Vector Calculus (H.1)
Curl

20151028

Curl (Circulation Density)

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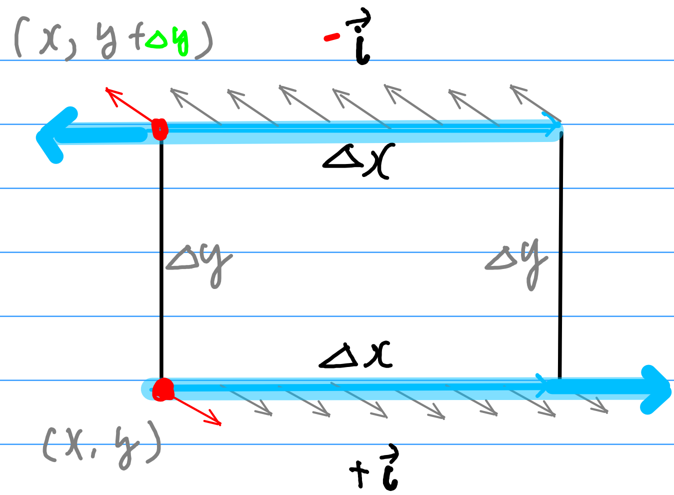
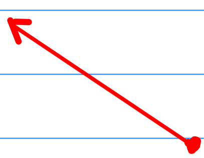
of a vector field $\vec{F} = M\vec{i} + N\vec{j}$

$$\operatorname{curl} \vec{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$$

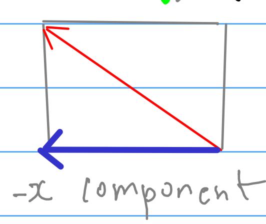
$\boxed{\vec{F}(x, y+\Delta y) \cdot (-\vec{i})\,\Delta x}$ $\boxed{\vec{F}(x, y) \cdot (+\vec{i})\,\Delta x}$



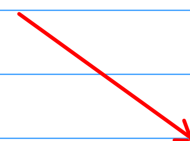
$\vec{F}(x, y+\Delta y)$



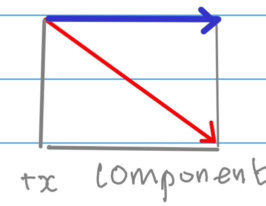
$\vec{F}(x, y+\Delta y) \cdot (-\vec{i})$

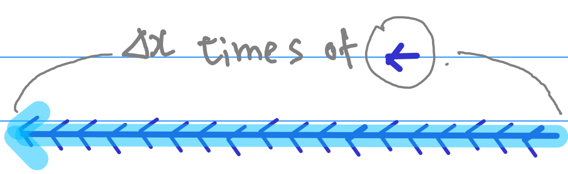


$\vec{F}(x, y)$



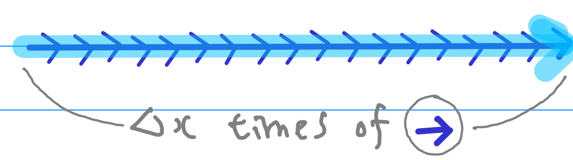
$\vec{F}(x, y) \cdot (+\vec{i})$



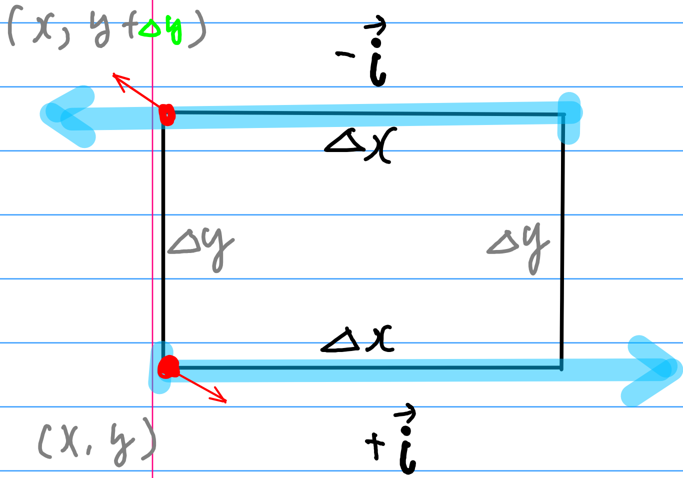


$\vec{F}(x, y+\Delta y) \cdot (-\vec{i})\,\Delta x$

$\vec{F}(x, y) \cdot (+\vec{i})\,\Delta x$



Consider the $+\vec{i}$ component of $\vec{F}$ only $\Rightarrow$ $M(x,y)$



$\vec{F}(x, y+\Delta y)\cdot(-\vec{i})\,\Delta x$

$\Rightarrow -M(x, y+\Delta y)\,\Delta x$

$\Rightarrow M(x, y)\,\Delta x$

$\vec{F}(x, y)\cdot(+\vec{i})\,\Delta x$

$$\vec{F}(x, y+\Delta y)\cdot(-\vec{i})\,\Delta x + \vec{F}(x, y)\cdot(+\vec{i})\,\Delta x$$

