Hamiltonian Cycle (3A)

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6/28/18

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Hamiltonian Cycles

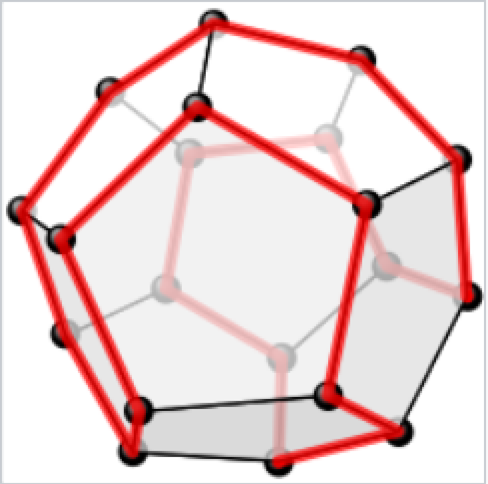
A Hamiltonian path is a path
in an undirected or directed graph
that visits **each vertex** exactly **once**.

A Hamiltonian cycle is
a Hamiltonian path that is a cycle.

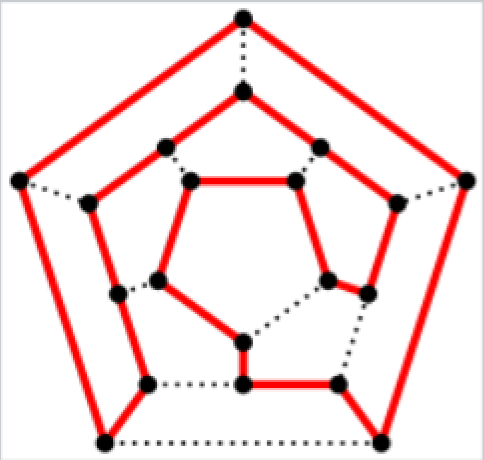
the Hamiltonian path problem is NP-complete.

https://en.wikipedia.org/wiki/Hamiltonian_path

Hamiltonian Cycles



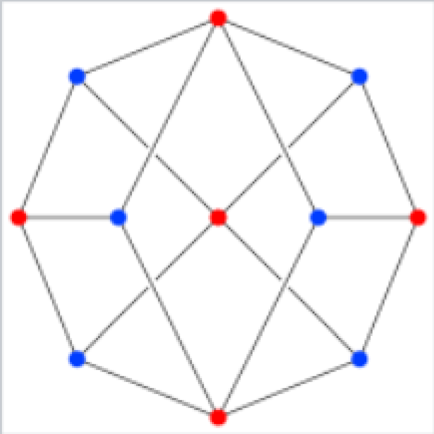
One possible Hamiltonian cycle through every vertex of a dodecahedron is shown in red – like all platonic solids, the dodecahedron is Hamiltonian



The above as a two-dimensional planar graph

https://en.wikipedia.org/wiki/Hamiltonian_path

Hamiltonian Cycles



The Herschel graph is the smallest possible polyhedral graph that does not have a Hamiltonian cycle.

https://en.wikipedia.org/wiki/Hamiltonian_path

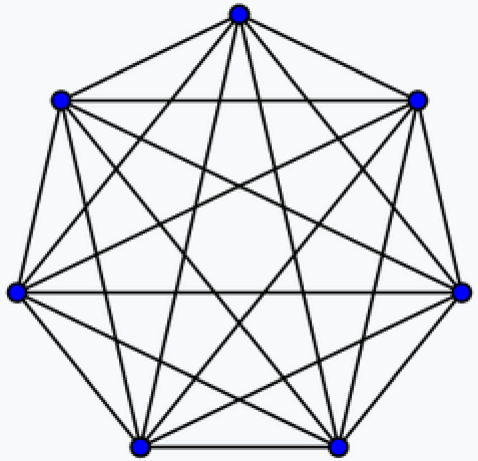
Hamiltonian Cycles

- a **complete graph** with more than two vertices is Hamiltonian
- every **cycle graph** is Hamiltonian
- every **tournament** has an odd number of Hamiltonian paths
- every **platonic solid**, considered as a graph, is Hamiltonian
- the **Cayley graph** of a finite **Coxeter** group is Hamiltonian

