we can calculate the specific value for x which is 3.
  - a **random variable** has a set of values, and any of those values *could be* the resulting outcome as seen in the example of the dice above.

https://www.investopedia.com/terms/r/random-variable.asp

Understanding Random Variables (3)

- A typical example of a random variable is the outcome of a coin toss.
  - Consider a probability distribution in which the outcomes of a random event are not equally likely to happen.
  - If the **random variable** Y is the number of heads we get from tossing two coins, then Y could be 0, 1, or 2. (no heads, one head, or both heads)

https://www.investopedia.com/terms/r/random-variable.asp

Understanding Random Variables (4)

- Let the **random variable** Y be the number of heads of tossing two coins
  - the two coins land in four different ways: TT, HT, TH, and HH.
  - the $P(Y = 0) = 1/4$ since we have one chance of getting no heads (i.e., two tails [TT] when the coins are tossed).
  - the $P(Y = 2) = 1/4$: the probability of getting two heads (HH)
  - getting one head has a likelihood of occurring twice: in HT and TH. In this case, $P(Y = 1) = 2/4 = 1/2$.

https://www.investopedia.com/terms/r/random-variable.asp

Formal definition of a random variable

A random variable X is a measurable function $X\colon \Omega \to E$ from a set of possible outcomes Ω to a measurable space E.

The technical axiomatic definition requires Ω to be a sample space of a probability triple $(\Omega, \mathscr{F}, \mathrm{P})$

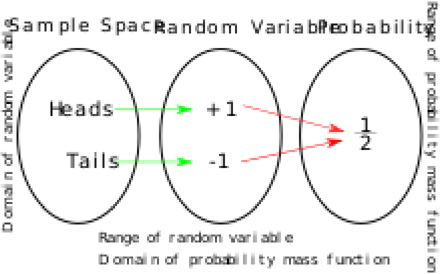
A random variable is often denoted by capital roman letters such as X, Y, Z, T.

The probability that X takes on a value in a measurable set $S \subseteq E$ is written as

$\mathrm{P}(X \in S) = \mathrm{P}(\{\omega \in \Omega \mid X(\omega) \in S\})$

https://en.wikipedia.org/wiki/Random_variable

Random variable example



This graph shows how random variable is a function from all possible outcomes to real values. It also shows how random variable is used for defining probability mass functions.

https://en.wikipedia.org/wiki/Random_variable

Probability Space (1)

In probability theory, a probability space or a probability triple $(\Omega, \mathscr{F}, P)$ is a mathematical construct

that provides a formal model of a random process or "experiment".

For example, one can define a probability space which models the throwing of a die

https://en.wikipedia.org/wiki/Probability_space

Probability Space (2)

A probability space consists of three elements

A sample space, Ω , which is the set of all possible outcomes.

An event space, which is a set of events $\mathscr{F}$,

an event being a set of outcomes in the sample space.

A probability function, which assigns each event in the event space a probability, which is a number between 0 and 1.

https://en.wikipedia.org/wiki/Probability_space

Probability Space (3)

In the example of the throw of a standard die, we would take the sample space to be $\{1,2,3,4,5,6\}$. For the event space, we could simply use the set of all subsets of the sample space, which would then contain simple events such as $\{5\}$ ("the die lands on 5"), as well as complex events such as $\{2,4,6\}$ ("the die lands on an even number"). Finally, for the probability function, we would map each event to the number of outcomes in that event divided by 6 — so for example, $\{5\}$ would be mapped to 1 / 6 1/6, and $\{2,4,6\}$ would be mapped to $3/6 = 1/2$.

https://en.wikipedia.org/wiki/Probability_space

Random Process (1)

A random process

a function of both **outcome** s and **time** t

$$X(t, s)$$

assigning a time function to every **outcome** s_i

$$s_i \rightarrow x(t, s_i)$$

Random Process (2)

A random process

the family of such time functions is called a random process

$$x(t, s_i) = X(t, s_i)$$

$$x(t, s) = X(t, s)$$

Random Process (3)

We have seen that a random variable X is
a rule which assigns a number
to every outcome e of an experiment.
The random variable is a function X(e)
that maps the set of experiment outcomes
to the set of numbers.
A random process is a rule
that maps every outcome e of an experiment
to a function X(t, e).
A random process is usually conceived of
as a function of time,
but there is no reason to not consider random processes
that are functions of other independent variables,
such as spatial coordinates.
The function X(u, v, e) would be a function
whose value depended on the location (u, v) and the outcome e

Ensemble of time functions

Time functions

A random process $X(t, s)$ represents
a family or ensemble of time functions

$X(t, s)$ represents

- a single time function $x(t, s)$
- when t is a variable and s is fixed at an outcome