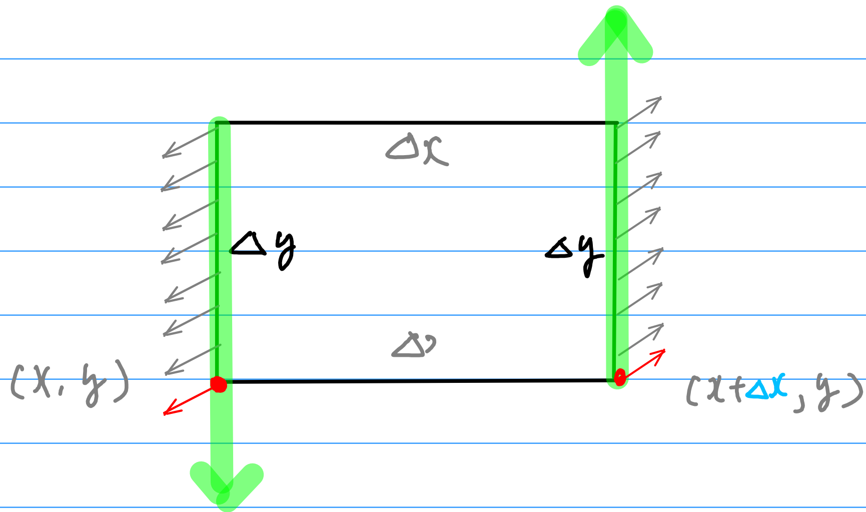
$$\Rightarrow -M(x, y+\Delta y)\,\Delta x + M(x, y)\,\Delta x$$

$$\Rightarrow -\big(M(x, y+\Delta y) - M(x, y)\big)\,\Delta x$$

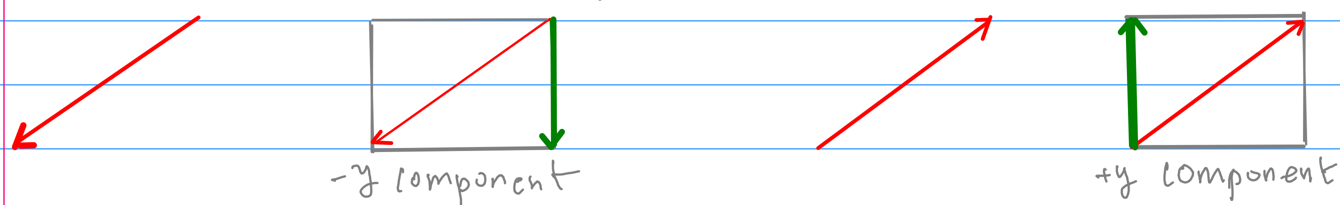
$$\Rightarrow \boxed{-\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x}$$

$$\boxed{\frac{\partial M}{\partial y} = \lim_{\Delta y \to 0} \frac{M(x, y+\Delta y) - M(x, y)}{\Delta y}}$$

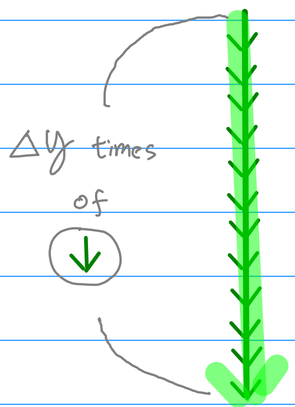
$\boxed{\vec{F}(x,y)\cdot(-\vec{j})\,\Delta y}$ $\boxed{\vec{F}(x+\Delta x,y)\cdot(+\vec{j})\,\Delta y}$

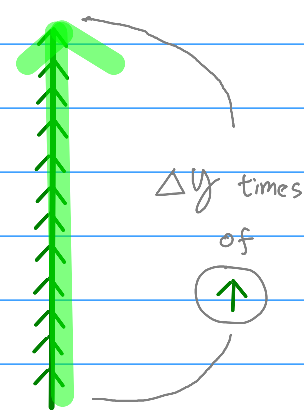


$\vec{F}(x,y)$ $\vec{F}(x,y)\cdot(-\vec{j})$ $\vec{F}(x+\Delta x,y)$ $\vec{F}(x+\Delta x,y)\cdot(+\vec{j})$



$\vec{F}(x,y)\cdot(-\vec{j})\,\Delta y$ $\vec{F}(x+\Delta x,y)\cdot(+\vec{j})\,\Delta y$

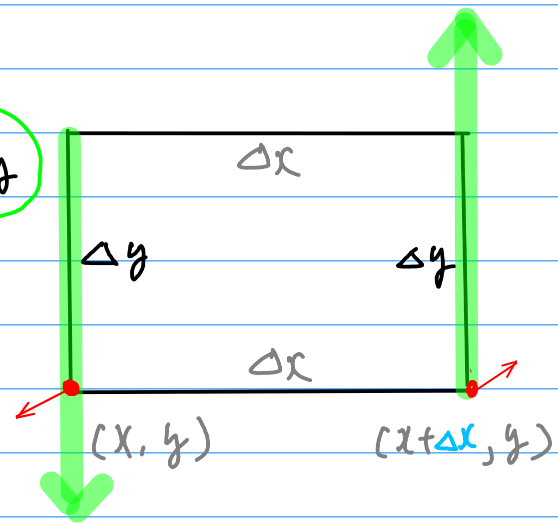




$\boxed{\text{Consider the } +\vec{j} \text{ component of } \vec{F} \text{ only } \Rightarrow N(x,y)}$