https://en.wikipedia.org/wiki/Hamiltonian_path

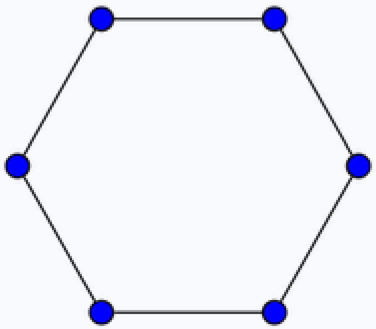
Complete Graphs and Cycle Graphs

Complete graph



K_7, a complete graph with 7 vertices

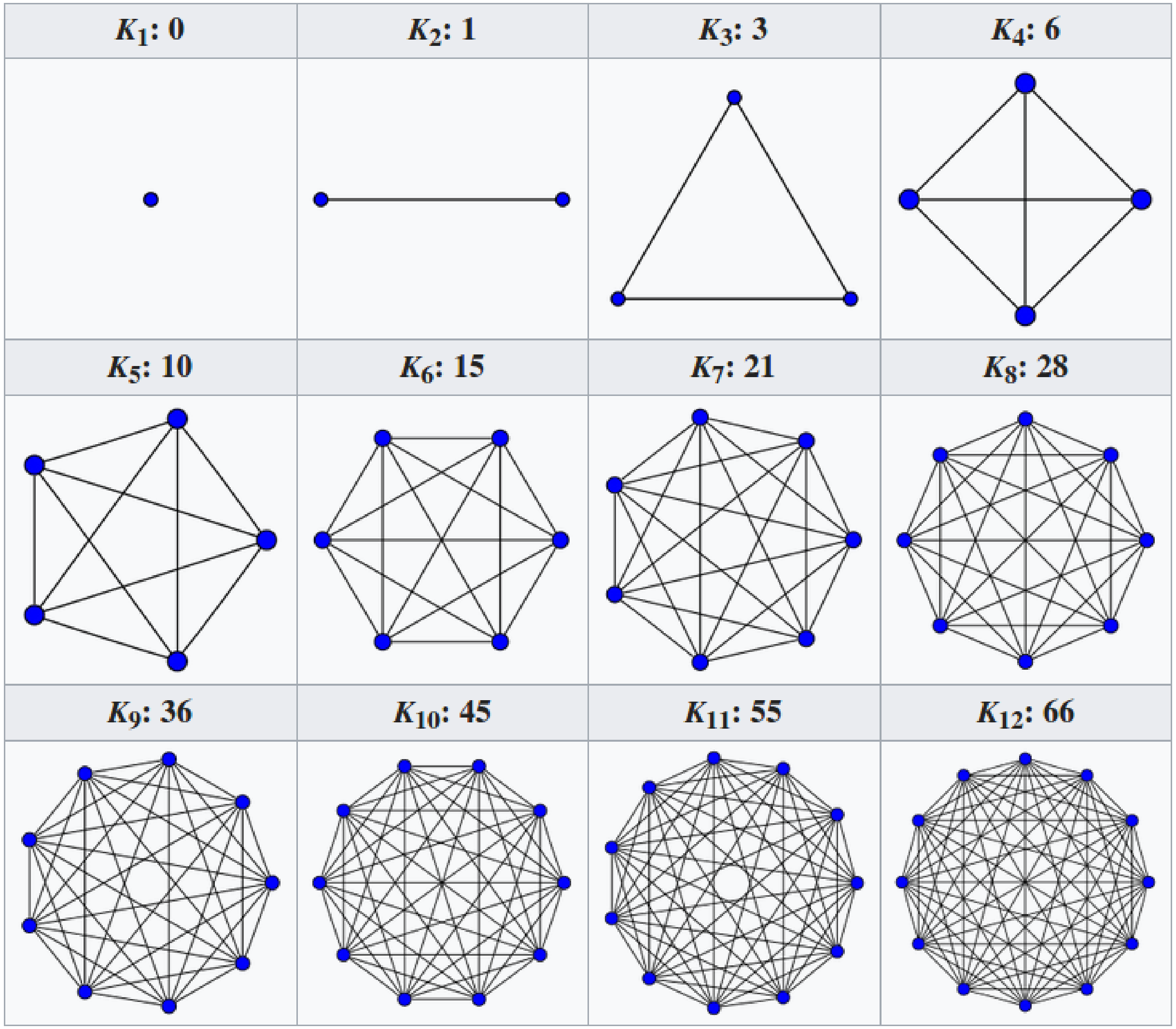
Cycle graph



A cycle graph of length 6

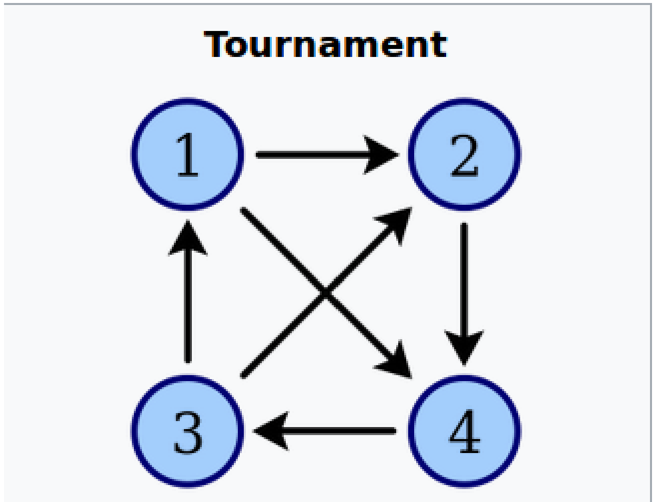
https://en.wikipedia.org/wiki/Complete_graph
https://en.wikipedia.org/wiki/Cycle_graph

Complete Graphs

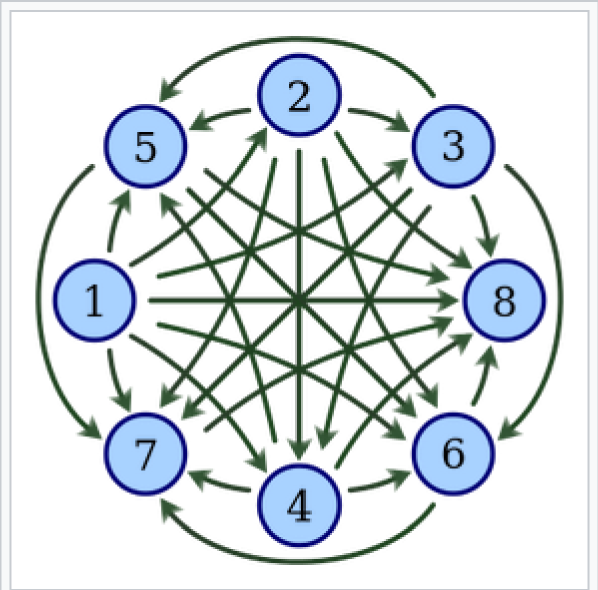


https://en.wikipedia.org/wiki/Complete_graph

Tournament Graphs



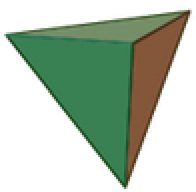
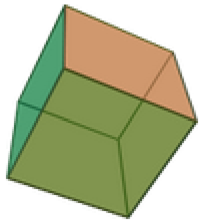
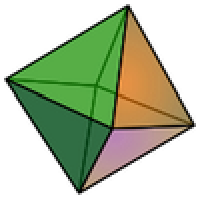
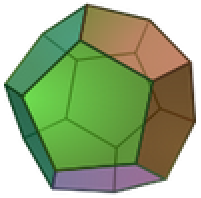
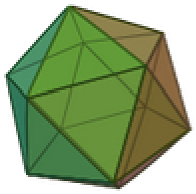
A tournament on 4 vertices



A transitive tournament on 8 vertices.

https://en.wikipedia.org/wiki/Tournament_(graph_theory

Platonic Solid Graphs

Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Four faces	Six faces	Eight faces	Twelve faces	Twenty faces
 (Animation) (3D model)	 (Animation) (3D model)	 (Animation) (3D model)	 (Animation) (3D model)	 (Animation) (3D model)

https://en.wikipedia.org/wiki/Platonic_solid

Hamiltonian Cycles – Properties (1)

Any **Hamiltonian cycle** can be converted
to a **Hamiltonian path** by removing one of its edges,

but a **Hamiltonian path** can be extended to
Hamiltonian cycle only if its endpoints are adjacent.