$x(t, s)$ represents

- a sample function,
- an ensemble member,
- a realization of the process

Short-form notation for time functions

The short-form notation $x(t)$

to represent a specific waveform of a random process $X(t)$
for a given **outcome** s_i

$$x(t) = x(t, s)$$

$$X(t) = X(t, s)$$

Random Process Example

Example

$$X(t, s_1) = x_1(t) \qquad s_1 \longrightarrow x_1(t)$$
$$X(t, s_2) = x_2(t) \qquad s_2 \longrightarrow x_2(t)$$
$$\ldots \qquad \ldots$$
$$X(t, s_n) = x_n(t) \qquad s_n \longrightarrow x_n(t)$$

$$S = \{s_1, s_2, s_3, \ldots, s_n\} \quad \text{a sample space}$$
$$X(t) = \{x_1(t), x_2(t), x_3(t), \ldots, x_n(t)\} \quad \text{a random process}$$

Random variables with time

a **random process** $X(t,s)$ represents a single time function when t is a variable and s is fixed at an outcome

a random process $X(t,s)$ represents a single **random variable** when both t and s are fixed at a time and an outcome, respectively

$$X_i = X(t_i, s) = X(t_i) \qquad \textit{\textbf{randomvariable}}$$

$$X(t,s) = X(t) \qquad \textit{\textbf{randomprocess}}$$

An alphabet

the alphabet of $X(t)$

the set of its possible values

- the values of **time** t for which a random process is defined
- the **alphabet** of the random variable $X = X(t)$ at time t

Classification of Random Processes

(1) Types of time and alphabet

- the values of **time** t for which a random process is defined
  - continuous time
  - discrete time
- the **alphabet** of the random variable $X = X(t)$ at time t
  - continuous alphabet
  - discrete alphabet

Classification of Random Processes

(2) types of the random variable $X(t)$ and the time t

- a continuous **alphabet** continuous **time** random process
  - $X(t)$ has continuous values and t has continuous values
- a discrete **alphabet** continuous **time** random process
  - $X(t)$ has discrete values and t has continuous values
- a continuous **alphabet** discrete **time** random process
  - $X(t)$ has continuous values and t has discrete values
- a discrete **alphabet** discrete **time** random process
  - $X(t)$ has discrete values and t has discrete values

Deterministic and Non-deterministic Random Processes

- A process is **non-deterministic**
  if future values of any sample function
  cannot be predicted exactly from observed past values
- A process is **deterministic**
  if future values of any sample function
  can be predicted from observed past values

Deterministic Random Process Example (1)

$$X(t) = A\cos(\omega_0 t + \Theta)$$

A, Θ, or ω_0(or all) can be random variables.

a sample function corresponds to the above equation with particular values of these random variables.

$$x_i(t) = A_i \cos(\omega_{0,i} t + \Theta_i)$$

Deterministic Random Process Example (2)

$$x_i(t) = A_i \cos(\omega_{0,i} t + \Theta_i)$$

the knowledge of the sample function
prior to any time instance fully allows
the prediction of the sample function's future values
because all the necessary information is known

$$x_i(t) \quad t \leq 0 \qquad \Longrightarrow \qquad x_i(t) \quad t > 0$$

Functions and variables of a random process $X(t, \theta)$ (1)

$X(t, \theta)$	a family of functions, an ensemble
$X(t, \theta_k)$	a single time function selected by the outcome θ_k
$X(t_1, \theta)$	a random variable at the time $t = t_1$
$X(t_1, \theta_k)$	a number at the time $t = t_1$, of the outcome θ_k

https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf

Functions and variables of a random process $X(t, \theta)$ (2)

- $X(t, \theta)$ is a family of functions. Imagine a giant strip chart recording in which each pen is identified with a different θ. This family of functions is traditionally called an ensemble.
- A single function $X(t, \theta_k)$ is selected by the outcome θ_k. This is just a time function that we could call $X_k(t)$. Different outcomes give us different time functions.
- If t is fixed, say $t = t_1$, then $X(t_1, \theta)$ is a random variable. Its value depends on the outcome θ.
- If both t_1 and θ_k are given then $X(t_1, \theta_k)$ is just a number.

https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf

Stochastic Process (1)

In probability theory and related fields,
a **stochastic** (/stoʊˈkæstɪk/) or **random** process is
a mathematical object usually defined as
a family of **random variables**.

The word stochastic in English was originally used as an adjective with the definition "pertaining to **conjecturing**", and stemming from a Greek word meaning "to aim at a mark, guess", and the Oxford English Dictionary gives the year 1662 as its earliest occurrence.