$\vec{F}(x,y)\cdot(-\vec{j})\,\Delta y$

$-N(x,y)\,\Delta y$



$\vec{F}(x+\Delta x,y)\cdot(+\vec{j})\,\Delta y$

$N(x+\Delta x,y)\,\Delta y$

$$\vec{F}(x+\Delta x,y)\cdot(+\vec{j})\,\Delta y + \vec{F}(x,y)\cdot(-\vec{j})\,\Delta y$$

$$\Rightarrow N(x+\Delta x,y)\,\Delta y - N(x,y)\,\Delta y$$

$$\Rightarrow \Big(N(x+\Delta x,y) - N(x,y)\Big)\,\Delta y$$

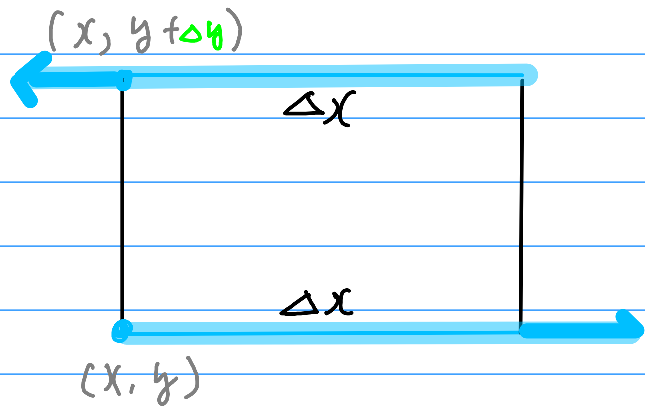
$$\Rightarrow \boxed{\left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y}$$

$$\boxed{\frac{\partial N}{\partial x} = \lim_{\Delta x \to 0} \frac{N(x+\Delta x,y) - N(x,y)}{\Delta x}}$$

Circulation Density along $\vec{i}$ axis

$$\vec{F}(x, y+\Delta y) \cdot (-\vec{i})\, \Delta x$$

the $-\vec{i}$ direction component of $\vec{F}$ multiplied by Δx



$$\vec{F}(x, y) \cdot (+\vec{i})\, \Delta x$$

the $+\vec{i}$ direction component of $\vec{F}$ multiplied by Δx

$$\vec{F}(x, y+\Delta y) \cdot (-\vec{i})\, \Delta x + \vec{F}(x, y) \cdot (+\vec{i})\, \Delta x$$

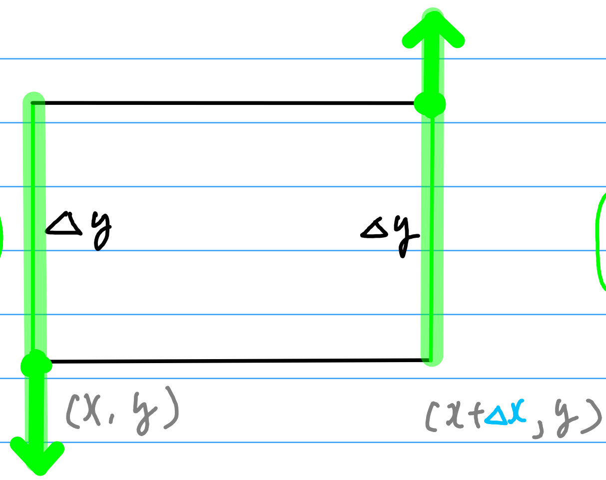
$$\Rightarrow \boxed{-\left(\frac{\partial M}{\partial y} \Delta y\right) \Delta x}$$

$$\vec{F}(x, y) = M(x, y)\vec{i} + N(x, y)\vec{j}$$

Velocity field of a fluid flow in a plane

Circulation Density along $\vec{j}$ axis

$\vec{F}(x,y) \cdot (-\vec{j})\,\Delta y$



$\vec{F}(x+\Delta x, y) \cdot (+\vec{j})\,\Delta y$

the $-\vec{j}$ direction component of $\vec{F}$ multiplied by Δy

the $-\vec{j}$ direction component of $\vec{F}$ multiplied by Δy

$$\vec{F}(x+\Delta x, y) \cdot (+\vec{j})\,\Delta y + \vec{F}(x,y) \cdot (-\vec{j})\,\Delta y$$

$$\Rightarrow \boxed{\left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y}$$

$$\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$$

Velocity field of a fluid flow in a plane

$$\leftarrow \quad \vec{F}(x, y+\Delta y)\cdot(-\vec{i})\,\Delta x = -M(x, y+\Delta y)\,\Delta x$$