All **Hamiltonian graphs** are **biconnected**, but a
biconnected graph need not be Hamiltonian

https://en.wikipedia.org/wiki/Hamiltonian_path

Biconnected Graph

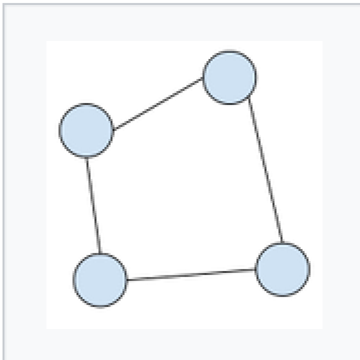
a biconnected graph is a connected and "nonseparable" graph, meaning that if any one **vertex** were to be removed, the graph will remain connected.

a biconnected graph has no articulation vertices.

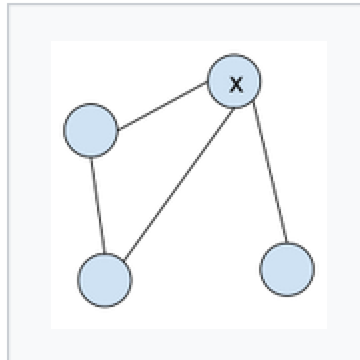
The property of being **2-connected** is equivalent to **biconnectivity**, with the caveat that the complete graph of two vertices is sometimes regarded as biconnected but not 2-connected.

https://en.wikipedia.org/wiki/Biconnected_graph

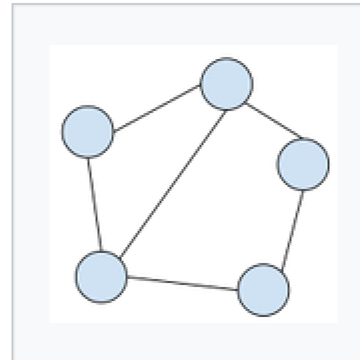
Biconnected Graph Examples



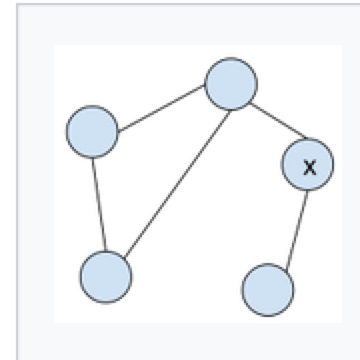
A biconnected graph on four vertices and four edges



A graph that is not biconnected. The removal of vertex x would disconnect the graph.



A biconnected graph on five vertices and six edges

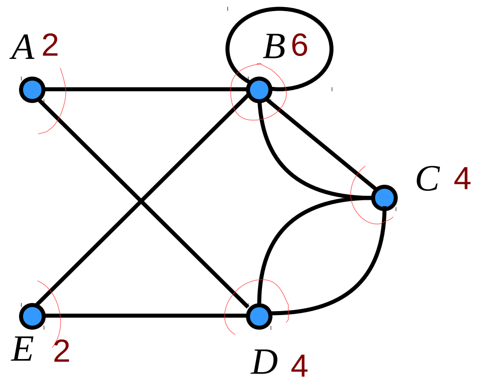


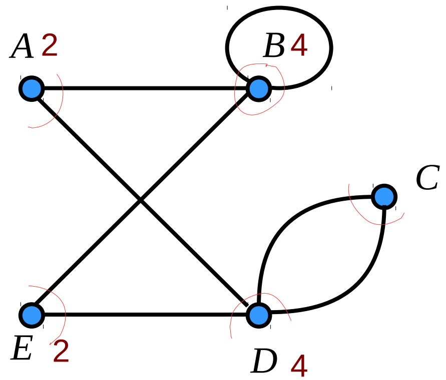
A graph that is not biconnected. The removal of vertex x would disconnect the graph.

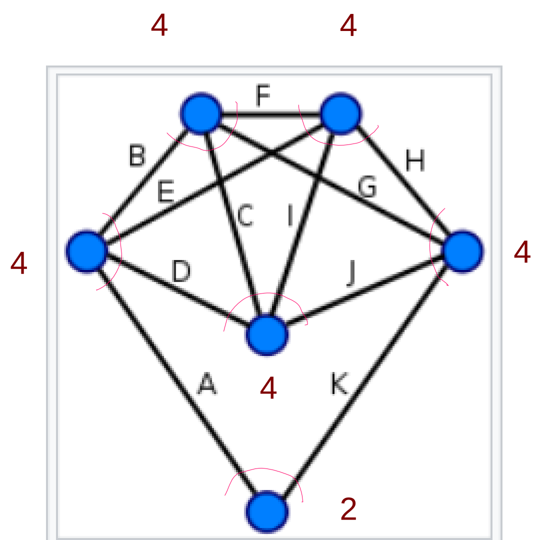
https://en.wikipedia.org/wiki/Biconnected_graph

An Eulerian graph G :
a **connected** graph in which
every **vertex** has **even degree**

An **Eulerian graph** G necessarily has an **Euler cycle**,
a closed walk passing through each **edge** of G exactly **once**.