From Ancient Greek στοχαστικός (stokhastikós), from στοχάζομαι (stokházomai, "aim at a target, guess"), from στόχος (stókhos, "an aim, a guess").

https://en.wikipedia.org/wiki/Stochastic
https://en.wiktionary.org/wiki/stochastic

Stochastic Process (2)

The definition of a **stochastic process** varies,
but a **stochastic process** is *traditionally* defined as
a collection of **random variables** indexed by some set.

The terms **random process** and **stochastic process**
are considered synonyms and are used interchangeably,
without the **index set** being precisely specified.

Both "**collection**", or "**family**" are used
while instead of "**index set**", sometimes
the terms "**parameter set**" or "**parameter space**" are used.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process (3)

The term **random function** is also used to refer to a **stochastic** or **random process**, though sometimes it is only used when the stochastic process takes real values.

This term is also used when the **index sets** are **mathematical spaces** other than the **real line**,

while the terms **stochastic process** and **random process** are usually used when the **index set** is interpreted as time,

and other terms are used such as **random field** when the **index set** is n-dimensional **Euclidean space** $\mathbb{R}^n$ or a manifold

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process (4)

A **stochastic process** can be denoted,
by $\{X(t)\}_{t \in T}$, $\{X_t\}_{t \in T}$, $\{X(t)\}$, $\{X_t\}$ or simply as X or $X(t)$,
although $X(t)$ is regarded as an abuse of function notation.

For example, $X(t)$ or X_t are used to refer to
the **random variable** with the **index** t,
and not the entire **stochastic process**.

If the **index set** is $T = [0, \infty)$, then one can write,
for example, $(X_t, t \geq 0)$ to denote the **stochastic process**.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (1)

A **stochastic process** is defined as
a collection of **random variables**
defined on a common **probability space** $(\Omega, \mathscr{F}, P)$,

- Ω is a **sample space**,
- $\mathscr{F}$ is a σ **-algebra**,
- P is a **probability measure**;
- the **random variables**, indexed by some set T,
- all take values in the same **mathematical space** S, which must be **measurable** with respect to some σ -algebra Σ

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (2)

In other words, for a given **probability space** $(\Omega, \mathscr{F}, P)$
and a **measurable space** (S, Σ),
a **stochastic process** is a
collection of S-valued **random variables**,
which can be written as:

$$\{X(t) : t \in T\}.$$

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (3)

Historically, in many problems from the natural sciences
a point $t \in T$ had the meaning of time,
so $X(t)$ is a **random variable** representing
a value observed at time t.

A **stochastic process** can also be written as $\{X(t,\omega) : t \in T\}$
to reflect that it is actually a function of two variables, $t \in T$ and $\omega \in \Omega$.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (4)

There are other ways to consider a stochastic process,
with the above definition being considered the traditional one.

For example, a stochastic process
can be interpreted or defined as
a S^T-valued **random variable**,
where S^T is the space of all the possible functions
from the set T into the space S.

However this alternative definition as
a "**function-valued random variable**"
in general requires additional regularity assumptions
to be **well-defined**.

https://en.wikipedia.org/wiki/Stochastic_process

Index set (1)

The set T is called the **index set** or **parameter set** of the **stochastic process**.

Often this set is some subset of the real line, such as the natural numbers or an interval, giving the set T the interpretation of time.

https://en.wikipedia.org/wiki/Stochastic_process

Index set (2)

In addition to these sets, the index set T can be another set with a **total order** or a more general set, such as the Cartesian plane R^2 or n-dimensional **Euclidean space**, where an element $t \in T$ can represent a point in space.

That said, many results and theorems are only possible for **stochastic processes** with a **totally ordered index set**.

https://en.wikipedia.org/wiki/Stochastic_process

State space

The **mathematical space** S of a **stochastic process** is called its **state space**.

This mathematical space can be defined using integers, real lines, n-dimensional Euclidean spaces, complex planes, or more abstract mathematical spaces.

The **state space** is defined using elements that reflect the different values that the **stochastic process** can take.

https://en.wikipedia.org/wiki/Stochastic_process

Sample function (1)

A **sample function** is a single **outcome** of a **stochastic process**,
so it is formed by taking a single possible value
of each **random variable** of the **stochastic process**.

More precisely, if $\{X(t,\omega) : t \in T\}$ is a **stochastic process**,
then for any point $\omega \in \Omega$, the mapping $X(\cdot,\omega) : T \to S$,
is called a **sample function**, a **realization**, or,
particularly when T is interpreted as time,
a **sample path** of the **stochastic process** $\{X(t,\omega) : t \in T\}$.

https://en.wikipedia.org/wiki/Stochastic_process

Sample function (2)

This means that for a fixed $\omega \in \Omega$,
there exists a **sample function**
that maps the **index set** T to the **state space** S.

Other names for a **sample function** of a **stochastic process**
include **trajectory**, **path function** or **path**

https://en.wikipedia.org/wiki/Stochastic_process