$$\rightarrow \quad \vec{F}(x,y)\cdot(+\vec{i})\,\Delta x = +M(x,y)\,\Delta x$$

$$\uparrow \quad \vec{F}(x+\Delta x, y)\cdot(+\vec{j})\,\Delta y = +N(x+\Delta x, y)\,\Delta y$$

$$\downarrow \quad \vec{F}(x,y)\cdot(-\vec{j})\,\Delta y = -N(x,y)\,\Delta y$$

scalar

$$-\Big(M(x, y+\Delta y) - M(x,y)\Big)\Delta x \approx \left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$$

$$\Big(N(x+\Delta x, y) - N(x,y)\Big)\Delta y \approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y$$

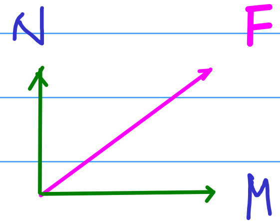
$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\underbrace{\Delta x\,\Delta y}_{R}$$

$$\frac{\text{Circulation around rectangle}}{\text{rectangle area}} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

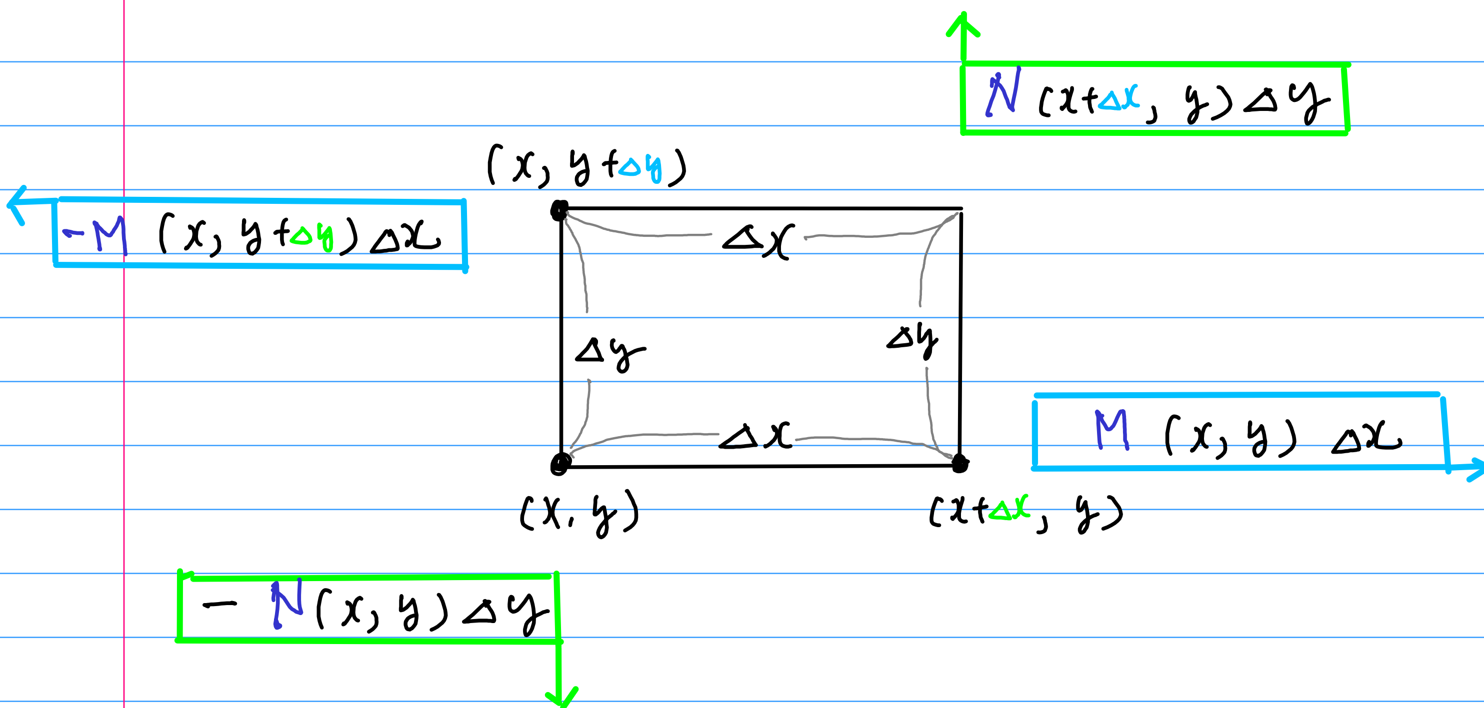
Curl (Circulation Density)