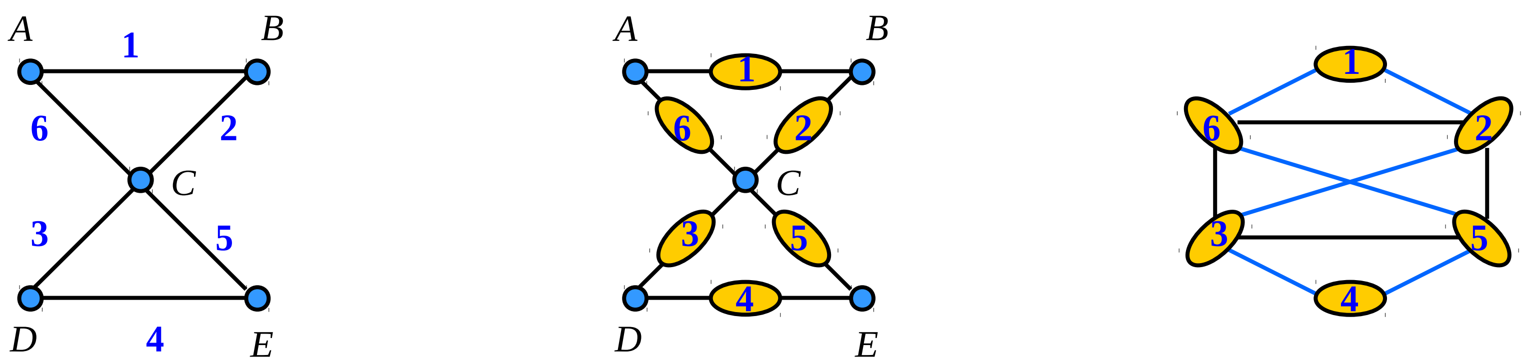


Every vertex of this graph has an even degree. Therefore, this is an Eulerian graph. Following the edges in alphabetical order gives an Eulerian circuit/cycle.

Eulerian Graph (1)

The **Eulerian cycle** corresponds to a **Hamiltonian cycle** in the **line graph L(G)**, so the **line graph** of every **Eulerian graph** is **Hamiltonian graph**.

G → **L(G)**

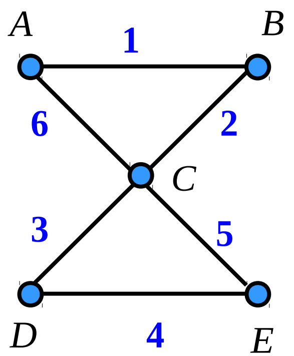


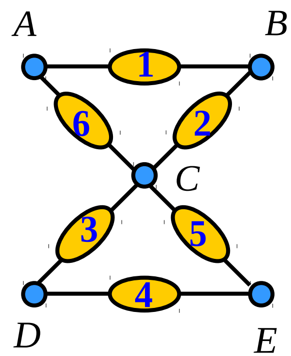
Eulerian Cycle
ABCDECA
→
Hamiltonian Cycle
1-2-3-4-5-6-1

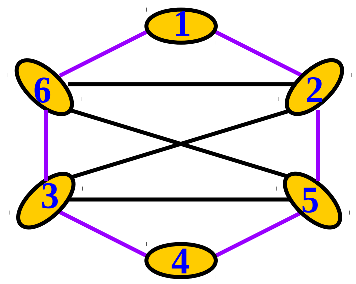
Eulerian Graph (2)

The **Eulerian cycle** corresponds to a **Hamiltonian cycle** in the **line graph L(G)**, so the **line graph** of every **Eulerian graph** is **Hamiltonian graph**.

G → **L(G)**







Eulerian Cycle
ABCEDCA

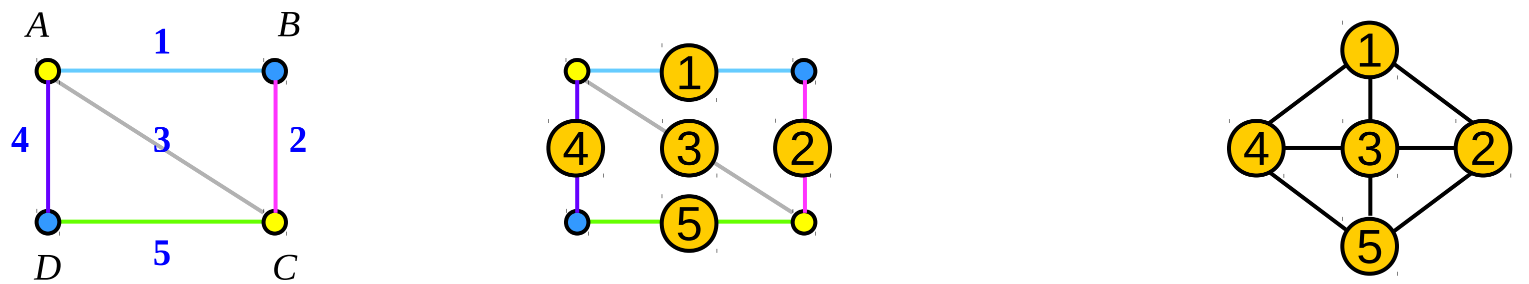
→

Hamiltonian Cycle
1-2-5-4-3-6-1

Eulerian Path (1)

The **Eulerian path** corresponds to a **Hamiltonian path** in the **line graph L(G)**

G → **L(G)**

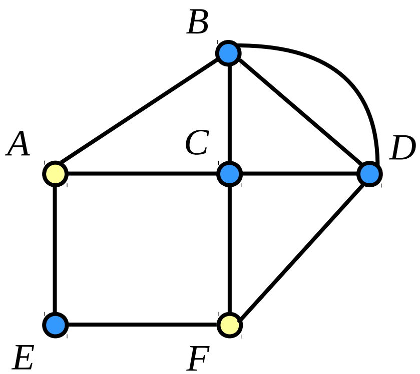


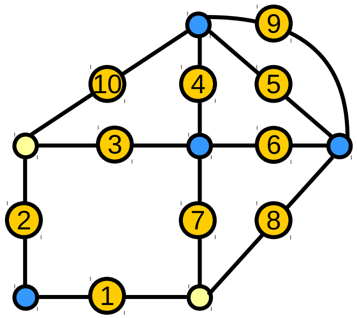
Eulerian Path
ABCADC
→
Hamiltonian Path
1-2-3-4-5

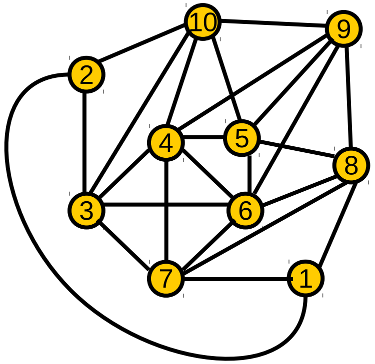
Eulerian Path (2)

Line graphs may have <u>other</u> **Hamiltonian cycles** that do <u>not</u> correspond to **Euler cycles**.

G → **L(G)**







Eulerian Path
FEACBDCFDBA

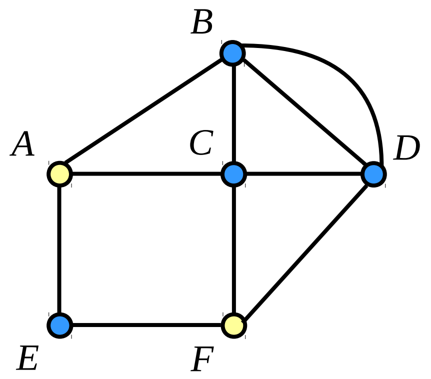
→

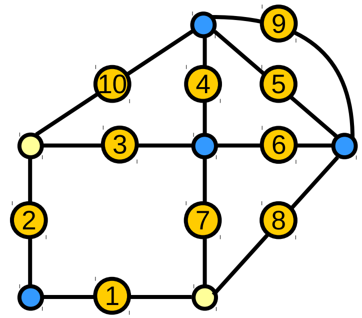
Hamiltonian Path
1-2-3-4-5-6-7-8-9-10

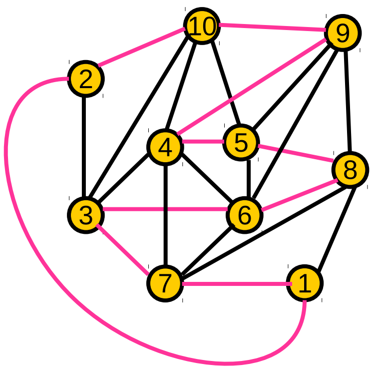
Eulerian Path (3)

Line graphs may have <u>other</u> **Hamiltonian cycles** that do <u>not</u> correspond to **Euler cycles**.

G → **L(G)**







not always

Eulerian Cycle X
Eulerian Path X

Hamiltonian Cycle
1-7-3-6-8-5-4-9-10-2-1

Hamiltonian Cycles – Properties (2)

This **Eulerian cycle** corresponds to a **Hamiltonian cycle** in the **line graph** L(G), so the **line graph** of every **Eulerian graph** is **Hamiltonian graph**.

Line graphs may have other Hamiltonian cycles that do not correspond to Euler paths.

The **line graph** L(G) of every **Hamiltonian graph** G is itself **Hamiltonian**, regardless of whether the graph G is **Eulerian**.

https://en.wikipedia.org/wiki/Hamiltonian_path

Line Graphs

In the mathematical discipline of graph theory, the line graph of an undirected graph G is another graph L(G) that represents the adjacencies between edges of G.

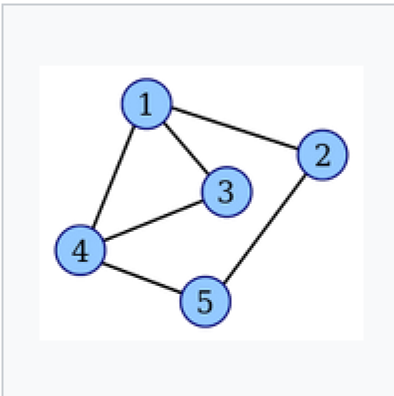
Given a graph G, its line graph L(G) is a graph such that

- each **vertex** of L(G) represents an **edge** of G; and
- two **vertices** of L(G) are **adjacent** if and only if their corresponding **edges** share a **common endpoint** ("are incident") in G.

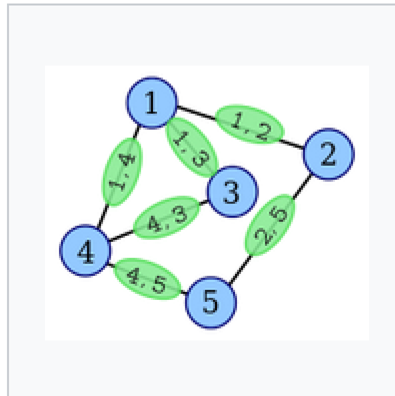
That is, it is the **intersection graph** of the **edges** of G, representing each edge by the set of its two endpoints.

https://en.wikipedia.org/wiki/Line_graph

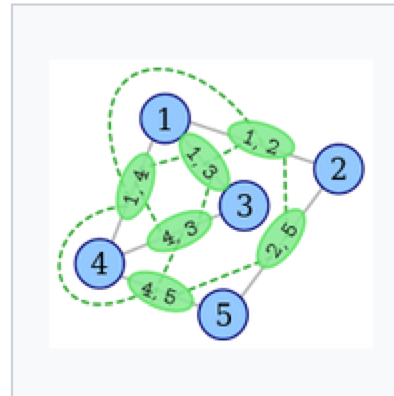
Line Graphs Examples



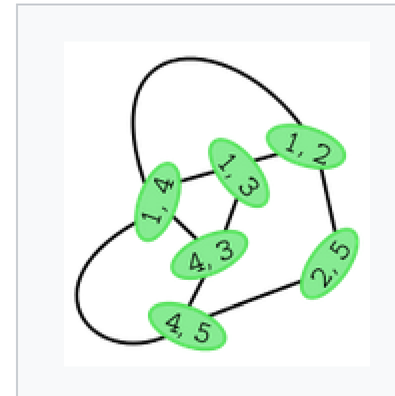
Graph G



Vertices in L(G) constructed from edges in G



Added edges in L(G)



The line graph L(G)

https://en.wikipedia.org/wiki/Line_graph

Hamiltonian Cycles – Properties (3)

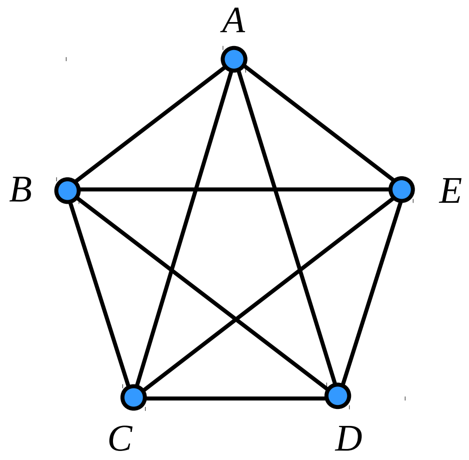
A tournament (with more than two vertices) is Hamiltonian if and only if it is **strongly connected**.

The number of different Hamiltonian cycles
in a **complete undirected** graph on **n** vertices is **(n − 1)! / 2**
in a complete directed graph on n vertices is (n − 1)!.