of a vector field $\vec{F} = M\vec{i} + N\vec{j}$

$$\text{curl } \vec{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$



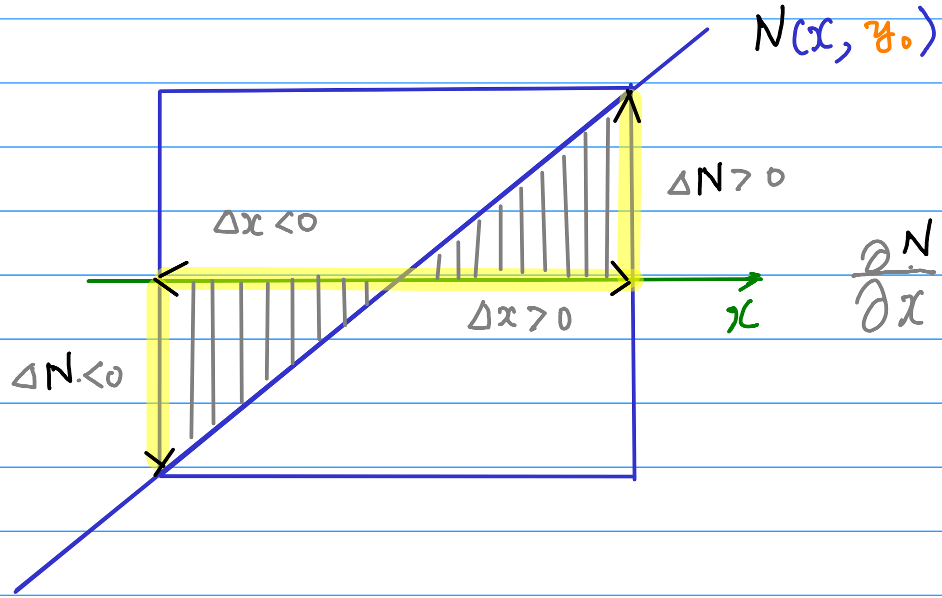
$$\vec{F} = M\vec{i} + N\vec{j} \implies \text{curl } \vec{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$



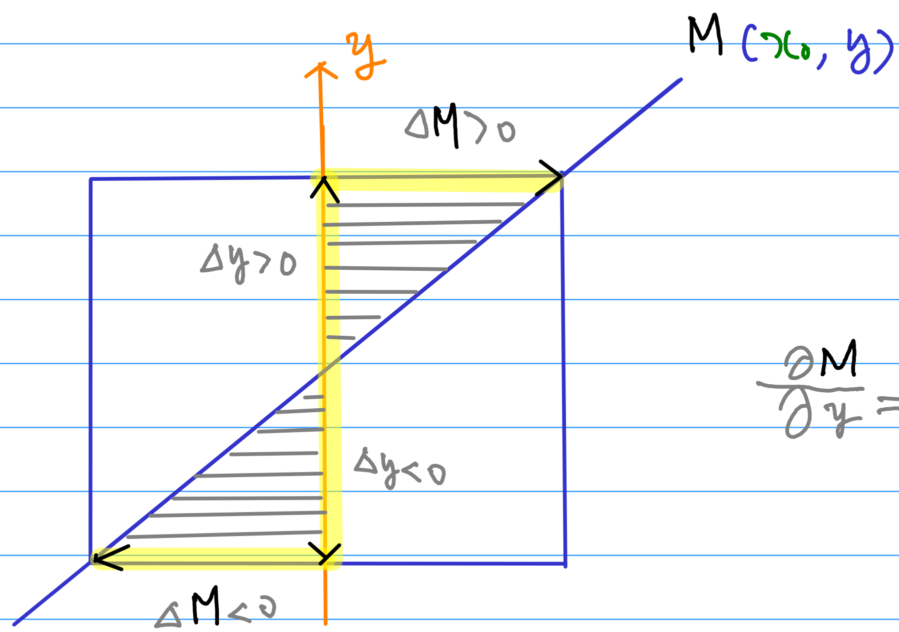
$$-\left[M(x, y+\Delta y) - M(x,y)\right]\Delta x \approx -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$$

$$\left[N(x+\Delta x, y) - N(x,y)\right]\Delta y \approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y$$