These counts assume that cycles that are the same apart from their starting point are not counted separately.

https://en.wikipedia.org/wiki/Hamiltonian_path

Number of Hamiltonian Cycles (1)

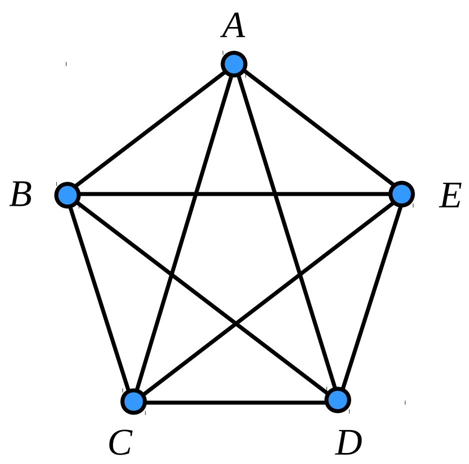


$$(5-1)! = 24$$

A	BCDE	AB	CDE	ABC	DE	ABCD	E	ABCDE
				ABD	CE	ABCE	D	ABCED
		AC	BDE	ABE	CD	ABDC	E	ABDCE
						ABDE	C	ABDEC
		AD	BCE	ACB	DE	ABEC	D	ABECD
				ACD	BE	ABED	C	ABEDC
		AE	BCD	ACE	BD			
						ACBD	E	ACBDE
				ADB	CE	ACBE	D	ACBED
				ADC	BE	ACDB	E	ACDBE
				ADE	BC	ACDE	B	ACDEB
						ACEB	D	ACEBD
				AEB	CD	ACED	B	ACEDB
				AEC	BD			
				AED	BC	ADBC	E	ADBCE
						ADBE	C	ADBEC
						ADCB	E	ADCBE
						ADCE	B	ADCEB
						ADEB	C	ADEBC
						ADEC	B	ADECB
						AEBC	D	AEBCD
						AEBD	C	AEBDC
						AECB	D	AECBD
						AECD	B	AECDB
						AEDB	C	AEDBC
						AEDC	B	AEDCB

https://en.wikipedia.org/wiki/Hamiltonian_path

Number of Hamiltonian Cycles (2)

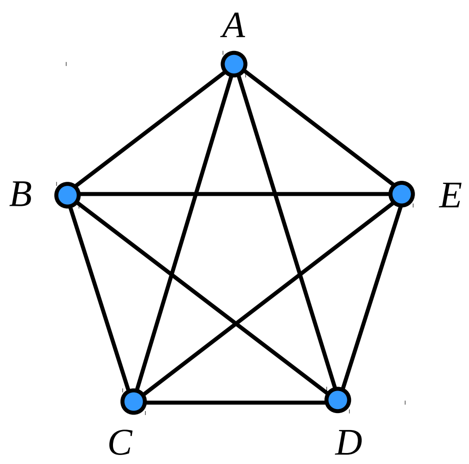


$$(5-1)! = 24$$

https://en.wikipedia.org/wiki/Hamiltonian_path

ABCDE	BACDE	CABDE	DABCE	EABCD
ABCED	BACED	CABED	DABEC	EABDC
ABDCE	BADCE	CADBE	DACBE	EACBD
ABDEC	BADEC	CADEB	DACEB	EACDB
ABECD	BAECD	CAEBD	DADBC	EADBC
ABEDC	BAEDC	CAEDB	DADCB	EADCB
ACBDE	BCADE	CBADE	DBACE	EBACD
ACBED	BCAED	CBAED	DBAEC	EBADC
ACDBE	BCDAE	CBDAE	DBCAE	EBCAD
ACDEB	BCDEA	CBDEA	DBCEA	EBCDA
ACEBD	BCEAD	CBEAD	DBEAC	EBDAC
ACEDB	BCEDA	CBEDA	DBECA	EBDCA
ADBCE	BDACE	CDABE	DCABE	ECABD
ADBEC	BDAEC	CDAEB	DCAEB	ECADB
ADCBE	BDCAE	CDBAE	DCBAE	ECBAD
ADCEB	BDCEA	CDBEA	DCBEA	ECBDA
ADEBC	BDEAC	CDEAB	DCEAB	ECDAB
ADECB	BDECA	CDEBA	DCEBA	ECDBA
AEBCD	BEACD	CEABD	DEABC	EDABC
AEBDC	BEADC	CEADB	DEACB	EDACB
AECBD	BECAD	CEBAD	DEBAC	EDBAC
AECDB	BECDA	CEBDA	DEBCA	EDBCA
AEDBC	BEDAC	CEDAB	DECAB	EDCAB
AEDCB	BEDCA	CEDBA	DECBA	EDCBA

Number of Hamiltonian Cycles (3)



$$\frac{(5-1)!}{2} = \frac{24}{2} = 12$$

https://en.wikipedia.org/wiki/Hamiltonian_path

ABCDE
ABCED
ABDCE
ABDEC
ABECD
ABEDC

ACBDE
ACBED
ACDBE
ACDEB
ACEBD
ACEDB

ADBCE
ADBEC
ADCBE
ADCEB
ADEBC
ADECB

AEBCD
AEBDC
AECBD
AECDB
AEDBC
AEDCB

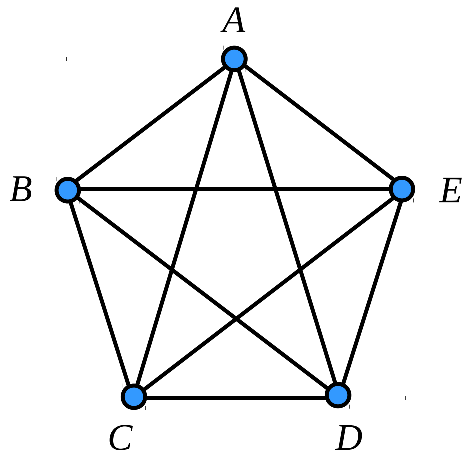
$A-B-C-D-E-A$

$A-B-C-E-D-A$

$A-B-D-C-E-A$

(n − 1)! / 2

Number of Hamiltonian Cycles (4)



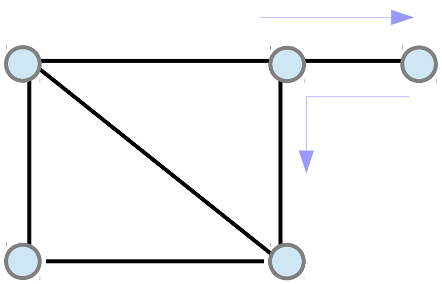
$$\frac{(5-1)!}{2} = \frac{24}{2} = 12$$

https://en.wikipedia.org/wiki/Hamiltonian_path

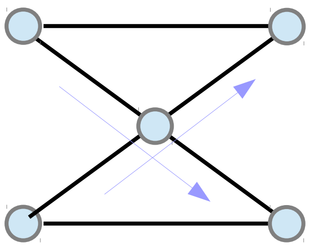
ABCDE	ABCDE	ABCDE
ABCED	ABCED	ABCED
ABDCE	ABDCE	ABDCE
ABDEC	ABDEC	ABDEC
ABECD	ABECD	ABECD
ABEDC	ABEDC	ABEDC
ACBDE	ACBDE	ACBDE
ACBED	ACBED	ACBED
ACDBE	ACDBE	ACDBE
ACDEB	ACDEB	ACEBD
ACEBD	ACEBD	ADBCE
ACEDB	ACEDB	ADCBE
ADBCE	ADBCE	
ADBEC	ADBEC	
ADCBE	ADCBE	
ADCEB	ADCEB	
ADEBC	ADEBC	
ADECB	ADECB	
AEBCD	AEBCD	
AEBDC	AEBDC	
AECBD	AECBD	
AECDB	AECDB	
AEDBC	AEDBC	
AEDCB	AEDCB	

(n − 1)! / 2

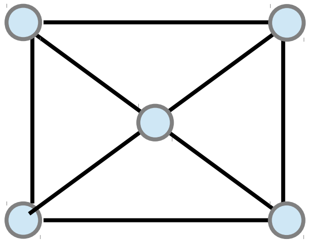
No Hamiltonian Cycle



No Hamiltonian Cycle



No Hamiltonian Cycle



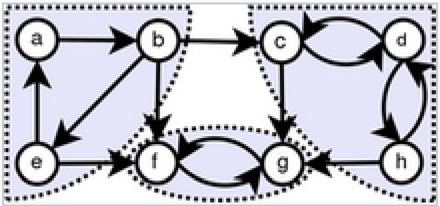
Hamiltonian Cycle

Ross

Strongly Connected Component

a directed graph is said to be **strongly connected** or **diconnected** if every **vertex** is reachable from every other **vertex**.

The **strongly connected components** or **diconnected components** of an arbitrary directed graph form a **partition** into **subgraphs** that are themselves **strongly connected**.



Graph with strongly connected components marked

https://en.wikipedia.org/wiki/Hamiltonian_path

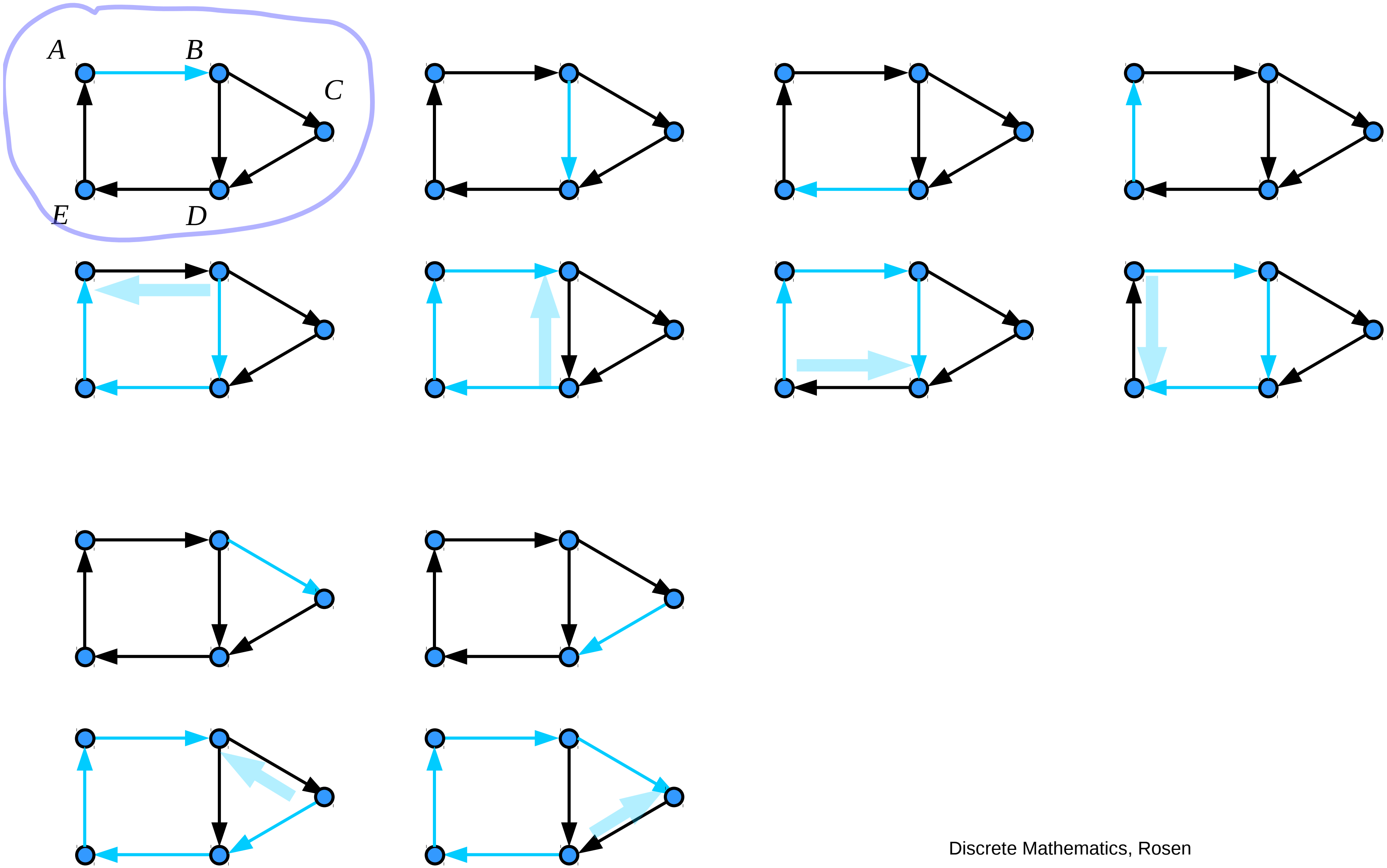
SCC and WCC

a directed graph is **strongly connected**
if there is a **path** from **a** to **b** and from **b** to **a**
whenever **a** and **b** are **vertices** in the graph

a directed graph is **weakly connected**
if there is a **path** between every two **vertices**
in the underlying undirected graph
(either way)
directions of edges are disregarded