$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x\,\Delta y$$

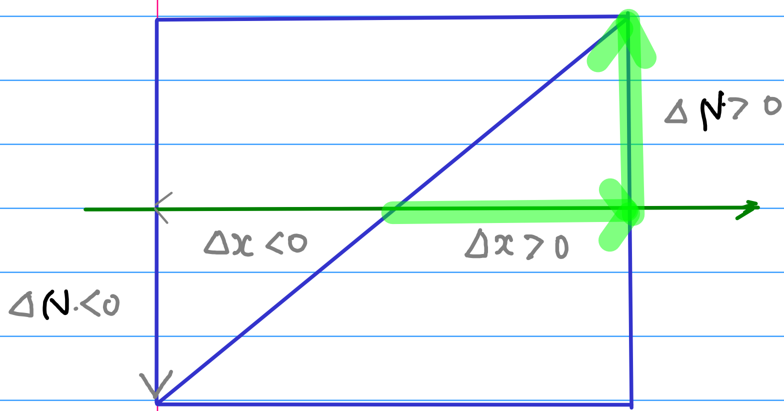


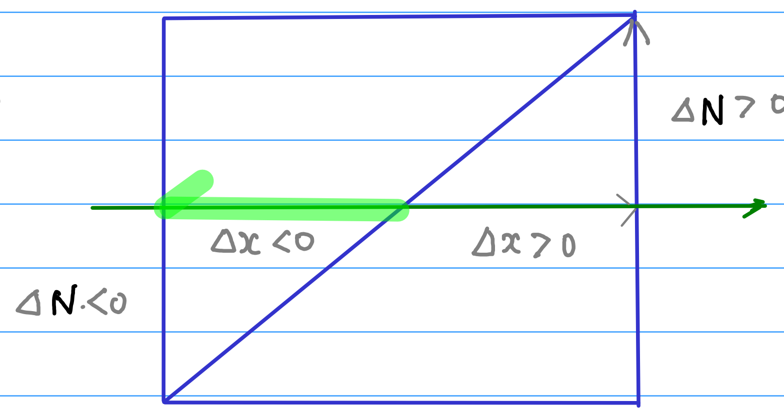
$$\frac{\partial N}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta N}{\Delta x}$$



$$\frac{\partial M}{\partial y} = \lim_{\Delta x \to 0} \frac{\Delta M}{\Delta y}$$

All the same $\frac{\partial N}{\partial x}$





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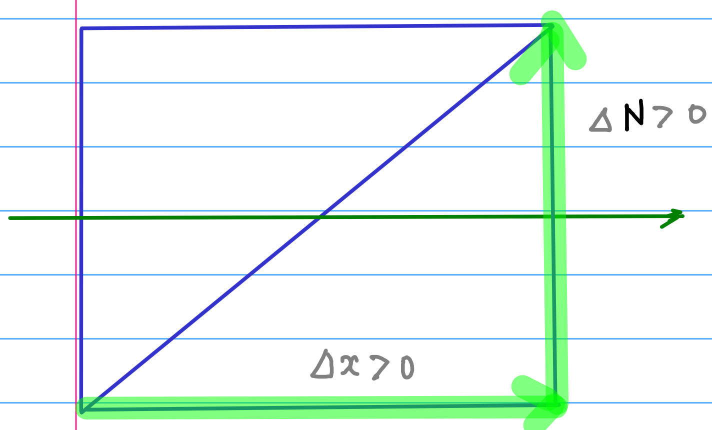

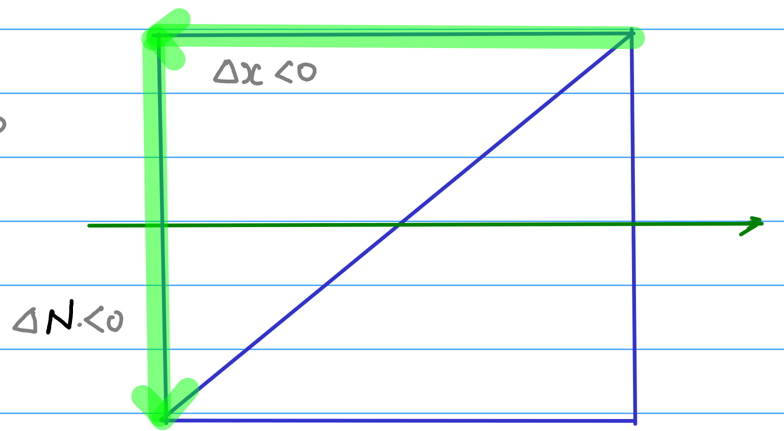

$$\frac{\partial N}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta N}{\Delta x}$$

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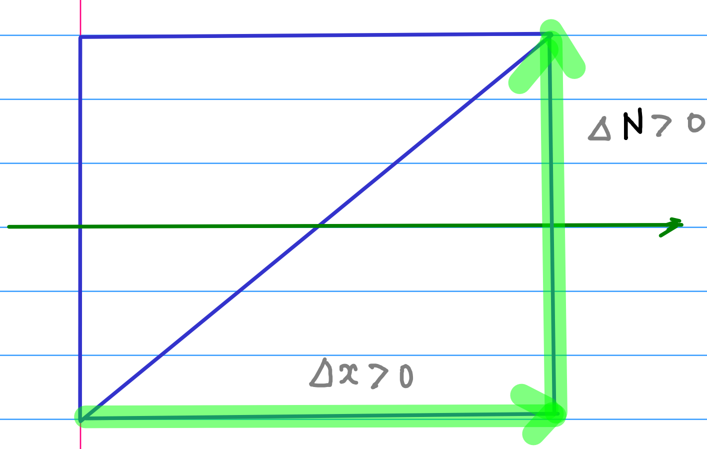
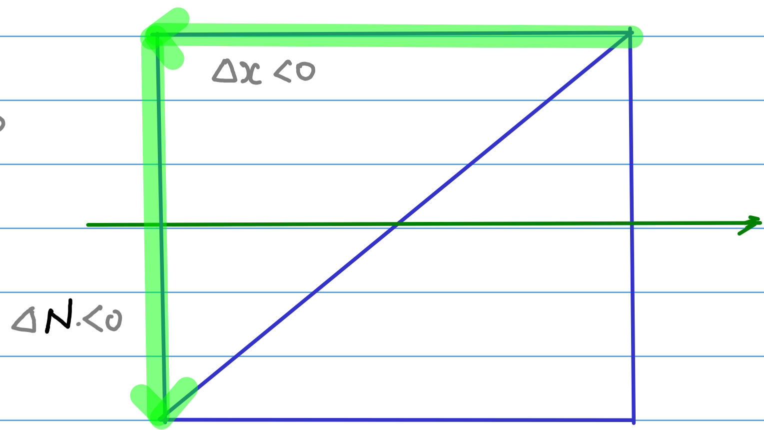
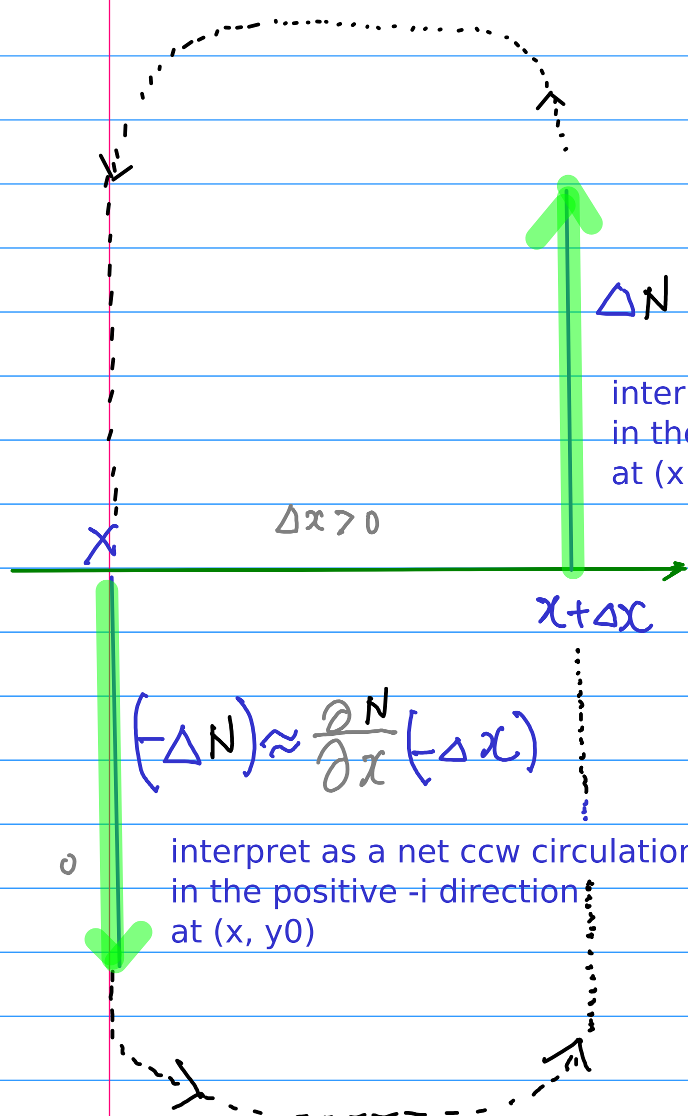
Circulation Interpretation





$$\frac{\partial N}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta N}{\Delta x} \qquad \frac{\partial N}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta N}{\Delta x}$$



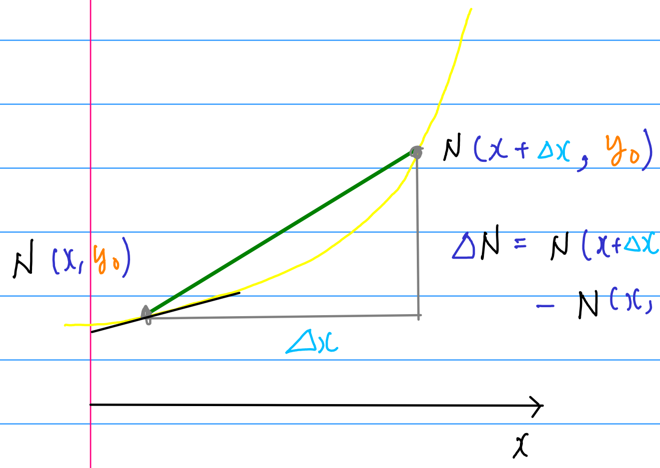
$$\Delta N \approx \frac{\partial N}{\partial x} \Delta x$$

interpret as a net ccw circulation
in the positive +i direction
at (x+dx, y0)

$$(-\Delta N) \approx \frac{\partial N}{\partial x}(-\Delta x)$$

interpret as a net ccw circulation
in the positive -i direction
at (x, y0)