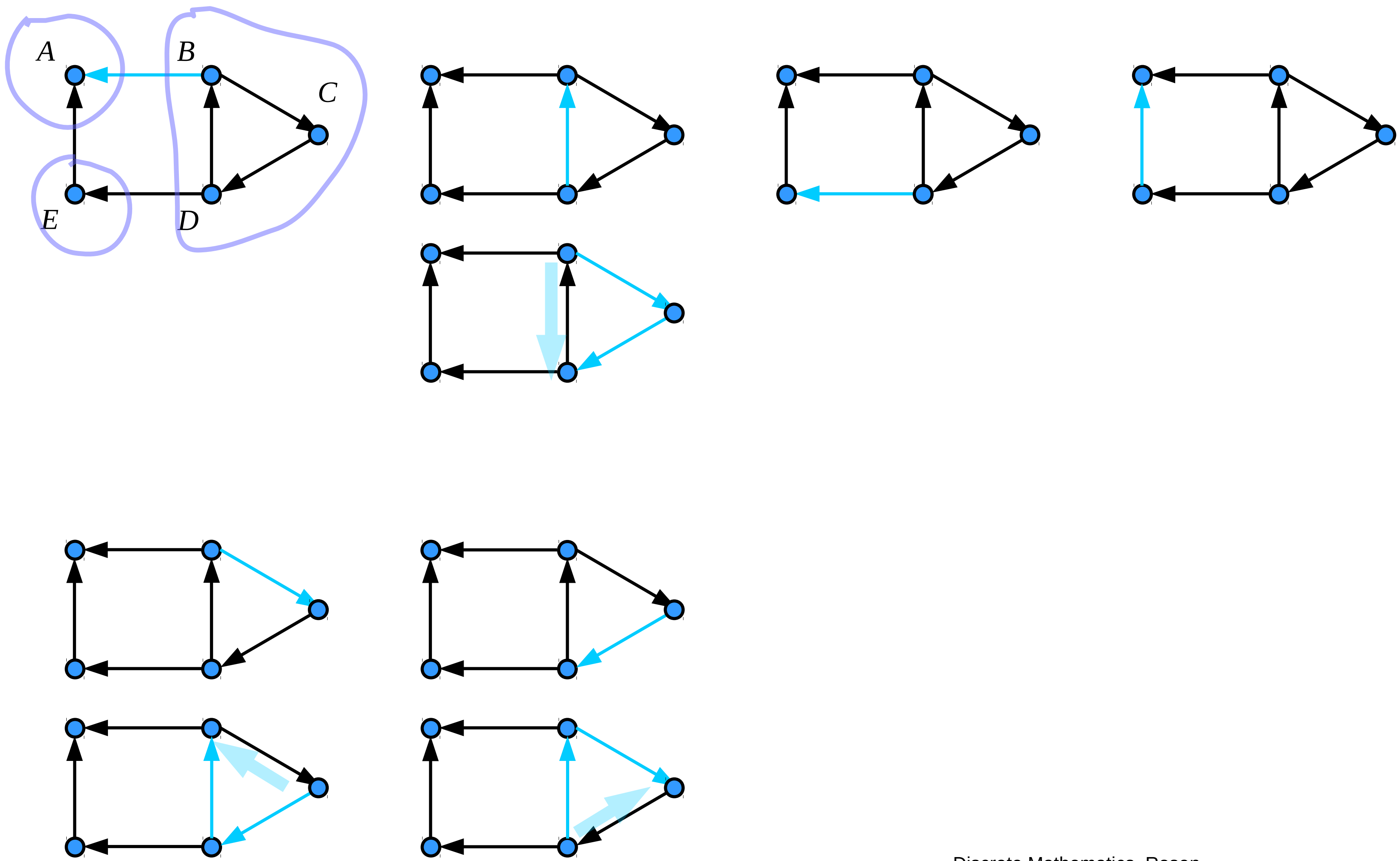
Discrete Mathematics, Rosen

SC examples (1)



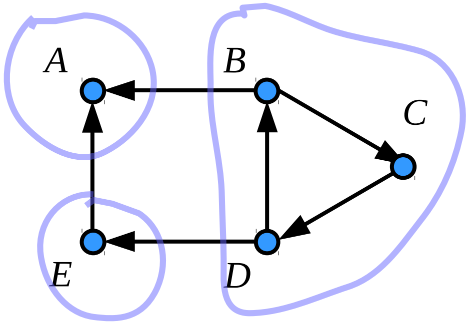
Discrete Mathematics, Rosen

SC examples (2)

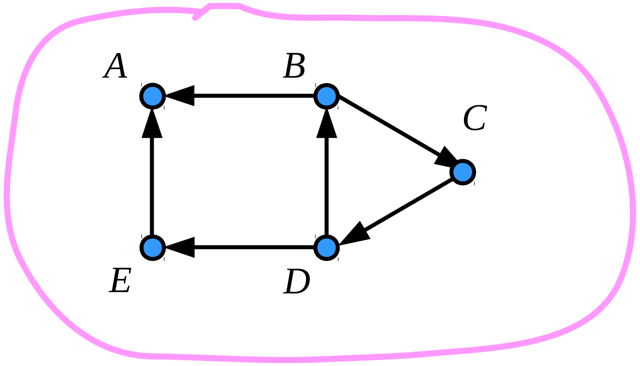


Discrete Mathematics, Rosen

SCC and WCC examples



three strongly connected components



one weakly connected components

Discrete Mathematics, Rosen

References

[1] http://en.wikipedia.org/
[2]