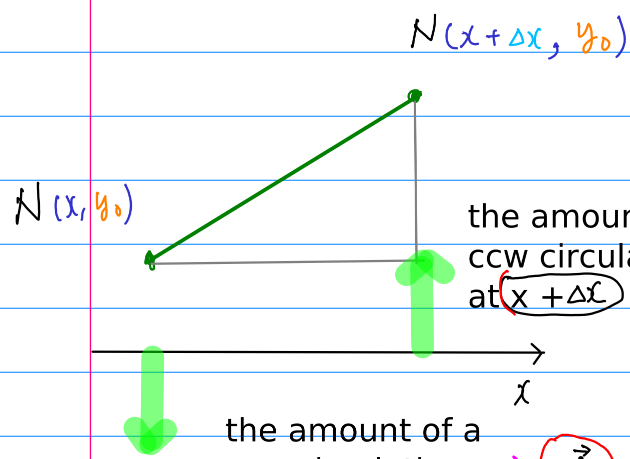
$\Delta N(x, y)$ Interpretation



$\Delta N = N(x+\Delta x, y_0) - N(x, y_0)$

the slope of a tangent :

$$\lim_{\Delta x \to 0} \frac{\Delta N}{\Delta x} = \frac{\partial N}{\partial x}$$



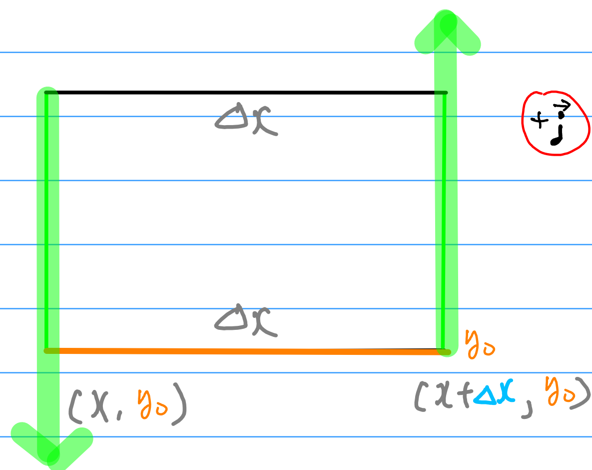
the amount of a ccw circulation $\Longrightarrow$ $+\vec{j}$ direction at $(x + \Delta x, y_0)$

the amount of a ccw circulation $\Longrightarrow$ $-\vec{j}$ direction at (x, y_0)

the net outward flux :

$\Delta M = M(x+\Delta x, y_0) - M(x, y_0)$

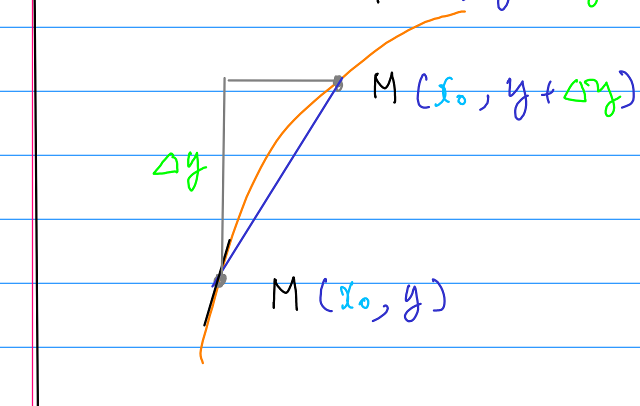
$-\vec{j}$ diretion



$+\vec{j}$ diretion

Δ M(x, y) Interpretation

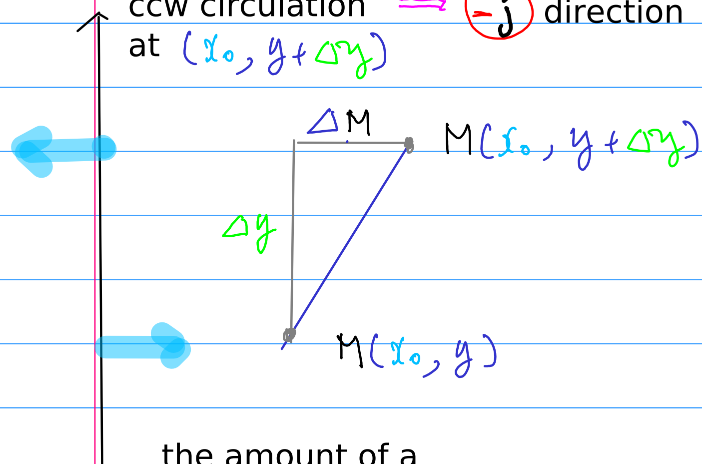
$$\Delta M = M(x_0, y+\Delta y) - M(x_0, y)$$



the $\color{red}{\text{slope}}$ of a tangent :

$$\lim_{\Delta y \to 0} \frac{\Delta M}{\Delta y} = \frac{\partial M}{\partial y}$$

the amount of a ccw circulation at $(x_0, y+\Delta y)$ $\Longrightarrow$ $-\vec{j}$ direction



the net outward flux :

$$\Delta M = N(x_0, y+\Delta y) - N(x_0, y)$$

the amount of a ccw circulation at (x_0, y) $\Longrightarrow$ $+\vec{j}$ direction